Methods in the Nonlinear Analysis for the Study of Boundary Value Problems

Metode de analiză neliniară în studiul problemelor la limită

Ph.D. Thesis defended by

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at the Babeş–Bolyai University, Cluj–Napoca, 2005

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Contents

Introduction

The strongest explosive is neither toluene nor the atomic bomb, but the human idea.

Grigore Moisil (1906-1973)

Partial differential equations are of crucial importance in the modeling and the description of natural phenomena. Many physical phenomena from fluid dynamics, continuum mechanics, aircraft simulation, computer graphics and weather prediction are modeled by various partial differential equations. The central equations of general relativity and quantum mechanics are also partial differential equations. The motion of planets, computers, electric light, the working of GPS (Global Positioning System) and the changing weather can all be described by differential equations.

The goal of this work is to apply some basic methods of the nonlinear analysis in order to develop a qualitative study of some classes of stationary partial differential equations. Their nonlinearities are essential for a realistic description of several natural questions, such as existence and uniqueness of solutions, asymptotic behaviour, approximation and so on. However, the tools for solving the equations, in particular the numerical tools, are rather general in this work, but they may have future relevance for other applied problems.

We discuss some classes of nonlinear elliptic equations from the perspective of three basic methods: the maximum principle, the calculus of variations, and nonlinear operator theory. Our starting point is related to the Laplace operator, but we emphasize various generalizations of the linear Laplace equation, including linear perturbations of the Laplace operator or quasilinear problems involving variable exponents. That is why we are concerned with classical solutions, but also with weak solutions either in classical Sobolev spaces or in generalized Sobolev spaces (functions spaces with variable exponent endowed with the Luxemburg norm). Our arguments and proofs rely essentially on one of the following basic results in nonlinear analysis:

The Maximum Principle. *Let Ω be a bounded domain in $\mathbb{R}^N$ and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $\Delta u \geq 0$ in Ω. Then*

$$\sup_{x\in\Omega} u(x) = \max_{x\in\partial\Omega} u(x)\,.$$

Moreover, the following alternative holds: either u is constant in $\overline{\Omega}$ or $u < \max_{x\in\partial\Omega} u(x)$ in Ω.

Ekeland's Variational Principle (weak form). *Let X be a Banach space and assume that $F : X \to \mathbb{R}$ is a functional of class C^1 which is bounded from below. Then, for any $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $F(x_\varepsilon) \leq \inf_{x\in X} F(x) + \varepsilon$ and $\|F'(x_\varepsilon)\|_{x\in X^*} \leq \varepsilon$.*

The Mountain Pass Theorem. *Let X be a real Banach space and let $F : X \to \mathbb{R}$ be a C^1–functional. Suppose that F satisfies the Palais-Smale condition: any sequence (u_n) in X such that*

$$\sup_n |F(u_n)| < \infty \qquad \text{and} \qquad \|F'(u_n)\|_{X^*} \to 0$$

has a convergent subsequence.

We also assume that F fulfills the following geometric assumptions:

$$\begin{cases} \exists\, R,\, c_0 > 0 \text{ such that } F(u) \geq c_0,\ \forall\, u \in X \text{ with } \|u\| = R\,; \\ F(0) < c_0 \text{ and there exists } v \in X \text{ such that } \|v\| > R \text{ and } F(v) < c_0. \end{cases}$$

Then the functional F possesses at least a critical value c, characterized by

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} F(p(t))\,,$$

where $\mathcal{P} := \{p \in C([0,1], X);\ p(0) = 0,\ p(1) = v\}$.

The above Palais-Smale "compactness condition" was introduced in [118] and is intensively used in many arguments related to the existence of critical points.

In the last chapter of this work we apply a **nonsmooth** version of the Mountain Pass Theorem (for locally Lipschitz functionals that are **not** necessarily of class C^1) which is due to Chang (see [26]). The name of the above result is a consequence of a simplified visualization for the objects involved in the theorem. Indeed, consider the set $\{0,\, v\}$, where 0 and v are two villages, and the set of all paths joining 0 and v. Then, assuming that $F(u)$ represents the altitude of point u, the hypotheses of the theorem are equivalent to say that the villages 0 and v are separated by a mountains chain. So, the conclusion of the theorem tells us that there exists a path between the villages with a minimal altitude. With other words, there exists a "mountain pass".

In Chapter 4 we apply the following $\mathbb{Z}_2$-symmetric version (that is, for even functionals) of the Mountain Pass Lemma (see Theorem 9.12 in Rabinowitz [126]).

Symmetric Mountain Pass Theorem. *Let X be an infinite dimensional real Banach space and let $F \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition and $F(0) = 0$. Suppose that*

(I1) There exist two constants ρ, $a > 0$ such that $F(x) \geq a$ if $\|x\| = \rho$.

(I2) For each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1;\ F(x) \geq 0\}$ is bounded.

Then F has an unbounded sequence of critical values.

The Saddle Point Theorem. *Let X be a real Banach space and let $F : X \to \mathbb{R}$ be a functional of class C^1 satisfying the Palais-Smale condition. Suppose that $X = V \oplus W$ with $\dim V < \infty$ and, for some $R > 0$,*

$$\max_{v\in V,\, \|v\|=R} F(v) \leq \alpha < \beta \leq \inf_{w\in W} F(w)\,.$$

Then F has a critical value $c \geq \beta$, characterized by

$$c = \inf_{p\in\mathcal{P}} \max_{v\in V,\, \|v\|\leq R} F(p(v))\,,$$

where $\mathcal{P} := \{p \in C(V \cap \bar{B}_R, X);\ p(v) = v,\ \text{for all } v \in \partial B_R\}$.

We also apply several times in the present work the following elementary results.

Hölder's Inequality. *Let p and p' be dual indices, that is, $1/p+1/p'=1$ with $1<p<\infty$. Assume that $f\in L^p(\Omega)$ and $g\in L^{p'}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. Then $fg\in L^1(\Omega)$ and*

$$\left|\int_\Omega f(x)g(x)dx\right|\le\left(\int_\Omega |f(x)|^p dx\right)^{1/p}\cdot\left(\int_\Omega |g(x)|^{p'}dx\right)^{1/p'}.$$

The special case $p=p'=2$ is known as the **Cauchy-Schwarz** *inequality.*

The Sobolev Embedding Theorem. *Let Ω be a bounded open subset of $\mathbb{R}^N$, with a C^1 boundary. Assume that $1\le p<N$ and $u\in W^{1,p}(\Omega)$. Then $u\in L^{p^*}(\Omega)$, where $p^*=Np/(N-p)$. We have in addition the estimate*

$$\|u\|_{L^{p^*}(\Omega)}\le C\,\|u\|_{W^{1,p}(\Omega)},$$

the constant C depending only on p, N and Ω.

*Moreover, $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, for each $1\le q<p^*$ (***Rellich-Kondrashov***).*

Hardy's Inequality. *Assume that $1<p<N$. Then*

$$\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^p}dx\le\frac{p^p}{(N-p)^p}\int_{\mathbb{R}^N}|\nabla u(x)|^p dx$$

for any $u\in W^{1,p}(\mathbb{R}^N)$ such that $u/|x|\in L^p(\mathbb{R}^N)$. Moreover, the constant $p^p(N-p)^{-p}$ is optimal.

Lebesgue's Dominated Convergence Theorem. *Let $f_n:\mathbb{R}^N\to\mathbb{R}$ be a sequence of functions in $L^1(\mathbb{R}^N)$. We assume that*

(i) $f_n(x)\to f(x)$ a.e. in $\mathbb{R}^N$,

(ii) there exists $g\in L^1(\mathbb{R}^N)$ such that, for all $n\ge 1$, $|f_n(x)|\le g(x)$ a.e. in $\mathbb{R}^N$.

Then $f\in L^1(\mathbb{R}^N)$ and $||f_n-f||_{L^1}\to 0$ as $n\to\infty$.

In this thesis we have also applied several results from the monographs Barbu [11], Drabek, Kufner and Nicolosi [42], Evans [51], Gilbarg and Trudinger [62], Hyers, Isac and Rassias [73], Jabri [75], Jebelean [76], Kristály and Varga [88], Kufner and Persson [89], Mawhin and Willem [104], Niculescu [111], O'Regan and Precup [115], Precup [123, 124], Renardy and Rogers [137], Struwe [144], Varga [150], and Willem [151, 152].

This work is divided into five chapters. The first three parts are devoted to the study of several classes of **semilinear** elliptic problems on bounded domains or on the whole space. We are concerned with: (i) classical solutions on the whole space (entire solutions) for nonlinear eigenvalue problems or logistic type equations in anisotropic media; (ii) weak solutions for a subcritical perturbation of a linear eigenvalue problem with sign-changing potential. Chapters 4 and 5 deal essentially with **quasilinear** partial differential equations. Chapter 4 is mainly devoted to the study of some classes of quasilinear eigenvalue problems in Sobolev spaces with variable exponent. In Chapter 5 we establish several existence results for a multivalued Schrödinger equation on the whole spaces and for a Schrödinger elliptic system with discontinuous nonlinearity. Our results in the last chapter extend a theorem a Rabinowitz for a singlevalued Schrödinger equation on $\mathbb{R}^N$. We give in what follows a more precise description of the main results contained in this work.

In Chapters 1 and 4 we are concerned with some classes of nonlinear eigenvalue problems associated to linear or quasilinear elliptic operators. Our interest for *spectral problems* can be motivated by the following quotation of S. H. Gould [65] which asserts, in fact, that the mathematical spectrum is partly made of "eigenvalues", a strange word which has not been immediately adopted: *The concept of an eigenvalue is of great importance in both pure and applied mathematics. The German word "eigen" means "characteristic" and the hybrid word eigenvalue is used for characteristic numbers in order to avoid confusion with the many other uses in English of the word "characteristic". There can be no doubt that "eigenvalue" will soon find its way into the standard dictionaries. The English language has many such hybrids: for example "liverwurst".* We conclude these historical comments with the following deep remarks which are due to M. Zworski [157]: *Eigenvalues describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life [...]. In most situation however, states do not exist for ever, and a more accurate model is given by a decaying state that oscillates at some rate. Eigenvalues are yet another expression of humanity's narcissist desire for immortality.*

Chapter 1 deals with a nonlinear perturbation of the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x) u \quad \text{in } \Omega\,, \\ u \in H_0^1(\Omega), \end{cases} \tag{1}$$

where Ω is an arbitrary open set in $\mathbb{R}^N$, $N \geq 3$. Problems of this type have a long history. If Ω is bounded and $V \equiv 1$, problem (1) is related to the Riesz-Fredholm theory of self-adjoint and compact operators (see, e.g., Theorem VI.11 in Brezis [18]). The case of a non-constant potential V has been first considered in the pioneering papers of Bocher [17], Hess and Kato [72], Minakshisundaran and Pleijel [105] and Pleijel [120]. For instance, Minakshisundaran and Pleijel [105], [120] studied the case where Ω is bounded, $V \in L^\infty(\Omega)$, $V \geq 0$ in Ω and $V > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. An important contribution in the study of (1) if Ω is not necessarily bounded has been given by Szulkin and Willem [146] under some additional assumptions on V. In our framework we assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a Hölder function that satisfies

(V) $V \in L^\infty(\mathbb{R}^N),\ V^+ = V_1 + V_2 \not\equiv 0,\ V_1 \in L^{N/2}(\mathbb{R}^N),\ \lim_{|x|\to\infty} |x|^2 V_2(x) = 0.$

For any $R > 0$, denote $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$ and set

$$\lambda_1(R) = \min\left\{ \int_{B_R} |\nabla u|^2 dx;\ u \in H_0^1(B_R),\ \int_{B_R} V(x) u^2 dx = 1 \right\}.$$

Consequently, the mapping $R \longmapsto \lambda_1(R)$ is decreasing and so, there exists $\Lambda := \lim_{R\to\infty} \lambda_1(R) \geq 0$. We first state a sufficient condition so that Λ is positive. For this aim we impose the additional assumptions

$$\text{there exist } A,\, \alpha > 0 \text{ such that } V^+(x) \leq A|x|^{-2-\alpha}, \quad \text{for all } x \in \mathbb{R}^N \tag{2}$$

and

$$\lim_{x\to 0} |x|^{2(N-1)/N} V_2(x) = 0. \tag{3}$$

Theorem 1. *Assume that V satisfies conditions (V), (2) and (3). Then $\Lambda > 0$.*

The next result asserts that Λ plays a crucial role for the nonlinear eigenvalue logistic problem

$$\begin{cases} -\Delta u = \lambda\,(V(x)u - f(u)) \quad \text{in } \mathbb{R}^N\,,\\ u > 0 \quad \text{in } \mathbb{R}^N\,,\\ \lim_{|x|\to\infty} u(x) = 0\,, \end{cases} \tag{4}$$

where the nonlinear absorption term $f : [0,\infty) \to [0,\infty)$ is a C^1–function such that

$(f1)$ $f(0) = f'(0) = 0$ and $\liminf_{u\searrow 0} \frac{f'(u)}{u} > 0$;

$(f2)$ the mapping $f(u)/u$ is increasing in $(0,+\infty)$;

$(f3)$ $\lim_{u\to+\infty} \frac{f(u)}{u} > \|V\|_{L^\infty}$.

