

# A REMARK ON 'SOME NUMERICAL RESULTS IN COMPLEX DIFFERENTIAL GEOMETRY'

KEFENG LIU AND XIAONAN MA

ABSTRACT. In this note we verify certain statement about the operator  $Q_K$  constructed by Donaldson in [3] by using the full asymptotic expansion of Bergman kernel obtained in [2] and [4].

In order to find explicit numerical approximation of Kähler-Einstein metric of projective manifolds, Donaldson introduced in [3] various operators with good properties to approximate classical operators. See the discussions in Section 4.2 of [3] for more details related to our discussion. In this note we verify certain statement of Donaldson about the operator  $Q_K$  in Section 4.2 by using the full asymptotic expansion of Bergman kernel derived in [2, Theorem 4.18] and [4, §3.4]. Such statement is needed for the convergence of the approximation procedure.

Let  $(X, \omega, J)$  be a compact Kähler manifold of  $\dim_{\mathbb{C}} X = n$ , and let  $(L, h^L)$  be a holomorphic Hermitian line bundle on  $X$ . Let  $\nabla^L$  be the holomorphic Hermitian connection on  $(L, h^L)$  with curvature  $R^L$ . We assume that

$$(1) \quad \frac{\sqrt{-1}}{2\pi} R^L = \omega.$$

Let  $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$  be the Riemannian metric on  $TX$  induced by  $\omega, J$ . Let  $dv_X$  be the Riemannian volume form of  $(TX, g^{TX})$ , then  $dv_X = \omega^n/n!$ . Let  $d\nu$  be any volume form on  $X$ . Let  $\eta$  be the positive function on  $X$  defined by

$$(2) \quad dv_X = \eta d\nu.$$

The  $L^2$ -scalar product  $\langle \cdot \rangle_{\nu}$  on  $\mathcal{C}^{\infty}(X, L^p)$ , the space of smooth sections of  $L^p$ , is given by

$$(3) \quad \langle \sigma_1, \sigma_2 \rangle_{\nu} := \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_{L^p} d\nu(x).$$

Let  $P_{\nu,p}(x, x')$  ( $x, x' \in X$ ) be the smooth kernel of the orthogonal projection from  $(\mathcal{C}^{\infty}(X, L^p), \langle \cdot \rangle_{\nu})$  onto  $H^0(X, L^p)$ , the space of the holomorphic sections of  $L^p$  on  $X$ , with respect to  $d\nu(x')$ . Note that  $P_{\nu,p}(x, x') \in L_x^p \otimes L_{x'}^{p*}$ . Following [3, §4], set

$$(4) \quad K_p(x, x') := |P_{\nu,p}(x, x')|_{h_x^p \otimes h_{x'}^{p*}}^2, \quad R_p := (\dim H^0(X, L^p))/\text{Vol}(X, \nu),$$

here  $\text{Vol}(X, \nu) := \int_X d\nu$ . Set  $\text{Vol}(X, dv_X) := \int_X dv_X$ .

Let  $Q_{K_p}$  be the integral operator associated to  $K_p$  which is defined for  $f \in \mathcal{C}^{\infty}(X)$ ,

$$(5) \quad Q_{K_p}(f)(x) := \frac{1}{R_p} \int_X K_p(x, y) f(y) d\nu(y).$$

Let  $\Delta$  be the (positive) Laplace operator on  $(X, g^{TX})$  acting on the functions on  $X$ . We denote by  $\|\cdot\|_{L^2}$  the  $L^2$ -norm on the function on  $X$  with respect to  $dv_X$ .

**Theorem 1.** *There exists a constant  $C > 0$  such that for any  $f \in \mathcal{C}^\infty(X)$ ,  $p \in \mathbb{N}$ ,*

$$(6) \quad \begin{aligned} & \left| \left( Q_{K_p} - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \eta \exp\left(-\frac{\Delta}{4\pi p}\right) \right) f \right|_{L^2} \leq \frac{C}{p} \|f\|_{L^2}, \\ & \left| \left( \frac{\Delta}{p} Q_{K_p} - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \frac{\Delta}{p} \eta \exp\left(-\frac{\Delta}{4\pi p}\right) \right) f \right|_{L^2} \leq \frac{C}{p} \|f\|_{L^2}. \end{aligned}$$

Moreover, (6) is uniform in that there is an integer  $s$  such that if all data  $h^L$ ,  $dv$  run over a set which are bounded in  $\mathcal{C}^s$  and that  $g^{TX}$ ,  $dv_X$  are bounded from below, then the constant  $C$  is independent of  $h^L$ ,  $dv$ .