The following existence and non-existence result shows that, in fact, Λ is as a bifurcation point in Problem (4).

Theorem 2. *Assume that V and f satisfy the assumptions (V), (2), $(f1)$, $(f2)$ and $(f3)$.*

Then the following hold:

(i) problem (4) has a unique solution for any $\lambda > \Lambda$;

(ii) problem (4) does not have any solution for all $\lambda \le \Lambda$.

Theorems 1 and 2 are the original results contained in the first chapter of this work. These theorems are included in Rădulescu [129].

In Chapter 2 we first consider the nonlinear problem

$$\begin{cases} -\Delta u = \rho(x) f(u) & \text{in } \mathbb{R}^N\\ u > \ell & \text{in } \mathbb{R}^N\\ u(x) \to \ell & \text{as } |x| \to \infty, \end{cases} \tag{5}$$

where $N \ge 3$, $\ell \ge 0$, $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \ge 0$ and $\rho \not\equiv 0$. The nonlinearity $f : (0,\infty) \to (0,\infty)$ satisfies $f \in C^{0,\alpha}_{\text{loc}}(0,\infty)$ $(0 < \alpha < 1)$ and has a sublinear growth, in the sense that

$(f1)$ the map $u \longmapsto f(u)/u$ is decreasing on $(0,\infty)$ and $\lim_{u\to\infty} f(u)/u = 0$.

We point out that condition (f1) does **not** require that f is smooth at the origin. A model example of such a nonlinearity is $f(u) = u^\alpha$, with $-\infty < \alpha < 1$. This function has a singularity at the origin, provided $\alpha < 0$.

Our study is motivated by the celebrated paper by Brezis and Kamin [19], where it is considered the sublinear elliptic equation *without condition at infinity*

$$-\Delta u = \rho(x)u^\alpha \qquad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{6}$$

with $0 < \alpha < 1$, $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \not\equiv 0$. Brezis and Kamin proved that the *nonlinear* problem (6) has a bounded solution $u > 0$ if and only if the *linear* problem

$$-\Delta u = \rho(x) \qquad \text{in } \mathbb{R}^N$$

has a bounded solution. In this case, Problem (6) has a minimal positive solution and this solution satisfies $\liminf_{|x|\to\infty} u(x) = 0$. Moreover, the minimal solution is the unique positive solution of (6) which tends to zero at infinity. Brezis and Kamin also showed that if the potential $\rho(x)$ decays fast enough at infinity then Problem (6) has a solution and, moreover, such a solution does not exist if $\rho(x)$ has a slow decay at infinity. For instance, if $\rho(x) = (1+|x|^p)^{-1}$, then (6) has a bounded solution if and only if $p > 2$. More generally, Brezis and Kamin have proved that Problem (6) has a bounded solution if and only if $\rho(x)$ is potentially bounded, that is, the mapping $x \longmapsto \int_{\mathbb{R}^N} \rho(y)|x-y|^{2-N}dy$ is in $L^\infty(\mathbb{R}^N)$.

In our first result in Chapter 2 we suppose that the growth at infinity of the anisotropic potential $\rho(x)$ is given by

(ρ1) $\int_0^\infty r\Phi(r)dr < \infty$, where $\Phi(r) := \max_{|x|=r} \rho(x)$.

Entire solutions of (5) decaying to zero at infinity have been studied in Cîrstea and Rădulescu [27] for $\ell = 0$, provided that: ρ satisfies (ρ1), there exists $\beta > 0$ such that the mapping $u \longmapsto f(u)/(u+\beta)$ is decreasing on $(0,\infty)$, $\lim_{u\searrow 0} f(u)/u = +\infty$ and f is bounded in a neighborhood of $+\infty$. Our main purpose in the present chapter is to consider both cases $\ell = 0$ and $\ell > 0$, under the **weaker** assumption that the mapping $u \longmapsto f(u)/u$ is decreasing on $(0,\infty)$. We point out that the assumption "$u \longmapsto f(u)/u$ is decreasing on $(0,\infty)$" has been introduced in Brezis and Oswald [20]. However, their framework is related to **bounded** domains, while our analysis is on the **whole space**.

The next two theorems are the main results contained in Rădulescu [134].

Theorem 3. *Assume that* $\ell > 0$. *Then Problem (5) has a unique classical solution.*

In the case $\ell = 0$ we impose the stronger condition

$(\rho 2)$ $\int_0^\infty r^{N-1}\Phi(r)dr < \infty$.

Additionally, we suppose that

$(f2)$ f is increasing in $(0, \infty)$ and $\lim_{u\searrow 0} f(u)/u = +\infty$.

Theorem 4. *Assume that* $\ell = 0$ *and assumptions* $(\rho 2)$, *(f1) and (f2) are fulfilled. Then Problem (5) has a unique classical solution.*

Our arguments are related to some ideas found in Cîrstea and Rădulescu [27], Edelson [43], Lair and Shaker [91], and Lazer and McKenna [94]. In contrast with the approach on bounded domains (as developed in Brezis and Oswald [20]), our novelties are the following:

(i) Our proofs of existence combine the analysis on bounded domains with a comparison argument, while Brezis and Oswald use minimization techniques.

(ii) Our proofs of uniqueness rely essentially on the maximum principle, while Brezis and Oswald introduce a subtle energy method which is reminiscent of the device used in the theory of monotone operators.

Next, we study the problem

$$\begin{cases} -\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma} & \text{in } \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{7}$$

where $N \geq 3$, $a > 0$, $\gamma > 0$, p, $q \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$, $p > 0$ and $q \geq 0$ in $\mathbb{R}^N$. Set $\Phi(r) := \max_{|x|=r} p(x)$. We impose no growth hypothesis on q but we suppose that p satisfies the following decay condition to zero at infinity:

$$\int_0^\infty r\Phi(r)dr < \infty.$$

In particular, potentials $p(x)$ which behave like $|x|^{-\alpha}$ as $|x| \to \infty$, with $\alpha > 2$, satisfy this assumption.

We prove the following theorem which is the main result in Rădulescu [130].

Theorem 5. *Under the above hypotheses, the problem (7) has a unique classical solution.*

In Chapter 3 we study a boundary value problem related to the linear eigenvalue problem (1), but under different hypotheses on the sign-changing potential $V(x)$. More precisely, we assume that

$$(H) \qquad V^+ \not\equiv 0 \text{ and } V \in L^s(\Omega)\,,$$

where $s > N/2$ if $N \geq 2$ and $s = 1$ if $N = 1$. As usually, we have denoted $V^+(x) = \max\{V(x), 0\}$. Obviously, $V = V^+ - V^-$, where $V^-(x) = \max\{-V(x), 0\}$.

In order to study the main properties (isolation, simplicity) of the principal eigenvalue of (1), Cuesta [32] proved that the minimization problem

$$\min\left\{\int_\Omega |\nabla u|^2 dx;\ u \in H_0^1(\Omega),\ \int_\Omega V(x)u^2 dx = 1\right\}$$

has a positive solution $\varphi_1 = \varphi_1(\Omega)$ which is an eigenfunction of (1) corresponding to the eigenvalue $\lambda_1 := \lambda_1(\Omega) = \int_\Omega |\nabla \varphi_1|^2 dx$.

In Chapter 3 we consider the perturbed nonlinear boundary value problem

$$\begin{cases} -\Delta u = \lambda_1 V(x)u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{8}$$

After multiplication with φ_1 and integration by parts we obtain that problem (8) does not have any solution if g has a constant sign in Ω. Our main purpose in Chapter 3 is to provide sufficient conditions in order to obtain at least one solution. The original results are stated in Theorems 6 and 7 and are contained in Rădulescu [131]. Throughout this chapter we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $g(x, 0) = 0$ and with subcritical growth, that is,

$$|g(x,s)| \le a_0 \cdot |s|^{r-1} + b_0, \qquad \text{for all } s\in\mathbb{R}, \quad \text{a.e. } x\in\Omega,$$

for some constants a_0, $b_0 > 0$, where $2 \le r < 2^*$. We recall that 2^* denotes the critical Sobolev exponent, that is, $2^* := \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = +\infty$ if $N \in \{1,2\}$. Set $G(x,s) := \int_0^s g(x,t)dt$.

In the next two theorems, we prove the existence of a solution under the following assumptions on the potential G:

$(G_1)_q$ $$\limsup_{|s|\to\infty} \frac{G(x,s)}{|s|^q} \le b < \infty \quad \text{uniformly a.e. } x\in\Omega\,,\ q>2;$$

$(G_2^+)_\mu$ $$\liminf_{|s|\to\infty} \frac{g(x,s)s - 2G(x,s)}{|s|^\mu} \ge a > 0 \quad \text{uniformly a.e. } x\in\Omega;$$

$(G_2^-)_\mu$ $$\limsup_{|s|\to\infty} \frac{g(x,s)s - 2G(x,s)}{|s|^\mu} \le -a < 0 \quad \text{uniformly a.e. } x\in\Omega\,.$$

Theorem 6. *Assume that G satisfies conditions $(G_1)_q$, $(G_2^+)_\mu$ [or $(G_2^-)_\mu$] and*

(G_3)
$$\limsup_{s\to 0} \frac{2G(x,s)}{s^2} \le \alpha < \lambda_1 < \beta \le \liminf_{|s|\to\infty} \frac{2G(x,s)}{s^2} \quad \textit{uniformly a.e. } x\in\Omega\,,$$

with $\mu > 2N/(q-2)$ if $N \ge 3$ or $\mu > q-2$ if $1 \le N \le 2$. Then Problem (8) has at least one solution.

Theorem 7. *Assume that $G(x,s)$ satisfies $(G_2^-)_\mu$ [or $(G_2^+)_\mu$], for some $\mu > 0$, and*

(G_4) $$\lim_{|s|\to\infty} \frac{G(x,s)}{s^2} = 0 \quad \textit{uniformly a.e. } x\in\Omega\,.$$

Then Problem (8) has at least one solution.

A pioneering result of Ambrosetti and Rabinowitz [5] asserts that the semilinear boundary value problem

$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

has at least a nontrivial solution in $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2 < p < 2N/(N-2)$ if $N \geq 3$ and $p \in (2, \infty)$ if $N = 1$ or $N = 2$. The proof relies on the Mountain Pass Theorem.

This equation is called the *Kazdan–Warner equation* and the existence results are related not only to the values of p, but also to the geometry of Ω. For instance, problem (9) has no solution if $p \geq 2N/(N-2)$ and if Ω is a *starshaped domain* with respect to a certain point (the proof uses the *Pohozaev identity*, which is obtained after multiplication in (9) with $x \cdot \nabla u$ and integration by parts). If Ω is **not** starshaped, Kazdan and Warner proved in [82] that problem (9) has a solution for **any** $p > 2$, where Ω is an **annulus** in $\mathbb{R}^N$.

Under the same assumptions on p, similar arguments based on the Mountain Pass Theorem show that the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution for any $\lambda < \lambda_1$, where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. Moreover, by multiplication with φ_1 and integration on Ω we deduce that there is no solution if $\lambda \geq \lambda_1$, where φ_1 stands for the first eigenfunction of the Laplace operator. Our first purpose in Chapter 4 is to study a related problem, but for a more general differential operator, the so-called $p(x)$–Laplace operator. This degenerate differential operator is defined by $\Delta_{p(x)} u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ (where $p(x)$ is a certain function whose properties will be stated in what follows) and that generalizes the celebrated p–Laplace operator, defined by $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a constant. The $p(x)$–Laplace operator possesses more complicated nonlinearity than the p–Laplacian, for example, it is inhomogeneous.

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary. Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue and Sobolev spaces

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\}\,.$$

On these spaces we define, respectively, the following norms

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\} \qquad \text{(called Luxemburg norm)}$$

and

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}\,.$$

Consider the boundary value problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda u^{p(x)-1} + u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \ge 0,\ u \not\equiv 0 & \text{in } \Omega\,, \end{cases} \tag{10}$$

where λ, $q \in \mathbb{R}$ and $p \in C_+(\overline{\Omega})$ such that $p^+ < N$.

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a solution of Problem (10) if $u \ge 0$, $u \not\equiv 0$ in Ω and

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \lambda \int_\Omega u^{p(x)-1} v dx + \int_\Omega u^{q-1} v dx, \qquad \forall v \in W_0^{1,p(x)}(\Omega)\,.$$

A crucial role in the statement of the next result is played by the nonlinear eigenvalue problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{11}$$

Let Λ denote the set of eigenvalues of (11), that is,

$$\Lambda = \Lambda_{p(x)} = \{\lambda \in \mathbb{R};\ \lambda \text{ is an eigenvalue of Problem (11)}\}\,.$$

Set

$$\lambda^* = \lambda^*_{p(x)} = \inf \Lambda \,.$$

The following existence property is the main result contained in Rădulescu [132].

Theorem 8. *Assume that $\lambda < \lambda^*$ and $p^+ < q < Np^-/(N-p^-)$. Then Problem (10) has at least a solution.*

Next, we consider the problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = -\lambda |u|^{m(x)-2}u + |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega\,, \end{cases} \tag{12}$$

where

$$m(x) := \max\{p_1(x), p_2(x)\} < q(x) < \begin{cases} \dfrac{N \cdot m(x)}{N - m(x)} & \text{if } m(x) < N \\ +\infty & \text{if } m(x) \geq N\,, \end{cases}$$

for any $x \in \overline{\Omega}$ and all $\lambda > 0$.

Under these assumptions, we prove the following multiplicity result which is contained in Rădulescu [133].

Theorem 9. *For every $\lambda > 0$ problem (12) has infinitely many weak solutions, provided that $2 \leq p_i^-$ for $i \in \{1,2\}$, $m^+ < q^-$ and $q^+ < \frac{N \cdot m^-}{N - m^-}$.*

Our next result asserts that we can not expect to obtain infinitely many solutions, provided the signs are reversed in the right hand-side of (12). Indeed, consider the problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = \lambda |u|^{m(x)-2}u - |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{13}$$

We prove (see Rădulescu [133])

Theorem 10. *There exists $\lambda^\star > 0$ such that for any $\lambda \geq \lambda^\star$, problem (13) has a nontrivial weak solution, provided that $m^+ < q^-$ and $q^+ < \frac{N \cdot m^-}{N - m^-}$.*

There are strong similarities but also differences between problems (12) and (13). We first observe that the signs are reversed in the right hand-sides. Next, Problem (12) admits infinitely many solutions for **any** $\lambda > 0$. In contrast, Problem (13) admits **at least** one solution, provided λ is **sufficiently large**.

In a celebrated paper, Rabinowitz [126] studied the semilinear elliptic equation

$$-\Delta u + a(x)u = f(x,u) \qquad \text{in } \mathbb{R}^N .$$

Rabinowitz proved the existence of a ground-state solution (mountain-pass solution), under suitable conditions on a and assuming that f is smooth, superlinear and subcritical. Our purpose in the last chapter is to provide two generalizations of this result. We are first concerned with the multivalued problem

$$\begin{cases} -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) + a(x)|u|^{p(x)-2}u \in [\underline{f}(x,u), \overline{f}(x,u)] & \text{in } \mathbb{R}^N \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{14}$$

where

$$\underline{f}(x,t) := \lim_{\varepsilon\searrow 0} \text{ essinf } \{f(x,s);\ |t-s| < \varepsilon\}\,;$$

$$\overline{f}(x,t) = \lim_{\varepsilon\searrow 0} \text{ esssup } \{f(x,s);\ |t-s| < \varepsilon\}\,.$$

We assume that $a \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a variable potential such that, for some $a_0 > 0$,

$$a(x) \geq a_0 \qquad \text{a.e. } x \in \mathbb{R}^N \quad \text{ and } \quad \text{ess} \lim_{|x|\to\infty} a(x) = +\infty\,. \tag{15}$$

In (14) we suppose that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that, for some $C > 0$, $q \in \mathbb{R}$ with $p^+ < q+1 \leq Np^-/(N-p^-)$ if $p^- < N$ and $p^+ < q+1 < +\infty$ if $p^- \geq N$, and $\mu > p^+$, we have

$$|f(x,t)| \leq C(|t| + |t|^q) \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}\,; \tag{16}$$

$$\lim_{\varepsilon\searrow 0} \text{ esssup } \left\{\left|\frac{f(x,t)}{t^{p^+-1}}\right|;\ (x,t) \in \mathbb{R}^N \times (-\varepsilon,\varepsilon)\right\} = 0\,; \tag{17}$$

$$0 \leq \mu \int_0^t f(x,s)ds \leq t\underline{f}(x,t) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0,+\infty)\,. \tag{18}$$

The following original result is contained in Rădulescu [135].

Theorem 11. *Assume that hypotheses (15)–(18) are fulfilled. Then Problem (14) has at least one solution.*

Next, we consider the multivalued system

$$\begin{cases} -\Delta u_1 + a(x)u_1 \in [\underline{f}(x,u_1(x),u_2(x)), \overline{f}(x,u_1(x),u_2(x))] & \text{a.e. } x \in \mathbb{R}^N \\ -\Delta u_2 + b(x)u_2 \in [\underline{g}(x,u_1(x),u_2(x)), \overline{g}(x,u_1(x),u_2(x))] & \text{a.e. } x \in \mathbb{R}^N\,, \end{cases} \tag{19}$$

where $a(x) \geq \underline{a} > 0$, $b(x) \geq \underline{b} > 0$. We assume that $f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are nontrivial measurable functions satisfying the following hypotheses:

$$\begin{cases} |f(x,t)| \leq C(|t| + |t|^p) \text{ for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2 \\ |g(x,t)| \leq C(|t| + |t|^p) \text{ for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2\,, \end{cases} \tag{20}$$

where $p < 2^*$;

$$\begin{cases} \lim\limits_{\delta\to 0} \operatorname{esssup}\left\{\dfrac{|f(x,t)|}{|t|};\ (x,t) \in \mathbb{R}^N \times (-\delta,+\delta)^2\right\} = 0 \\ \lim\limits_{\delta\to 0} \operatorname{esssup}\left\{\dfrac{|g(x,t)|}{|t|};\ (x,t) \in \mathbb{R}^N \times (-\delta,+\delta)^2\right\} = 0. \end{cases} \tag{21}$$

We suppose that the mapping $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,t_1,t_2) := \int_0^{t_1} f(x,\tau,t_2)\,d\tau + \int_0^{t_2} g(x,0,\tau)\,d\tau$ satisfies

$$\begin{cases} F(x,t_1,t_2) = \displaystyle\int_0^{t_2} g(x,t_1,\tau)\,d\tau + \int_0^{t_1} f(x,\tau,0)\,d\tau \\ \text{and } F(x,t_1,t_2) = 0 \text{ if and only if } t_1 = t_2 = 0; \end{cases} \tag{22}$$

there exists $\mu > 2$ such that for any $x \in \mathbb{R}^N$

$$0 \leq \mu F(x,t_1,t_2) \leq \begin{cases} t_1\underline{f}(x,t_1,t_2) + t_2\underline{g}(x,t_1,t_2);\ t_1,\ t_2 \geq 0 \\ t_1\underline{f}(x,t_1,t_2) + t_2\overline{g}(x,t_1,t_2);\ t_1 \geq 0,\ t_2 \leq 0 \\ t_1\overline{f}(x,t_1,t_2) + t_2\overline{g}(x,t_1,t_2);\ t_1,\ t_2 \leq 0 \\ t_1\overline{f}(x,t_1,t_2) + t_2\underline{g}(x,t_1,t_2);\ t_1 \leq 0,\ t_2 \geq 0\,. \end{cases} \tag{23}$$

We prove the following existence result which is contained in Rădulescu [136].

Theorem 12. *Assume that conditions (20)-(23) are fulfilled. Then Problem (19) has at least a nontrivial solution.*

It is a big pleasure to thank Professor Radu Precup for his constant support and for his high level guidance during the preparation of this dissertation.

I would like to express my sincere thanks to Professor Ioan Vrabie, Professor Csaba Varga, and Professor Petru Jebelean, who kindly accepted to act as referees for this Ph. D. thesis. I am particularly indebted to Professor Ioan Vrabie for the very careful reading of this work and for his remarks which substantially improved a preliminary version of my thesis.

December 9, 2005

Chapter 1

Entire solutions of nonlinear eigenvalue problems

Data aequatione quotcunque
fluentes quantitae involvente
fluxiones invenire et vice versa.
[“It is useful to differentiate
functions and to solve
differential equations.”]

Sir Isaac Newton to Leibniz,
1676

Abstract. In this chapter we are concerned with positive solutions decaying to zero at infinity for the logistic equation $-\Delta u = \lambda\,(V(x)u - f(u))$ in $\mathbb{R}^N$, where $V(x)$ is a variable potential that may change sign, λ is a real parameter, and f is an absorption term such that the mapping $f(t)/t$ is increasing in $(0,\infty)$. We prove that there exists a bifurcation non-negative number Λ such that the above problem has exactly one solution if $\lambda > \Lambda$, but no such a solution exists provided $\lambda \leq \Lambda$.

1.1 A class of nonlinear eigenvalue logistic problems with sign-changing potential and absorption

In this chapter we are concerned with the existence, uniqueness or the non-existence of positive solutions of the eigenvalue logistic problem with absorption

$$-\Delta u = \lambda\,(V(x)u - f(u)) \qquad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{1.1}$$

where V is a smooth sign-changing potential and $f : [0, \infty) \to [0, \infty)$ is a smooth function. Equations of this type arise in the study of population dynamics. In this case, the unknown u corresponds to the density of a population, the potential V describes the birth rate of the population, while the term $-f(u)$ in (1.1) signifies the fact that the population is self-limiting. In the region where V is positive (resp., negative) the population has positive (resp., negative) birth rate. Since u describes a population density, we are interested in investigating only positive solutions of problem (1.1).