*Proof.* We explain at first the full asymptotic expansion of  $P_{\nu,p}(x, x')$  from [2, Theorem 4.18'] and [4, §3.4]. For more details on our approach we also refer the readers to the recent book [5].

Let  $E = \mathbb{C}$  be the trivial holomorphic line bundle on  $X$ . Let  $h^E$  the metric on  $E$  defined by  $|1|_{h^E}^2 = 1$ , here 1 is the canonical unity element of  $E$ . We identify canonically  $L^p$  to  $L^p \otimes E$  by Section 1.

As in [4, §3.4], let  $h_\omega^E$  be the metric on  $E$  defined by  $|1|_{h_\omega^E}^2 = \eta^{-1}$ , here 1 is the canonical unity element of  $E$ . Let  $\langle \cdot, \cdot \rangle_\omega$  be the Hermitian product on  $\mathcal{C}^\infty(X, L^p \otimes E) = \mathcal{C}^\infty(X, L^p)$  induced by  $h^L$ ,  $h_\omega^E$ ,  $dv_X$  as in (3). Then by (2),

$$(7) \quad (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega) = (\mathcal{C}^\infty(X, L^p), \langle \cdot, \cdot \rangle_\nu).$$

Observe that  $H^0(X, L^p \otimes E)$  does not depend on  $g^{TX}$ ,  $h^L$  or  $h^E$ . If  $P_{\omega,p}(x, x')$ ,  $(x, x' \in X)$  denotes the smooth kernel of the orthogonal projection  $P_{\omega,p}$  from  $(\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega)$  onto  $H^0(X, L^p \otimes E) = H^0(X, L^p)$  with respect to  $dv_X(x)$ , from (2), as in [4, (3.38)], we have

$$(8) \quad P_{\nu,p}(x, x') = \eta(x') P_{\omega,p}(x, x').$$

For  $f \in \mathcal{C}^\infty(X)$ , set

$$(9) \quad \begin{aligned} K_{\omega,p}(x, x') &= |P_{\omega,p}(x, x')|_{(h^{L^p} \otimes h_\omega^E)_x \otimes (h^{L^p*} \otimes h_\omega^{E*})_{x'}}^2, \\ (K_{\omega,p}f)(x) &= \int_X K_{\omega,p}(x, y) f(y) dv_X(y). \end{aligned}$$

By the definition of the metric  $h^E$ ,  $h_\omega^E$ , if we denote by  $1^*$  the dual of the section 1 of  $E$ , we know

$$(10) \quad 1 = |1 \otimes 1^*|_{h^E \otimes h^{E*}}^2(x, x') = |1 \otimes 1^*|_{h_\omega^E \otimes h_\omega^{E*}}^2(x, x') \eta(x) \eta^{-1}(x').$$

Recall that we identified  $(L^p, h^{L^p})$  to  $(L^p \otimes E, h^{L^p} \otimes h^E)$  by Section 1. Thus from (4), (8) and (10), we get

$$(11) \quad K_p(x, x') = |P_{\nu,p}(x, x')|_{(h^{L^p} \otimes h^E)_x \otimes (h^{L^p*} \otimes h^{E*})_{x'}}^2 = \eta(x) \eta(x') K_{\omega,p}(x, x'),$$

and from (2), (5) and (11),

$$(12) \quad Q_{K_p}(f)(x) = \frac{1}{R_p} \int_X K_{\omega,p}(x, y) \eta(y) f(y) dv_X(y).$$

Now for the kernel  $P_{\omega,p}(x, x')$ , we can apply the full asymptotic expansion [2, Theorem 4.18']. In fact let  $\bar{\partial}^{L^p \otimes E, * \omega}$  be the formal adjoint of the Dolbeault operator  $\bar{\partial}^{L^p \otimes E}$  on the Dolbeault complex  $\Omega^{0,\bullet}(X, L^p \otimes E)$  with the scalar product induced by  $g^{TX}$ ,  $h^L$ ,  $h_\omega^E$ ,  $dv_X$  as in (3), and set

$$(13) \quad D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, * \omega}).$$

Then  $H^0(X, L^p \otimes E) = \text{Ker } D_p$  for  $p$  large enough, and  $D_p$  is a Dirac operator, as  $g^{TX}(\cdot, \cdot) = \omega(\cdot, J \cdot)$  is a Kähler metric on  $TX$ .