Our results are related to a certain linear eigenvalue problem. We recall in what follows the results that we need in the sequel. Let Ω be an arbitrary open set in $\mathbb{R}^N$, $N \geq 3$. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x) u & \text{in } \Omega\,, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

Problems of this type have a long history. If Ω is bounded and $V \equiv 1$, problem (1.2) is related to the Riesz-Fredholm theory of self-adjoint and compact operators (see, e.g., Theorem VI.11 in Brezis [18]). The case of a non-constant potential V has been first considered in the pioneering papers of Bocher [17], Hess and Kato [72], Minakshisundaran and Pleijel [105] and Pleijel [120]. For instance, Minakshisundaran and Pleijel [105], [120] studied the case where Ω is bounded, $V \in L^\infty(\Omega)$, $V \geq 0$ in Ω and $V > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. An important contribution in the study of (1.2) if Ω is not necessarily bounded has been given by Szulkin and Willem [146] under the assumption that the sign-changing potential V satisfies

$$(H) \qquad \begin{cases} V \in L^1_{\text{loc}}(\Omega),\ V^+ = V_1 + V_2 \neq 0,\ V_1 \in L^{N/2}(\Omega), \\ \displaystyle\lim_{\substack{x \to y \\ x \in \Omega}} |x - y|^2 V_2(x) = 0 \text{ for every } y \in \overline{\Omega},\ \lim_{\substack{|x| \to \infty \\ x \in \Omega}} |x|^2 V_2(x) = 0. \end{cases}$$

We have denoted $V^+(x) = \max\{V(x), 0\}$. Obviously, $V = V^+ - V^-$, where $V^-(x) = \max\{-V(x), 0\}$.

In order to find the principal eigenvalue of (1.2), Szulkin and Willem [146] proved that the minimization problem

$$\min\left\{\int_\Omega |\nabla u|^2 dx;\ u \in H_0^1(\Omega),\ \int_\Omega V(x) u^2 dx = 1\right\}$$

has a solution $\varphi_1 = \varphi_1(\Omega) \geq 0$ which is an eigenfunction of (1.2) corresponding to the eigenvalue $\lambda_1(\Omega) = \int_\Omega |\nabla\varphi_1|^2 dx$.

Throughout this chapter the sign-changing potential $V : \mathbb{R}^N \to \mathbb{R}$ is assumed to be a Hölder function that satisfies

$$(V)\;\; V \in L^\infty(\mathbb{R}^N),\; V^+ = V_1 + V_2 \neq 0,\; V_1 \in L^{N/2}(\mathbb{R}^N),\; \lim_{|x|\to\infty} |x|^2 V_2(x) = 0.$$

We suppose that the nonlinear absorption term $f : [0,\infty) \to [0,\infty)$ is a C^1–function such that

$(f1)$ $\qquad f(0) = f'(0) = 0$ and $\liminf_{u\searrow 0} \dfrac{f'(u)}{u} > 0$;

$(f2)$ $\qquad$ the mapping $f(u)/u$ is increasing in $(0,+\infty)$.

This assumption implies $\lim_{u\to+\infty} f(u) = +\infty$. We impose that f does not have a sublinear growth at infinity. More precisely, we assume

$(f3)$ $\qquad \lim_{u\to+\infty} \dfrac{f(u)}{u} > \|V\|_{L^\infty}$.

Our framework includes the following cases: (i) $f(u) = u^2$ that corresponds to the Fisher equation (see Fisher [57]) and the Kolmogoroff-Petrovsky-Piscounoff equation [83] (see also Kazdan and Warner [81] for a comprehensive treatment of these equations); (ii) $f(u) = u^{(N+2)/(N-2)}$ (for $N \geq 6$) which is related to the conform scalar curvature equation, cf. Li and Ni [96].

For any $R > 0$, denote $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$ and set

$$\lambda_1(R) = \min\left\{\int_{B_R} |\nabla u|^2 dx;\ u \in H_0^1(B_R),\ \int_{B_R} V(x)u^2 dx = 1\right\}. \qquad (1.3)$$

Consequently, the mapping $R \longmapsto \lambda_1(R)$ is decreasing and so, there exists

$$\Lambda := \lim_{R\to\infty} \lambda_1(R) \geq 0\,.$$

We first state a sufficient condition so that Λ is positive. For this aim we impose the additional assumptions

$$\text{there exist } A,\, \alpha > 0 \text{ such that } V^+(x) \leq A|x|^{-2-\alpha}, \quad \text{for all } x \in \mathbb{R}^N \qquad (1.4)$$

and

$$\lim_{x\to 0} |x|^{2(N-1)/N} V_2(x) = 0. \qquad (1.5)$$

Theorem 1. *Assume that V satisfies conditions (V), (1.4) and (1.5).*

Then $\Lambda > 0$.

Our main result asserts that Λ plays a crucial role for the nonlinear eigenvalue logistic problem

$$\begin{cases} -\Delta u = \lambda\,(V(x)u - f(u)) \quad \text{in } \mathbb{R}^N\,, \\ u > 0 \quad \text{in } \mathbb{R}^N\,, \\ \lim_{|x|\to\infty} u(x) = 0\,. \end{cases} \tag{1.6}$$

The following existence and non-existence result shows that Λ serves as a bifurcation point in our problem (1.6).

Theorem 2. *Assume that V and f satisfy the assumptions (V), (1.4), $(f1)$, $(f2)$ and $(f3)$.*

Then the following hold:

(i) problem (1.6) has a unique solution for any $\lambda > \Lambda$;

(ii) problem (1.6) does not have any solution for all $\lambda \leq \Lambda$.

The additional condition (1.4) implies that $V^+ \in L^{N/2}(\mathbb{R}^N)$, which does not follow from the basic hypothesis (V). As we shall see in the next section, this growth assumption is essential in order to establish the existence of positive solutions of (1.1) *decaying to zero* at infinity.

In particular, Theorem 2 shows that if $V(x) < 0$ for sufficiently large $|x|$ (that is, if the population has negative birth rate) then any positive solution (that is, the population density) of (1.1) tends to zero as $|x| \to \infty$.

We also refer to the recent papers Alves, Carrião and Miyagaki [4], Ambrosetti and Wang [6], Cabré [21], Dall'Aqua [33], deFigueiredo [35], Delgado and Suárez [36], Grossi, Magrone and Matzeu [68], Kabeya, Yanagida and Yotsutani [78], Oruganti, Shi and Shivaji [117], Shi and Shivaji [143], Taira [147] for further results related to problems of this type.

1.2 Proof of Theorem 1

For any $R > 0$, fix arbitrarily $u \in H_0^1(B_R)$ such that $\int_{B_R} V(x)u^2dx = 1$. We have

$$1 = \int_{B_R} V(x)u^2dx \le \int_{B_R} V^+(x)u^2dx = \int_{B_R} V_1(x)u^2dx + \int_{B_R} V_2(x)u^2dx.$$

Since $V_1 \in L^{N/2}(\mathbb{R}^N)$, using the Cauchy-Schwarz inequality and Sobolev embeddings we obtain

$$\int_{B_R} V_1(x)u^2dx \le \|V_1\|_{L^{N/2}(B_R)}\|u\|^2_{L^{2^*}(B_R)} \le C_1\|V_1\|_{L^{N/2}(\mathbb{R}^N)}\int_{B_R}|\nabla u|^2dx, \tag{1.7}$$

where 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$.

Fix $\epsilon > 0$. By our assumption (V), there exist positive numbers δ, R_1 and R such that $R^{-1} < \delta < R_1 < R$ such that for all $x \in B_R$ satisfying $|x| \ge R_1$ we have

$$|x|^2V_2(x) \le \epsilon\,. \tag{1.8}$$

On the other hand, by (V), for any $x \in B_R$ with $|x| \le \delta$ we have

$$|x|^{2(N-1)/N}V_2(x) \le \epsilon. \tag{1.9}$$

Define $\Omega := \omega_1 \cup \omega_2$, where $\omega_1 := B_R \backslash \overline{B}_{R_1}$, $\omega_2 := B_\delta \backslash \overline{B}_{1/R}$, and $\omega := B_{R_1} \backslash \overline{B}_\delta$.

By (1.8) and Hardy's inequality (see [70]) we find

$$\int_{\omega_1} V_2(x)u^2dx \le \epsilon\int_{\omega_1}\frac{u^2}{|x|^2}dx \le C_2\epsilon\int_{B_R}|\nabla u|^2dx. \tag{1.10}$$

Using now (1.9) and Hölder's inequality we obtain

$$\begin{aligned}\int_{\omega_2} V_2(x)u^2dx &\le \epsilon\int_{\omega_2}\frac{u^2}{|x|^{2(N-1)/N}}dx\\ &\le \epsilon\left[\int_{\omega_2}\left(\frac{1}{|x|^{2(N-1)/N}}dx\right)^{N/2}dx\right]^{2/N}\|u\|^2_{L^{2^*}(B_R)}\\ &\le C\epsilon\left(\int_{1/R}^{\delta}\frac{1}{s^{N-1}}s^{N-1}\omega_N ds\right)^{2/N}\int_{B_R}|\nabla u|^2dx\\ &\le C_3\left(\delta-\frac{1}{R}\right)^{2/N}\int_{B_R}|\nabla u|^2dx.\end{aligned} \tag{1.11}$$

By compactness and our assumption (V), there exists a finite covering of $\overline{\omega}$ by the closed balls $\overline{B}_{r_1}(x_1), ..., \overline{B}_{r_k}(x_k)$ such that, for all $1 \le j \le k$

$$\text{if } \; |x - x_j| \le r_j \; \text{ then } \; |x - x_j|^{2(N-1)/N} V_2(x) \le \epsilon. \tag{1.12}$$

There exists $r > 0$ such that, for any $1 \le j \le k$

$$\text{if } \; |x - x_j| \le r \; \text{ then } \; |x - x_j|^{2(N-1)/N} V_2(x) \le \frac{\epsilon}{k}.$$

Define $A := \cup_{j=1}^{k} B_r(x_j)$. The above estimate, Hölder's inequality and Sobolev embeddings yield

$$\begin{aligned}
\int_{B_r(x_j)} V_2(x) u^2 dx &\le \frac{\epsilon}{k} \int_{B_r(x_j)} \frac{u^2}{|x - x_j|^{2(N-1)/N}} dx \\
&\le \frac{\epsilon}{k} \left[\int_{B_r(x_j)} \left(|x - x_j|^{-2(N-1)/N} \right)^{N/2} dx \right]^{2/N} \|u\|^2_{L^{2^\star}(B_R)} \\
&\le C \frac{\epsilon}{k} \left(\int_{B_r} \frac{1}{|x|^{N-1}} dx \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\
&= C \frac{\epsilon}{k} \left(\int_0^r \frac{1}{s^{N-1}} s^{N-1} \omega_N ds \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\
&= C' \int_{B_R} |\nabla u|^2 dx,
\end{aligned}$$

for any $j = 1, \ldots, k$. By addition we find

$$\int_A V_2(x) u^2 dx \le C_4 \int_{B_R} |\nabla u|^2 dx. \tag{1.13}$$

It follows from (1.12) that $V_2 \in L^\infty(\omega \setminus A)$. Actually, if $x \in \omega \setminus A$ it follows that there exists $j \in \{1, ..., k\}$ such that $r_j > |x - x_j| > r > 0$. Thus,

$$V_2(x) \le r^{-2(N-1)/N} \epsilon \, .$$

Hence

$$\int_{\omega \setminus A} V_2(x) u^2 dx \le \epsilon r^{-2(N-1)/N} \int_{\omega \setminus A} u^2 dx \le C_5 \int_{B_R} |\nabla u|^2 dx. \tag{1.14}$$

Now, by inequalities (1.7), (1.10), (1.11), (1.13) and (1.14) we have

$$\lambda_1(R) \ge \left\{ C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} + C_2 \epsilon + C_3 \left(\delta - R^{-1} \right)^{2/N} + C_4 + C_5 \right\}^{-1}$$

and passing to the limit as $R \to \infty$ we conclude that

$$\Lambda \geq \left(C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} + C_2\epsilon + C_3\delta^{2/N} + C_4 + C_5\right)^{-1} > 0.$$

This completes the proof of Theorem 1. □

1.3 An auxiliary result

We show in this section that the logistic equation (1.1) has entire positive solutions if λ is sufficiently large. However, we are not able to establish that this solution decays to zero at infinity. This will be proved in the next section by means of the additional assumption (1.4). More precisely, we have

Proposition 1. *Assume that the functions V and f satisfy conditions (V), $(f1)$, $(f2)$ and $(f3)$. Then the problem*

$$\begin{cases} -\Delta u = \lambda\,(V(x)u - f(u)) \quad \text{in } \mathbb{R}^N\,, \\ u > 0 \quad \text{in } \mathbb{R}^N \end{cases} \tag{1.15}$$

has at least one solution, for any $\lambda > \Lambda$.

Proof. For any $R > 0$, consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda\,(V(x)u - f(u)) \quad \text{in } B_R\,, \\ u > 0 \quad \text{in } B_R\,, \\ u = 0 \quad \text{on } \partial B_R\,. \end{cases} \tag{1.16}$$

We first prove that problem (1.16) has at least one solution, for any $\lambda > \lambda_1(R)$. Indeed, the function $\overline{u}(x) = M$ is a supersolution of (1.16), for any M large enough. This follows from $(f3)$ and the boundedness of V. Next, in order to find a positive subsolution, we consider the minimization problem

$$\min_{u \in H_0^1(B_R)} \int_{B_R} \left(|\nabla u|^2 - \lambda V(x)u^2\right) dx\,.$$

Since $\lambda > \lambda_1(R)$, it follows that the least eigenvalue μ_1 is negative. Moreover, the corresponding eigenfunction e_1 satisfies

$$\begin{cases} -\Delta e_1 - \lambda V(x) e_1 = \mu_1 e_1 & \text{in } B_R\,, \\ e_1 > 0 & \text{in } B_R\,, \\ e_1 = 0 & \text{on } \partial B_R\,. \end{cases} \tag{1.17}$$

Then the function $\underline{u}(x) = \varepsilon e_1(x)$ is a subsolution of the problem (1.16). Indeed, it is enough to check that

$$-\Delta(\varepsilon e_1) - \lambda \varepsilon V e_1 + \lambda f(\varepsilon e_1) \leq 0 \qquad \text{in } B_R\,,$$

that is, by (1.17),

$$\varepsilon \mu_1 e_1 + \lambda f(\varepsilon e_1) \leq 0 \qquad \text{in } B_R\,. \tag{1.18}$$

But

$$f(\varepsilon e_1) = \varepsilon f'(0) e_1 + \varepsilon e_1 o(1), \qquad \text{as } \varepsilon \to 0.$$

So, since $f'(0) = 0$, relation (1.18) becomes

$$\varepsilon e_1 \left(\mu_1 + o(1)\right) \leq 0$$

which is true, provided $\varepsilon > 0$ is small enough, due to the fact that $\mu_1 < 0$.

Fix $\lambda > \Lambda$ and an arbitrary sequence $R_1 < R_2 < \ldots < R_n < \ldots$ of positive numbers such that $R_n \to \infty$ and $\lambda_1(R_1) < \lambda$. Let u_n be the solution of (1.16) on B_{R_n}. Fix a positive number M such that $f(M)/M > \|V\|_{L^\infty(\mathbb{R}^N)}$. The above arguments show that we can assume $u_n \leq M$ in B_{R_n}, for any $n \geq 1$. Since u_{n+1} is a supersolution of (1.16) for $R = R_n$, we can also assume that $u_n \leq u_{n+1}$ in B_{R_n}. Thus the function $u(x) := \lim_{n\to\infty} u_n(x)$ exists and is well-defined and positive in $\mathbb{R}^N$. Standard elliptic regularity arguments imply that u is a solution of problem (1.15). □

The above result shows the importance of the assumption (1.4) in the statement of Theorem 2. Indeed, assuming that V satisfies only the hypothesis (V), it is not clear whether or not the solution constructed in the proof of Proposition 1 tends to 0 as $|x| \to \infty$. However, it is easy to observe that if $\lambda > \Lambda$ and V satisfies (1.4) then problem (1.6) has at least one solution.

Indeed, we first observe that

$$\underline{u}(x) = \begin{cases} \varepsilon e_1(x), & \text{if } x \in B_R \\ 0, & \text{if } x \notin B_R \end{cases} \tag{1.19}$$

is a subsolution of problem (1.6), for some fixed $R > 0$, where e_1 satisfies (1.17). Next, we observe that $\overline{u}(x) = n/(1+|x|^2)$ is a supersolution of (1.6). Indeed, $\overline{u}$ satisfies

$$-\Delta \overline{u}(x) = \frac{2[n(1+|x|^2) - 4|x|^2]}{(1+|x|^2)^2} u(x), \qquad x \in \mathbb{R}^N.$$

It follows that $\overline{u}$ is a supersolution of (1.6) provided

$$\frac{2[n(1+|x|^2) - 4|x|^2]}{(1+|x|^2)^2} \geq \lambda V(x) - \lambda f\left(\frac{n}{1+|x|^2}\right), \qquad x \in \mathbb{R}^N.$$

This inequality follows from $(f3)$ and (1.4), provided that n is large enough.

1.4 Proof of Theorem 2

We split the proof of our main result into several steps. We will assume the conditions (V), (1.4), $(f1$-$f3)$ are satisfied by V, f throughout this section.

Proposition 2. *Let u be an arbitrary solution of problem (1.6). Then there exists $C > 0$ such that $|u(x)| \leq C|x|^{2-N}$ for all $x \in \mathbb{R}^N$.*

Proof. Let ω_N be the surface area of the unit sphere in $\mathbb{R}^N$. Consider the function V^+u as a Newtonian potential and define

$$v(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{V^+(y)u(y)}{|x-y|^{N-2}} dy.$$

A straightforward computation shows that

$$-\Delta v = V^+(x)u \qquad \text{in } \mathbb{R}^N. \tag{1.20}$$

But, by (1.4) and since u is bounded,

$$V^+(y)u(y) \leq C|y|^{-2-\alpha}, \qquad \text{for all } y \in \mathbb{R}^N.$$

So, by Lemma 2.3 in Li and Ni [96],

$$v(x) \le C|x|^{-\alpha}, \qquad \text{for all } x \in \mathbb{R}^N,$$

provided that $\alpha < N-2$. Set $w(x) = Cv(x) - u(x)$. Hence $w(x) \to 0$ as $|x| \to \infty$. Let us choose C sufficiently large so that $w(0) > 0$. We claim that this implies

$$w(x) > 0, \qquad \text{for all } x \in \mathbb{R}^N. \tag{1.21}$$

Indeed, if not, let $x_0 \in \mathbb{R}^N$ be a local minimum point of w. This means that $w(x_0) < 0$, $\nabla w(x_0) = 0$ and $\Delta w(x_0) \ge 0$. But

$$\Delta w(x_0) = -CV^+(x_0)u(x_0) + \lambda\left(V(x_0)u(x_0) - f(u(x_0))\right) < 0,$$

provided that $C > \lambda$. This contradiction implies (1.21). Consequently,

$$u(x) \le Cv(x) \le C|x|^{-\alpha}, \qquad \text{for any } x \in \mathbb{R}^N.$$

So, using again (1.4),

$$V^+(x)u(x) \le C|x|^{-2-2\alpha}, \qquad \text{for all } x \in \mathbb{R}^N.$$

Lemma 2.3 in Li and Ni [96] yields the improved estimate

$$v(x) \le C|x|^{-2\alpha}, \qquad \text{for all } x \in \mathbb{R}^N,$$

provided that $2\alpha < N-2$, and so on. Let n_α be the largest integer such that $n_\alpha \alpha < N-2$. Repeating $n_\alpha + 1$ times the above argument based on Lemma 2.3 (i) and (iii) in Li and Ni [96] we obtain

$$u(x) \le C|x|^{2-N}, \qquad \text{for all } x \in \mathbb{R}^N.$$

□

Proposition 3. *Let u be a solution of problem (1.6). Then V^+u, V^-u, $f(u) \in L^1(\mathbb{R}^N)$, and $u \in H^1(\mathbb{R}^N)$.*

Proof. For any $R > 0$ consider the average function

$$\overline{u}(R) = \frac{1}{\omega_N R^{N-1}} \int_{\partial B_R} u(x)d\sigma = \frac{1}{\omega_N} \int_{\partial B_1} u(rx)d\sigma,$$

where ω_N denotes the surface area of S^{N-1}. Then

$$\overline{u}'(R) = \frac{1}{\omega_N}\int_{\partial B_1}\frac{\partial u}{\partial \nu}(rx)d\sigma = \frac{1}{\omega_N R^{N-1}}\int_{\partial B_R}\frac{\partial u}{\partial \nu}(x)d\sigma = \frac{1}{\omega_N R^{N-1}}\int_{B_R}\Delta u(x)dx.$$

Hence

$$\begin{aligned}\omega_N R^{N-1}\overline{u}'(R) = \quad & -\lambda\int_{B_R}(V(x)u - f(u))\,dx = \\ & -\lambda\int_{B_R}V^+(x)udx + \lambda\int_{B_R}\big(V^-(x)u + f(u)\big)\,dx.\end{aligned} \tag{1.22}$$

By Proposition 2, there exists $C > 0$ such that $|\overline{u}(r)| \le Cr^{-N+2}$, for any $r > 0$. So, by (1.4),

$$\int_{1\le|x|\le r}V^+(x)udx \le CA\int_{1\le|x|\le r}|x|^{-N-\alpha}dx \le C,$$

where C does not depend on r. This implies $V^+u \in L^1(\mathbb{R}^N)$.

By contradiction, assume that $V^-u + f(u) \notin L^1(\mathbb{R}^N)$. So, by (1.22), $\overline{u}'(r) > 0$ if r is sufficiently large. It follows that $\overline{u}(r)$ does not converge to 0 as $r \to \infty$, which contradicts Proposition 2. So, $V^-u + f(u) \in L^1(\mathbb{R}^N)$. Next, in order to establish that $u \in L^2(\mathbb{R}^N)$, we observe that our assumption $(f1)$ implies the existence of some positive numbers a and δ such that $f'(t) > at$, for any $0 < t < \delta$. This implies $f(t) > at^2/2$, for any $0 < t < \delta$. Since u decays to 0 at infinity, it follows that the set $\{x \in \mathbb{R}^N;\ u(x) \ge \delta\}$ is compact. Hence

$$\int_{\mathbb{R}^N}u^2dx = \int_{[u\ge\delta]}u^2dx + \int_{[u<\delta]}u^2dx \le \int_{[u\ge\delta]}u^2dx + \frac{2}{a}\int_{[u<\delta]}f(u)dx < +\infty,$$

since $f(u) \in L^1(\mathbb{R}^N)$.