Let  $\nabla^E$  be the holomorphic Hermitian connection on  $(E, h_\omega^E)$ . Let  $\nabla^{TX}$  be the Levi-Civita connection on  $(TX, g^{TX})$ . Let  $R^E, R^{TX}$  be the corresponding curvatures.

Let  $a^X$  be the injectivity radius of  $(X, g^{TX})$ . We fix  $\varepsilon \in ]0, a^X/4[$ . We denote by  $B^X(x, \varepsilon)$  and  $B^{T_x X}(0, \varepsilon)$  the open balls in  $X$  and  $T_x X$  with center  $x$  and radius  $\varepsilon$ . We identify  $B^{T_x X}(0, \varepsilon)$  with  $B^X(x, \varepsilon)$  by using the exponential map of  $(X, g^{TX})$ .

We fix  $x_0 \in X$ . For  $Z \in B^{T_{x_0} X}(0, \varepsilon)$  we identify  $(L_Z, h_Z^L)$ ,  $(E_Z, h_Z^E)$  and  $(L^p \otimes E)_Z$  to  $(L_{x_0}, h_{x_0}^L)$ ,  $(E_{x_0}, h_{x_0}^E)$  and  $(L^p \otimes E)_{x_0}$  by parallel transport with respect to the connections  $\nabla^L, \nabla^E$  and  $\nabla^{L^p \otimes E}$  along the curve  $\gamma_Z : [0, 1] \ni u \rightarrow \exp_{x_0}^X(uZ)$ . Then under our identification,  $P_{\omega,p}(Z, Z')$  is a function on  $Z, Z' \in T_{x_0} X$ ,  $|Z|, |Z'| \leq \varepsilon$ , we denote it by  $P_{\omega,p,x_0}(Z, Z')$ . Let  $\pi : TX \times_X TX \rightarrow X$  be the natural projection from the fiberwise product of  $TX$  on  $X$ . Then we can view  $P_{\omega,p,x_0}(Z, Z')$  as a smooth function on  $TX \times_X TX$  (which is defined for  $|Z|, |Z'| \leq \varepsilon$ ) by identifying a section  $S \in \mathcal{C}^\infty(TX \times_X TX, \pi^* \text{End}(E))$  with the family  $(S_x)_{x \in X}$ , where  $S_x = S|_{\pi^{-1}(x)}$ ,  $\text{End}(E) = \mathbb{C}$ .

We choose  $\{w_j\}_{j=1}^n$  an orthonormal basis of  $T_{x_0}^{(1,0)} X$ , then  $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j)$  and  $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j)$ ,  $j = 1, \dots, n$  forms an orthonormal basis of  $T_{x_0} X$ . We use the coordinates on  $T_{x_0} X \simeq \mathbb{R}^{2n}$  where the identification is given by

$$(14) \quad (Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_i Z_i e_i \in T_{x_0} X.$$

In what follows we also introduce the complex coordinates  $z = (z_1, \dots, z_n)$  on  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ . By [2, (4.114)] (cf. [4, (1.91)]), set

$$(15) \quad P^N(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_i (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i)\right).$$

Then  $P^N$  is the classical Bergman kernel on  $\mathbb{C}^n$  (cf. [4, Remark 1.14]) and

$$(16) \quad |P^N(Z, Z')|^2 = e^{-\pi|Z-Z'|^2}.$$

By [2, Proposition 4.1], for any  $l, m \in \mathbb{N}$ ,  $\varepsilon > 0$ , there exists  $C_{l,m,\varepsilon} > 0$  such that for  $p \geq 1$ ,  $x, x' \in X$ ,

$$(17) \quad |P_{\omega,p}(x, x')|_{\mathcal{C}^m(X \times X)} \leq C_{l,m,\varepsilon} p^{-l} \quad \text{if } d(x, x') \geq \varepsilon.$$

Here the  $\mathcal{C}^m$ -norm is induced by  $\nabla^L, \nabla^E, \nabla^{TX}$  and  $h^L, h^E, g^{TX}$ .