It remains to prove that $\nabla u \in L^2(\mathbb{R}^N)^N$. We first observe that after multiplication by u in (1.1) and integration we find

$$\int_{B_R}|\nabla u|^2dx - \int_{\partial B_R}u(x)\frac{\partial u}{\partial \nu}(x)d\sigma = \lambda\int_{B_R}(V(x)u - f(u))\,dx,$$

for any $r > 0$. Since $Vu - f(u) \in L^1(\mathbb{R}^N)$, it follows that the left hand-side has a finite limit as $r \to \infty$. Arguing by contradiction and assuming that $\nabla u \notin L^2(\mathbb{R}^N)^N$, it follows that there exists $R_0 > 0$ such that

$$\int_{\partial B_R}u(x)\frac{\partial u}{\partial \nu}(x)d\sigma \ge \frac{1}{2}\int_{B_R}|\nabla u|^2dx, \qquad \text{for any } R \ge R_0. \tag{1.23}$$

Define the functions

$$A(R) = \int_{\partial B_R} u(x)\frac{\partial u}{\partial \nu}(x)d\sigma, \quad B(R) = \int_{\partial B_R} u^2(x)d\sigma, \quad C(R) = \int_{B_R} |\nabla u(x)|^2 dx.$$

Relation (1.23) can be rewritten as

$$A(R) \geq \frac{1}{2}\,C(R), \qquad \text{for any } R \geq R_0. \tag{1.24}$$

On the other hand, by the Cauchy-Schwarz inequality,

$$A^2(R) \leq \left(\int_{\partial B_R} u^2 d\sigma\right)\left(\int_{\partial B_R} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma\right) \leq B(R)C'(R).$$

Using now (1.24) we obtain

$$C'(R) \geq \frac{C^2(R)}{4B(R)}, \qquad \text{for any } \ R \geq R_0.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{4}{C(r)} + \int_0^r \frac{dt}{B(t)}\right]_{r=R} \leq 0, \qquad \text{for any } \ R \geq R_0. \tag{1.25}$$

But, since $u \in L^2(\mathbb{R}^N)$, it follows that $\int_0^\infty B(t)dt$ converges, so

$$\lim_{R\to\infty} \int_0^R \frac{dt}{B(t)} = +\infty. \tag{1.26}$$

On the other hand, our assumption $|\nabla u| \notin L^2(\mathbb{R}^N)$ implies

$$\lim_{R\to\infty} \frac{1}{C(R)} = 0. \tag{1.27}$$

Relations (1.25), (1.26) and (1.27) yield a contradiction, so our proof is complete. □

Proposition 4. *Let u and v be two distinct solutions of problem (1.6). Then*

$$\lim_{R\to\infty} \int_{\partial B_R} u(x)\frac{\partial v}{\partial \nu}(x)d\sigma = 0.$$

Proof. By multiplication with v in (1.6) and integration on B_R we find

$$\int_{B_R} \nabla u \cdot \nabla v dx - \int_{\partial B_R} u\frac{\partial v}{\partial \nu}d\sigma = \lambda \int_{B_R} \left(V(x)uv - f(u)v\right) dx.$$

So, by Proposition 3, there exists and is finite $\lim_{R\to\infty}\int_{\partial B_R} u\frac{\partial v}{\partial\nu}d\sigma$. But, by the Cauchy-Schwarz inequality,

$$\left|\int_{\partial B_R} u\frac{\partial v}{\partial\nu}d\sigma\right| \leq \left(\int_{\partial B_R} u^2 d\sigma\right)^{1/2}\left(\int_{\partial B_R}|\nabla v|^2 d\sigma\right)^{1/2}. \tag{1.28}$$

Since u, $|\nabla v|\in L^2(\mathbb{R}^N)$, it follows that $\int_0^\infty\left(\int_{\partial B_R}(u^2+|\nabla v|^2)d\sigma\right)dx$ is convergent. Hence

$$\lim_{R\to\infty}\int_{\partial B_R}(u^2+|\nabla v|^2)d\sigma=0. \tag{1.29}$$

Our conclusion now follows by (1.28) and (1.29). □

Proof of Theorem 2. (i) The existence of a solution follows with the arguments given in the preceding section. In order to establish the uniqueness, let u and v be two solutions of (1.6). We can assume without loss of generality that $u\leq v$. This follows from the fact that $\overline{u}=\min\{u,v\}$ is a supersolution of (1.6) and $\underline{u}$ defined in (1.19) is an arbitrary small subsolution. So, it sufficient to consider the ordered pair consisting of the corresponding solution and v.

Since u and v are solutions we have, by Green's formula,

$$\int_{\partial B_R}\left(u\frac{\partial v}{\partial\nu}-v\frac{\partial u}{\partial\nu}\right)d\sigma=\lambda\int_{B_R}uv\left(\frac{f(v)}{v}-\frac{f(u)}{u}\right)dx.$$

By Proposition 4, the left hand-side converges to 0 as $R\to\infty$. So, $(f1)$ and our assumption $u\leq v$ force $u=v$ in $\mathbb{R}^N$.

(ii) By contradiction, let $\lambda\leq\Lambda$ be such that problem (1.6) has a solution for this λ. So

$$\int_{B_R}|\nabla u|^2dx-\int_{\partial B_R}u\frac{\partial u}{\partial\nu}d\sigma=\lambda\int_{B_R}\big(V(x)u^2-f(u)u\big)\,dx.$$

By Propositions 3 and 4 and letting $R\to\infty$ we find

$$\int_{\mathbb{R}^N}|\nabla u|^2dx<\lambda\int_{\mathbb{R}^N}V(x)u^2dx. \tag{1.30}$$

On the other hand, using the definition of Λ and (1.3) we obtain

$$\Lambda\int_{\mathbb{R}^N}V\zeta^2dx\leq\int_{\mathbb{R}^N}|\nabla\zeta|^2dx, \tag{1.31}$$

for any $\zeta\in C_0^2(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}V\zeta^2dx>0$.

Fix $\zeta \in C_0^2(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$, $\zeta(x) = 1$ if $|x| \le 1$, and $\zeta(x) = 0$ if $|x| \ge 2$. For any $n \ge 1$ define $\Psi_n(x) = \zeta_n(x)u(x)$, where $\zeta_n(x) = \zeta(|x|/n)$. Thus $\Psi_n(x) \to u(x)$ as $n \to \infty$, for any $x \in \mathbb{R}^N$. Since $u \in H^1(\mathbb{R}^N)$, it follows by Corolarry IX.13 in Brezis [18] that $u \in L^{2N/(N-2)}(\mathbb{R}^N)$. So, the Lebesgue Dominated Convergence Theorem yields

$$\Psi_n \to u \qquad \text{in } L^{2N/(N-2)}(\mathbb{R}^N).$$

We claim that

$$\nabla\Psi_n \to \nabla u \qquad \text{in } L^2(\mathbb{R}^N)^N. \tag{1.32}$$

Indeed, let $\Omega_n := \{x \in \mathbb{R}^N;\ n < |x| < 2n\}$. Applying Hölder's inequality we find

$$\begin{aligned}\|\nabla\Psi_n - \nabla u\|_{L^2(\mathbb{R}^N)} \le \ & \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\nabla\zeta_n\|_{L^2(\Omega_n)} \le \\ & \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^{2N/(N-2)}(\Omega_n)} \cdot \|\nabla\zeta_n\|_{L^N(\mathbb{R}^N)}.\end{aligned} \tag{1.33}$$

But, since $|\nabla u| \in L^2(\mathbb{R}^N)$, it follows by Lebesgue's Dominated Convergence Theorem that

$$\lim_{n\to\infty} \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} = 0. \tag{1.34}$$

Next, we observe that

$$\|\nabla\zeta_n\|_{L^N(\mathbb{R}^N)} = \|\nabla\zeta\|_{L^N(\mathbb{R}^N)}. \tag{1.35}$$

Since $u \in L^{2N/(N-2)}(\mathbb{R}^N)$ then

$$\lim_{n\to\infty} \|u\|_{L^{2N/(N-2)}(\Omega_n)} = 0. \tag{1.36}$$

Relations (1.33)–(1.36) imply our claim (1.32).

Since $V^\pm u^2 \in L^1(\mathbb{R}^N)$ and $V^\pm\Psi_n^2 \le V^\pm u^2$, it follows by Lebesgue's Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V^\pm\Psi_n^2 dx = \int_{\mathbb{R}^N} V^\pm u^2 dx.$$

Consequently

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V\Psi_n^2 dx = \int_{\mathbb{R}^N} V u^2 dx. \tag{1.37}$$

So, by (1.30) and (1.37), it follows that there exists $n_0 \geq 1$ such that

$$\int_{\mathbb{R}^N} V\Psi_n^2 dx > 0, \qquad \text{for any } \; n \geq n_0.$$

This means that we can write (1.31) for ζ replaced by $\Psi_n \in C_0^2(\mathbb{R}^N)$. Using then (1.32) and (1.37) we find

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \Lambda \int_{\mathbb{R}^N} V u^2 dx. \tag{1.38}$$

Relations (1.30) and (1.38) yield a contradiction, so problem (1.6) has no solution if $\lambda \leq \Lambda$. □

Chapter 2

Entire solutions of nonlinear elliptic equations

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
>
> Albert Einstein (1879-1955)

Abstract. In the first part of this chapter we study the nonlinear elliptic problem $-\Delta u = \rho(x)f(u)$ in $\mathbb{R}^N$ $(N \geq 3)$, $\lim_{|x|\to\infty} u(x) = \ell$, where $\ell \geq 0$ is a real number, $\rho(x)$ is a nonnegative potential belonging to a certain Kato class, and $f(u)$ has a sublinear growth. We distinguish the cases $\ell > 0$ and $\ell = 0$ and we prove existence and uniqueness results if the potential $\rho(x)$ decays fast enough at infinity. Our arguments rely on comparison techniques and on a theorem of Brezis and Oswald for sublinear elliptic equations. Next, we consider the Emden-Fowler equation $-\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma}$ in $\mathbb{R}^N$, where a and γ are positive numbers, p and q are locally Hölder functions in $\mathbb{R}^N$, with $p > 0$ and $q \geq 0$. In the last section of this chapter we prove that the above equation has a unique positive solutions decaying to zero at infinity. Our proof is elementary and it combines the maximum principle for elliptic equations with a theorem of Crandall, Rabinowitz and Tartar.

2.1 Entire solutions of sublinear elliptic equations in anisotropic media

In their celebrated paper [19], Brezis and Kamin have been concerned with various questions related to the existence of bounded solutions of the sublinear elliptic equation without condition at infinity

$$-\Delta u = \rho(x)u^{\alpha} \qquad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{2.1}$$

where $0 < \alpha < 1$, $\rho \in L^{\infty}_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \not\equiv 0$. We summarize in what follows the main results obtained in Brezis and Kamin [19]. Brezis and Kamin proved that the *nonlinear* problem (2.1) has a bounded solution $u > 0$ if and only if the *linear* problem

$$-\Delta u = \rho(x) \qquad \text{in } \mathbb{R}^N$$

has a bounded solution. In this case, Problem (2.1) has a minimal positive solution and this solution satisfies $\liminf_{|x|\to\infty} u(x) = 0$. Moreover, the minimal solution is the unique positive solution of (2.1) which tends to zero at infinity. Brezis and Kamin also showed that if the potential $\rho(x)$ decays fast enough at infinity then Problem (2.1) has a solution and, moreover, such a solution does not exist if $\rho(x)$ has a slow decay at infinity. For instance, if $\rho(x) = (1 + |x|^p)^{-1}$, then (2.1) has a bounded solution if and only if $p > 2$. More generally, Brezis and Kamin have proved that Problem (2.1) has a bounded solution if and only if $\rho(x)$ is potentially bounded, that is, the mapping $x \longmapsto \int_{\mathbb{R}^N} \rho(y)|x-y|^{2-N}dy \in L^{\infty}(\mathbb{R}^N)$. We refer to Brezis and Oswald [20] and Krasnoselskii [85] for various results on bounded domains for sublinear elliptic equations with zero Dirichlet boundary condition. Problem (2.1) in the whole space has been considered in Badiale and Dobarro [8], Edelson [43], Egnell [48], Fukagai [58], Kawano [80], Lair and Shaker [91], Mabrouk [101], Naito [108], Rădulescu [129], Wu and Yang [155], under various assumptions on ρ. Sublinear problems (either stationary or evolution ones) appear in the study of population dynamics, of reaction-diffusion processes, of filtration in porous media with absorption, as well as in the study of the scalar curvature of warped products of semi-Riemannian manifolds (see, e.g., Bandle, Pozio and Tesei [10], Dobarro and Lami Dozo [41], Eidus [49], O'Neill [113]).

Our purpose in the first part of this chapter is to study the problem

$$\begin{cases} -\Delta u = \rho(x) f(u) & \text{in } \mathbb{R}^N \\ u > \ell & \text{in } \mathbb{R}^N \\ u(x) \to \ell & \text{as } |x| \to \infty, \end{cases} \tag{2.2}$$

where $N \geq 3$ and $\ell \geq 0$ is a real number.

Throughout the chapter we assume that the variable potential $\rho(x)$ satisfies $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \not\equiv 0$.

In our first result we suppose that the growth at infinity of the anisotropic potential $\rho(x)$ is given by

(ρ1) $\int_0^\infty r\Phi(r)dr < \infty$, where $\Phi(r) := \max_{|x|=r} \rho(x)$.

Assumption (ρ1) has been first introduced in Naito [108].

The nonlinearity $f : (0, \infty) \to (0, \infty)$ satisfies $f \in C^{0,\alpha}_{\text{loc}}(0, \infty)$ $(0 < \alpha < 1)$ and has a sublinear growth, in the sense that

(f1) the mapping $u \longmapsto f(u)/u$ is decreasing on $(0, \infty)$ and $\lim_{u\to\infty} f(u)/u = 0$.

We point out that condition (f1) does not require that f is smooth at the origin. The standard example of such a nonlinearity is $f(u) = u^p$, where $-\infty < p < 1$. We also observe that we study an equation of the same type as in Brezis and Kamin [19]. The main difference is that we require a certain asymptotic behaviour at infinity of the solution.

Entire solutions of (2.2) decaying to zero at infinity have been studied in Cîrstea and Rădulescu [27] for $\ell = 0$, provided that: ρ satisfies (ρ1), there exists $\beta > 0$ such that the mapping $u \longmapsto f(u)/(u+\beta)$ is decreasing on $(0, \infty)$, $\lim_{u\searrow 0} f(u)/u = +\infty$ and f is bounded in a neighborhood of $+\infty$. Our main purpose in the present chapter is to consider both cases $\ell = 0$ and $\ell > 0$, under the weaker assumption that the mapping $u \longmapsto f(u)/u$ is decreasing on $(0, \infty)$.

Under the above hypotheses (ρ1) and (f1), our first result concerns the case $\ell > 0$. We have

Theorem 3. *Assume that $\ell > 0$. Then Problem (2.2) has a unique classical solution.*

Next, consider the case $\ell = 0$. Instead of (ρ1) we impose the stronger condition

(ρ2) $\int_0^\infty r^{N-1}\Phi(r)dr < \infty$.

We remark that in Edelson [44] it is used the stronger assumption

$$\int_0^\infty r^{N-1+\lambda(N-2)}\Phi(r)dr < \infty\,, \qquad \text{for some } \lambda \in (0,1)\,.$$

Additionally, we suppose that

(f2) f is increasing in $(0,\infty)$ and $\lim_{u\searrow 0} f(u)/u = +\infty$.

A nonlinearity satisfying both (f1) and (f2) is $f(u) = u^p$, where $0 < p < 1$.

Our result in the case $\ell = 0$ is the following.

Theorem 4. *Assume that $\ell = 0$ and assumptions (ρ2), (f1) and (f2) are fulfilled. Then Problem (2.2) has a unique classical solution.*

A major role in our arguments is played by the Maximum Principle for elliptic equations (see Gilbarg and Trudinger [66], resp. Rus [138, 139] for a variant corresponding to elliptic systems).

We point out that assumptions (ρ1) and (ρ2) are related to a celebrated class introduced by Kato, with wide and deep applications in Potential Theory and Brownian Motion. We recall (see Aizenman and Simon [3]) that a real-valued measurable function ψ on $\mathbb{R}^N$ belongs to the Kato class $\mathcal{K}$ provided that

$$\lim_{\alpha\to 0}\sup_{x\in\mathbb{R}^N}\int_{|x-y|\le\alpha} E(y)|\psi(y)|dy = 0,$$

where E denotes the fundamental solution of the Laplace equation. According to this definition and our assumption (ρ1) (resp., (ρ2)), it follows that $\psi = \psi(|x|) \in \mathcal{K}$, where $\psi(|x|) := |x|^{N-3}\Phi(|x|)$ (resp., $\psi(|x|) := |x|^{-1}\Phi(|x|)$), for all $x \neq 0$.

2.2 Proof of Theorem 3

In order to prove the existence of a solution to Problem (2.2), we use a result established by Brezis and Oswald (see [20, Theorem 1]) for bounded domains.

Consider the problem

$$\begin{cases} -\Delta u = g(x,u) \quad \text{in } \Omega \\ u \geq 0, \quad u \not\equiv 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{2.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $g(x,u)$: $\Omega \times [0,\infty) \to \mathbb{R}$. Assume that

$$\begin{cases} \text{for a.e. } x \in \Omega \text{ the function } u \longmapsto g(x,u) \text{ is continuous on } [0,\infty) \\ \text{and the mapping } u \longmapsto g(x,u)/u \text{ is decreasing on } (0,\infty)\,; \end{cases} \tag{2.4}$$

$$\text{for each } u \geq 0 \text{ the function } x \longmapsto g(x,u) \text{ belongs to } L^\infty(\Omega)\,; \tag{2.5}$$

$$\exists C > 0 \text{ such that } g(x,u) \leq C(u+1) \text{ a.e. } x \in \Omega, \quad \forall\, u \geq 0. \tag{2.6}$$

Set

$$a_0(x) = \lim_{u \searrow 0} g(x,u)/u \qquad \text{and} \qquad a_\infty(x) = \lim_{u \to \infty} g(x,u)/u\,,$$

so that $-\infty < a_0(x) \leq +\infty$ and $-\infty \leq a_\infty(x) < +\infty$.

Under these hypotheses, Brezis and Oswald proved in [20] that Problem (2.3) has at most one solution. Moreover, a solution of (2.3) exists if and only if

$$\lambda_1(-\Delta - a_0(x)) < 0 \tag{2.7}$$

and

$$\lambda_1(-\Delta - a_\infty(x)) > 0, \tag{2.8}$$

where $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of the operator $-\Delta - a(x)$ with zero Dirichlet condition. The precise meaning of $\lambda_1(-\Delta - a(x))$ is

$$\lambda_1(-\Delta - a(x)) = \inf_{\varphi \in H_0^1(\Omega),\ \|\varphi\|_{L^2(\Omega)}=1} \left(\int |\nabla \varphi|^2 - \int_{[\varphi \neq 0]} a\varphi^2 \right).$$

Note that $\displaystyle\int_{[\varphi \neq 0]} a\varphi^2$ makes sense if $a(x)$ is any measurable function such that either $a(x) \leq C$ or $a(x) \geq -C$ a.e. on Ω.

For any positive integer k we consider the problem

$$\begin{cases} -\Delta u_k = \rho(x) f(u_k) \ , & \text{if } |x| < k \\ u_k > \ell, & \text{if } |x| < k \\ u_k(x) = \ell, & \text{if } \ |x| = k. \end{cases} \tag{2.9}$$

Equivalently, the above boundary value problem can be rewritten

$$\begin{cases} -\Delta v_k = \rho(x) f(v_k + \ell) \ , & \text{if } |x| < k \\ v_k(x) = 0, & \text{if } \ |x| = k. \end{cases} \tag{2.10}$$

In order to obtain a solution of the problem (2.10), it is enough to check the hypotheses of the Brezis-Oswald theorem.

- Since $f \in C(0,\infty)$ and $\ell > 0$, it follows that the mapping $v \longmapsto \rho(x) f(v+\ell)$ is continuous in $[0,\infty)$.
- From $\rho(x)\frac{f(v+\ell)}{v} = \rho(x)\frac{f(v+\ell)}{v+\ell}\frac{v+\ell}{v}$, using positivity of ρ and (f1) we deduce that the function $v \longmapsto \rho(x)\frac{v+\ell}{v}$ is decreasing on $(0,\infty)$.
- For all $v \geq 0$, since $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, we obtain that $\rho \in L^\infty(B(0,k))$, so the condition (2.5) is satisfied.
- By $\lim_{v\to\infty} f(v+\ell)/(v+1) = 0$ and $f \in C(0,\infty)$, there exists $M > 0$ such that $f(v+\ell) \leq M(v+1)$ for all $v \geq 0$. Therefore $\rho(x) f(v+\ell) \leq ||\rho||_{L^\infty(B(0,k))} M(v+1)$ for all $v \geq 0$.
- We have
$$a_0(x) = \lim_{v \searrow 0} \frac{\rho(x) f(v+\ell)}{v} = +\infty$$
and
$$a_\infty(x) = \lim_{v\to\infty} \frac{\rho(x) f(v+\ell)}{v} = \lim_{v\to\infty} \rho(x) \frac{f(v+\ell)}{v+\ell} \cdot \frac{v+\ell}{v} = 0\,.$$

Thus, by Theorem 1 in Brezis and Oswald [20], Problem (2.10) has a unique solution v_k which, by the maximum principle, is positive in $|x| < k$. Then $u_k = v_k + \ell$ satisfies (2.9). Define $u_k = \ell$ for $|x| > k$. The maximum principle implies that $\ell \leq u_k \leq u_{k+1}$ in $\mathbb{R}^N$.

We now justify the existence of a continuous function $v:\mathbb{R}^N\to\mathbb{R}$, $v>\ell$, such that $u_k\le v$ in $\mathbb{R}^N$. As in Lair and Shaker [91], we first construct a positive radially symmetric function w such that $-\Delta w=\Phi(r)$ $(r=|x|)$ in $\mathbb{R}^N$ and $\lim_{r\to\infty}w(r)=0$. A straightforward computation shows that

$$w(r)=K-\int_0^r\zeta^{1-N}\int_0^\zeta\sigma^{N-1}\Phi(\sigma)d\sigma d\zeta,$$

where

$$K=\int_0^\infty\zeta^{1-N}\int_0^\zeta\sigma^{N-1}\Phi(\sigma)d\sigma d\zeta,$$

provided the integral is finite. An integration by parts yields

$$\begin{aligned}\int_0^r\zeta^{1-N}\int_0^\zeta\sigma^{N-1}\Phi(\sigma)d\sigma d\zeta = & -\frac{1}{N-2}\int_0^r\frac{d}{d\zeta}\zeta^{2-N}\int_0^\zeta\sigma^{N-1}\Phi(\sigma)d\sigma d\zeta\\ &=\frac{1}{N-2}\left(-r^{2-N}\int_0^r\sigma^{N-1}\Phi(\sigma)d\sigma+\int_0^r\zeta\Phi(\zeta)\right)\\ &<\frac{1}{N-2}\int_0^\infty\zeta\Phi(\zeta)<+\infty\,.\end{aligned}$$

Moreover, w is decreasing and satisfies $0<w(r)<K$ for all $r\ge0$. Let $v>\ell$ be a function such that $w(r)=m^{-1}\int_0^{v(r)-\ell}\frac{t}{f(t+\ell)}dt$, where $m>0$ is chosen such that $Km\le\int_0^m\frac{t}{f(t+\ell)}dt$.