By [2, Theorem 4.18'], there exist  $J_r(Z, Z')$  polynomials in  $Z, Z'$ , such that for any  $k, m, m' \in \mathbb{N}$ , there exist  $N \in \mathbb{N}, C > 0, C_0 > 0$  such that for  $\alpha, \alpha' \in \mathbb{N}^n$ ,  $|\alpha| + |\alpha'| \leq m$ ,

$Z, Z' \in T_{x_0}X$ ,  $|Z|, |Z'| \leq \varepsilon$ ,  $x_0 \in X$ ,  $p \geq 1$ ,

$$(18) \quad \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \frac{1}{p^n} P_{\omega,p,x_0}(Z, Z') - \sum_{r=0}^k (J_r P^N)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \right) \right|_{\mathcal{C}^{m'}(X)} \\ \leq C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathcal{O}(p^{-\infty}).$$

Here  $\mathcal{C}^{m'}(X)$  is the  $\mathcal{C}^{m'}$  norm for the parameter  $x_0 \in X$ . The term  $\mathcal{O}(p^{-\infty})$  means that for any  $l, l_1 \in \mathbb{N}$ , there exists  $C_{l,l_1} > 0$  such that its  $\mathcal{C}^{l_1}$ -norm is dominated by  $C_{l,l_1} p^{-l}$ . (In fact, by [2, Theorems 4.6 and 4.17, (4.117)] (cf. [4, Theorem 1.18, (1.31)]), the polynomials  $J_r(Z, Z')$  have the same parity as  $r$  and  $\deg J_r(Z, Z') \leq 3r$ , whose coefficients are polynomials in  $R^{TX}$ ,  $R^E$  and their derivatives of order  $\leq r-1$ ).

Now we claim that in (18),

$$(19) \quad J_0 = 1, \quad J_1(Z, Z') = 0.$$

In fact, let  $dv_{T_{x_0}X}$  be the Riemannian volume form on  $(T_{x_0}X, g^{T_{x_0}X})$ , and  $\kappa$  be the function defined by

$$(20) \quad dv_X(Z) = \kappa(x_0, Z) dv_{T_{x_0}X}(Z).$$

Then (also cf. [4, (1.31)])

$$(21) \quad \kappa(x_0, Z) = 1 + \frac{1}{6} \langle R_{x_0}^{TX}(Z, e_i)Z, e_i \rangle_{x_0} + \mathcal{O}(|Z|^3).$$

As we only work on  $\mathcal{C}^\infty(X, L^p \otimes E)$ , by [2, (4.115)], we get the first equation in (19).

Recall that in the normal coordinate, after the rescaling  $Z \rightarrow Z/t$  with  $t = \frac{1}{\sqrt{p}}$ , we get an operator  $\mathcal{L}_t$  from the restriction of  $D_p^2$  on  $\mathcal{C}^\infty(X, L^p \otimes E)$  which has the following formal expansion (cf. [2, (1.104)], [4, Theorem 1.4]),

$$(22) \quad \mathcal{L}_t = \mathcal{L} + \sum_{r=1}^{\infty} \mathcal{Q}_r t^r.$$

Now, from [2, Theorem 5.1] (or [4, (1.87), (1.97)]),

$$(23) \quad \mathcal{L} = \sum_{j=1}^n (-2 \frac{\partial}{\partial z_i} + \pi \bar{z}_i)(2 \frac{\partial}{\partial \bar{z}_i} + \pi z_i), \quad \mathcal{Q}_1 = 0.$$

(In fact,  $P^N(Z, Z')$  is the smooth kernel of the orthogonal projection from  $L^2(\mathbb{R})$  onto  $\text{Ker}(\mathcal{L})$ ). Thus from [2, (4.107)] (cf. [4, (1.111)]), (21) and (23) we get the second equation of (19).

Note that  $|P_{\omega,p,x_0}(Z, Z')|^2 = P_{\omega,p,x_0}(Z, Z') \overline{P_{\omega,p,x_0}(Z, Z')}$ , thus from (9), (18) and (19), there exist  $J'_r(Z, Z')$  polynomials in  $Z, Z'$  such that

$$(24) \quad \left| \frac{1}{p^{2n+1}} \Delta_Z \left( K_{\omega,p,x_0}(Z, Z') - \left( 1 + \sum_{r=2}^k p^{-r/2} J'_r(\sqrt{p}Z, \sqrt{p}Z') \right) e^{-\pi p|Z-Z'|^2} \right) \right| \\ \leq C p^{-(k+1)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathcal{O}(p^{-\infty}).$$

For a function  $f \in \mathcal{C}^\infty(X)$ , we denote it as  $f(x_0, Z)$  a family (with parameter  $x_0$ ) of function on  $Z$  in the normal coordinate near  $x_0$ . Now, for any polynomial  $Q_{x_0}(Z')$ , we

define the operator

$$(25) \quad (\mathcal{Q}_p f)(x_0) = p^n \int_{|Z'| \leq \varepsilon} Q_{x_0}(\sqrt{p}Z') e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z').$$

Then we observe that there exists  $C_1 > 0$  such that for any  $p \in \mathbb{N}$ ,  $f \in \mathcal{C}^\infty(X)$ , we have

$$(26) \quad |\mathcal{Q}_p f|_{L^2} \leq C_1 |f|_{L^2}.$$

In fact,

$$(27) \quad \begin{aligned} |\mathcal{Q}_p f|_{L^2}^2 &\leq \int_X dv_X(x_0) \left\{ p^n \left( \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} dv_X(x_0, Z') \right) \right. \\ &\quad \left. \times p^n \left( \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \right) \right\} \\ &\leq C' \int_X dv_X(x_0) p^n \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \\ &\leq C_1 |f|_{L^2}^2. \end{aligned}$$

Observe that in the normal coordinate, at  $Z = 0$ ,  $\Delta_Z = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial Z_j^2}$ . Thus

$$(28) \quad (\Delta_Z e^{-\pi p|Z-Z'|^2})|_{Z=0} = 4\pi p(n - \pi p|Z'|^2) e^{-\pi p|Z'|^2}.$$

Thus from (16), (18), (19), (24) and (26), we get

$$(29) \quad \begin{aligned} \left| p^{-n} K_{\omega,p} f - p^n \int_{|Z'| \leq \varepsilon} e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} &\leq \frac{C}{p} |f|_{L^2}, \\ \left| p^{-n-1} \Delta K_{\omega,p} f - 4\pi p^n \int_{|Z'| \leq \varepsilon} (n - \pi p|Z'|^2) e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} &\leq \frac{C}{p} |f|_{L^2}. \end{aligned}$$

Set

$$(30) \quad \begin{aligned} K_{\eta,\omega,p}(x, y) &= \langle d\eta(x), d_x K_{\omega,p}(x, y) \rangle_{g^{T^*X}}, \\ (K_{\eta,\omega,p} f)(x) &= \int_X K_{\eta,\omega,p}(x, y) f(y) dv_X(y). \end{aligned}$$

Then from (18), (19) and (26), we get

$$(31) \quad \left| p^{-n-1} K_{\eta,\omega,p} f - 2\pi p^n \int_{|Z'| \leq \varepsilon} \sum_{i=1}^{2n} \left( \frac{\partial}{\partial Z_i} \eta \right)(x_0, 0) Z'_i e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.$$

Let  $e^{-u\Delta}(x, x')$  be the smooth kernel of the heat operator  $e^{-u\Delta}$  with respect to  $dv_X(x')$ . Let  $d(x, y)$  be the Riemannian distance from  $x$  to  $y$  on  $(X, g^{TX})$ . By the heat kernel expansion in [1, Theorems 2.23, 2.26], there exist  $\Phi_i(x, y)$  smooth functions on  $X \times X$  such that when  $u \rightarrow 0$ , we have the following asymptotic expansion

$$(32) \quad \left| \frac{\partial^l}{\partial u^l} \left( e^{-u\Delta}(x, y) - (4\pi u)^{-n} \sum_{i=0}^k u^i \Phi_i(x, y) e^{-\frac{1}{4u}d(x,y)^2} \right) \right|_{\mathcal{C}^m(X \times X)} = \mathcal{O}(u^{k-n-l-\frac{m}{2}+1}),$$

and

$$(33) \quad \Phi_0(x, y) = 1.$$

If we still use the normal coordinate, then by (32), there exist  $\phi_{i,x_0}(Z') := \Phi_i(0, Z')$  such that uniformly for  $x_0 \in X$ ,  $Z' \in T_{x_0}X$ ,  $|Z'| \leq \varepsilon$ , we have the following asymptotic expansion when  $u \rightarrow 0$ ,

$$(34) \quad \left| \frac{\partial^l}{\partial u^l} \left( e^{-u\Delta}(0, Z') - (4\pi u)^{-n} \left( 1 + \sum_{i=1}^k u^i \phi_{i,x_0}(Z') \right) e^{-\frac{1}{4u}|Z'|^2} \right) \right| = \mathcal{O}(u^{k-n-l+1}),$$

and

$$(35) \quad \left| \langle d\eta(x_0), d_{x_0} e^{-u\Delta} \rangle_{g^{T^*X}}(0, Z') \right. \\ \left. - (4\pi u)^{-n} \sum_{i=1}^{2n} \left( \frac{\partial}{\partial Z_i} \eta \right)(x_0, 0) \frac{Z'_i}{2u} \left( 1 + \sum_{i=1}^k u^i \phi_{i,x_0}(Z') \right) \right. \\ \left. - (4\pi u)^{-n} \sum_{i=1}^k u^i \langle d\eta(x_0), (d_{x_0} \Phi_i)(0, Z') \rangle e^{-\frac{1}{4u}|Z'|^2} \right| = \mathcal{O}(u^{k-n+\frac{1}{2}}).$$

Observe that

$$(36) \quad \frac{1}{p} \Delta \exp \left( -\frac{\Delta}{4\pi p} \right) = -\frac{1}{p} \left( \frac{\partial}{\partial u} e^{-u\Delta} \right) \Big|_{u=\frac{1}{4\pi p}}.$$

Now from (26), (29)–(36), we get

$$(37) \quad \left| \left( p^{-n} K_{\omega,p} - \exp \left( -\frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\ \left| \frac{1}{p} \left( p^{-n} \Delta K_{\omega,p} - \Delta \exp \left( -\frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.$$

and

$$(38) \quad \left| \frac{1}{p} \left( p^{-n} K_{\eta,\omega,p} - \langle d\eta, d \exp \left( -\frac{\Delta}{4\pi p} \right) \rangle \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.$$

Note that

$$(39) \quad (\Delta \eta K_{\omega,p})(x, y) = (\Delta \eta)(x) K_{\omega,p}(x, y) + \eta(x) \Delta_x K_{\omega,p}(x, y) \\ - 2 \langle d\eta(x), d_x K_{\omega,p}(x, x') \rangle_{g^{T^*X}},$$

and  $R_p = \frac{\text{Vol}(X, dv_X)}{\text{Vol}(X, \nu)} p^n + \mathcal{O}(p^{n-1})$ . From (12), (37)–(39), we get (6).

To get the last part of Theorem 1, as we noticed in [2, §4.5], the constants in (18) will be uniformly bounded under our condition, thus we can take  $C$  in (6), (37) and (38) independent of  $h^L$ ,  $dv$ .  $\square$

**Acknowledgments.** We thank Professor Simon Donaldson for useful communications.

## REFERENCES

- [1] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundle Math. Wiss. Band 298, Springer-Verlag, Berlin, 1992.
- [2] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198. The full version: J. Differential Geom. to appear, math.DG/0404494.
- [3] S. K. Donaldson, *Some numerical results in complex differential geometry*, math.DG/0512625.
- [4] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498. The full version: math.DG/0411559.
- [5] ———, *Holomorphic Morse Inequalities and Bergman Kernels*, book in preparation, (2006).

CENTER OF MATHEMATICAL SCIENCE, ZHEJIANG UNIVERSITY AND DEPARTMENT OF MATHEMATICS, UCLA, CA 90095-1555, USA (LIU@MATH.UCLA.EDU)

CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ, UMR 7640 DU CNRS, ECOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE (MA@MATH.POLYTECHNIQUE.FR)