Next, by L'Hôpital's rule for the case $\frac{\cdot}{\infty}$ (see [110, Theorem 3, p. 319]) we have

$$\lim_{x\to\infty}\frac{\int_0^x\frac{t}{f(t+\ell)}dt}{x}=\lim_{x\to\infty}\frac{x}{f(x+\ell)}=\lim_{x\to\infty}\frac{x+\ell}{f(x+\ell)}\cdot\frac{x}{x+\ell}=+\infty\,.$$

This means that there exists $x_1>0$ such that $\int_0^x\frac{t}{f(t)}\ge Kx$ for all $x\ge x_1$. It follows that for any $m\ge x_1$ we have $Km\le\int_0^m\frac{t}{f(t)}dt$.

Since w is decreasing, we obtain that v is a decreasing function, too. Then

$$\int_0^{v(r)-\ell}\frac{t}{f(t+\ell)}dt\le\int_0^{v(0)-\ell}\frac{t}{f(t+\ell)}dt=mw(0)=mK\le\int_0^m\frac{t}{f(t+\ell)}dt\,.$$

It follows that $v(r)\le m+\ell$ for all $r>0$.

From $w(r)\to0$ as $r\to\infty$ we deduce that $v(r)\to\ell$ as $r\to\infty$.

By the choice of v we have

$$\nabla w=\frac{1}{m}\frac{v-\ell}{f(v)}\nabla v\quad\text{and}\quad\Delta w=\frac{1}{m}\frac{v-\ell}{f(v)}\Delta v+\frac{1}{m}\left(\frac{v-\ell}{f(v)}\right)'|\nabla v|^2\,.$$

Since the mapping $u \longmapsto f(u)/u$ is decreasing on $(0,\infty)$ we deduce that

$$\Delta v < \frac{m}{v-\ell} f(v)\Delta w = -\frac{m}{v-\ell} f(v)\Phi(r) \le -f(v)\Phi(r)\,. \tag{2.11}$$

By (2.9), (2.11) and our hypothesis (f1), we obtain that $u_k(x) \le v(x)$ for each $|x| \le k$ and so, for all $x \in \mathbb{R}^N$.

In conclusion,

$$u_1 \le u_2 \le \ldots \le u_k \le u_{k+1} \le \ldots \le v,$$

with $v(x) \to \ell$ as $|x| \to \infty$. Thus, there exists a function $u \le v$ such that $u_k \to u$ pointwise in $\mathbb{R}^N$. In particular, this shows that $u > \ell$ in $\mathbb{R}^N$ and $u(x) \to \ell$ as $|x| \to \infty$.

A standard bootstrap argument (with the same details as in Lair and Shaker [91]) shows that u is a classical solution of the problem (2.2).

To conclude the proof, it remains to show that the solution found above is unique. Suppose that u and v are solutions of (2.2). It is enough to show that $u \le v$ or, equivalently, $\ln u(x) \le \ln v(x)$, for any $x \in \mathbb{R}^N$. Arguing by contradiction, there exists $\overline{x} \in \mathbb{R}^N$ such that $u(\overline{x}) > v(\overline{x})$. Since $\lim_{|x|\to\infty}(\ln u(x) - \ln v(x)) = 0$, we deduce that $\max_{\mathbb{R}^N}(\ln u(x) - \ln v(x))$ exists and is positive. At this point, say x_0, we have

$$\nabla(\ln u(x_0) - \ln v(x_0)) = 0, \tag{2.12}$$

so

$$\frac{\nabla u(x_0)}{u(x_0)} = \frac{\nabla v(x_0)}{v(x_0)}. \tag{2.13}$$

By (f1) we obtain

$$\frac{f(u(x_0))}{u(x_0)} < \frac{f(v(x_0))}{v(x_0)}.$$

So, by (2.12) and (2.13),

$$\begin{aligned} 0 &\ge \Delta(\ln u(x_0) - \ln v(x_0)) \\ &= \frac{1}{u(x_0)} \cdot \Delta u(x_0) - \frac{1}{v(x_0)} \cdot \Delta v(x_0) - \frac{1}{u^2(x_0)} \cdot |\nabla u(x_0)|^2 + \frac{1}{v^2(x_0)} \cdot |\nabla v(x_0)|^2 \\ &= \frac{\Delta u(x_0)}{u(x_0)} - \frac{\Delta v(x_0)}{v(x_0)} = -\rho(x_0)\left(\frac{f(u(x_0))}{u(x_0)} - \frac{f(v(x_0))}{v(x_0)}\right) > 0, \end{aligned}$$

which is a contradiction. Hence $u \le v$ and the proof is concluded. □

2.3 Proof of Theorem 4

2.3.1 Existence

Since f is an increasing positive function on $(0, \infty)$, there exists and is finite $\lim_{u \searrow 0} f(u)$, so f can be extended by continuity at the origin. Consider the Dirichlet problem

$$\begin{cases} -\Delta u_k = \rho(x) f(u_k) \ , & \text{if } |x| < k \\ u_k(x) = 0, & \text{if } \ |x| = k. \end{cases} \tag{2.14}$$

Using the same arguments as in case $\ell > 0$ we deduce that conditions (2.4) and (2.5) are satisfied. In what concerns assumption (2.6), we use both assumptions (f1) and (f2). Hence $f(u) \leq f(1)$ if $u \leq 1$ and $f(u)/u \leq f(1)$ if $u \geq 1$. Therefore $f(u) \leq f(1)(u+1)$ for all $u \geq 0$, which proves (2.6) . The existence of a solution for (2.14) follows from (2.7) and (2.8). These conditions are direct consequences of our assumptions $\lim_{u\to\infty} f(u)/u = 0$ and $\lim_{u \searrow 0} f(u)/u = +\infty$. Thus, by the Brezis-Oswald theorem, Problem (2.14) has a unique solution. Define $u_k(x) = 0$ for $|x| > k$. Using the same arguments as in case $\ell > 0$, we obtain $u_k \leq u_{k+1}$ in $\mathbb{R}^N$.

Next, we prove the existence of a continuous function $v : \mathbb{R}^N \to \mathbb{R}$ such that $u_k \leq v$ in $\mathbb{R}^N$. As in Lair and Shaker [91], we first construct a positive radially symmetric function w satisfying $-\Delta w = \Phi(r)$ $(r = |x|)$ in $\mathbb{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. We obtain

$$w(r) = K - \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta \, ,$$

where

$$K = \int_0^\infty \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) \, d\sigma \, d\zeta \, , \tag{2.15}$$

provided the integral is finite. By integration by parts we have

$$\int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)\,d\sigma\,d\zeta = -\frac{1}{N-2}\int_0^r \frac{d}{d\zeta}\zeta^{2-N}\int_0^\zeta \sigma^{N-1}\Phi(\sigma)\,d\sigma\,d\zeta =$$

$$\frac{1}{N-2}\left(-r^{2-N}\int_0^r \sigma^{N-1}\Phi(\sigma)\,d\sigma + \int_0^r \zeta\Phi(\zeta)\,d\zeta\right) < \frac{1}{N-2}\int_0^\infty \zeta\Phi(\zeta)\,d\zeta < \infty\,. \tag{2.16}$$

Therefore

$$w(r) < \frac{1}{N-2}\cdot\int_0^\infty \zeta\Phi(\zeta)d\zeta, \qquad \text{for all } r>0.$$

Let v be a positive function such that $w(r) = c^{-1}\int_0^{v(r)} t/f(t)dt$, where $c>0$ is chosen such that $Kc \le \int_0^c t/f(t)dt$. We argue in what follows that we can find $c>0$ with this property. Indeed, by L'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\int_0^x \frac{t}{f(t)}dt}{x} = \lim_{x\to\infty}\frac{x}{f(x)} = +\infty\,.$$

This means that there exists $x_1>0$ such that $\int_0^x t/f(t)dt \ge Kx$ for all $x \ge x_1$. It follows that for any $c \ge x_1$ we have $Kc \le \int_0^c t/f(t)dt$.

On the other hand, since w is decreasing, we deduce that v is a decreasing function, too. Hence

$$\int_0^{v(r)} \frac{t}{f(t)}dt \le \int_0^{v(0)} \frac{t}{f(t)}dt = c\cdot w(0) = c\cdot K \le \int_0^c \frac{t}{f(t)}dt\,.$$

It follows that $v(r) \le c$ for all $r>0$.

From $w(r)\to 0$ as $r\to\infty$ we deduce that $v(r)\to 0$ as $r\to\infty$.

By the choice of v we have

$$\nabla w = \frac{1}{c}\cdot\frac{v}{f(v)}\nabla v \quad\text{and}\quad \Delta w = \frac{1}{c}\frac{v}{f(v)}\Delta v + \frac{1}{c}\left(\frac{v}{f(v)}\right)'|\nabla v|^2. \tag{2.17}$$

Combining the fact that $f(u)/u$ is a decreasing function on $(0,\infty)$ with relation (2.17), we deduce that

$$\Delta v < c\frac{f(v)}{v}\Delta w = -c\,\frac{f(v)}{v}\Phi(r) \le -f(v)\Phi(r)\,. \tag{2.18}$$

By (2.14) and (2.18) and using our hypothesis (f2), as already done for proving the uniqueness in the case $\ell > 0$, we obtain that $u_k(x) \le v(x)$ for each $|x| \le k$ and so, for all $x \in \mathbb{R}^N$.

We have obtained a bounded increasing sequence

$$u_1 \le u_2 \le \ldots \le u_k \le u_{k+1} \le \ldots \le v\,,$$

with v vanishing at infinity. Thus, there exists a function $u \le v$ such that $u_k \to u$ pointwise in $\mathbb{R}^N$. A standard bootstrap argument implies that u is a classical solution of the problem (2.2).

2.3.2 Uniqueness

We split the proof into two steps. Assume that u_1 and u_2 are solutions of Problem (2.2). We first prove that if $u_1 \le u_2$ then $u_1 = u_2$ in $\mathbb{R}^N$. In the second step we find a positive solution $u \le \min\{u_1, u_2\}$ and thus, using the first step, we deduce that $u = u_1$ and $u = u_2$, which proves the uniqueness.

Step I. We show that $u_1 \le u_2$ in $\mathbb{R}^N$ implies $u_1 = u_2$ in $\mathbb{R}^N$. Indeed, since

$$u_1\Delta u_2 - u_2\Delta u_1 = \rho(x)u_1u_2\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) \ge 0\,,$$

it is sufficient to check that

$$\int_{\mathbb{R}^N}(u_1\Delta u_2 - u_2\Delta u_1) = 0 \tag{2.19}$$

Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be such that $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$, and denote $\psi_n := \psi(x/n)$ for any positive integer n. Set

$$I_n := \int_{\mathbb{R}^N}(u_1\Delta u_2 - u_2\Delta u_1)\psi_n dx.$$

We claim that $I_n \to 0$ as $n \to \infty$. Indeed,

$$|I_n| \le \int_{\mathbb{R}^N}|u_1\Delta u_2|\psi_n dx + \int_{\mathbb{R}^N}|u_2\Delta u_1|\psi_n dx\,.$$

So, by symmetry, it is enough to prove that $J_n := \int_{\mathbb{R}^N} |u_1\Delta u_2|\psi_n dx \to 0$ as $n\to\infty$. But, from (2.2),

$$\begin{aligned} J_n &= \int_{\mathbb{R}^N} |u_1 f(u_2)\rho(x)|\psi_n dx = \int_n^{2n} \int_{|x|=r} |u_1(x)f(u_2(x))\rho(x)|dxdr \\ &\le \int_n^{2n} \Phi(r) \int_{|x|=r} |u_1(x)f(u_2(x))|dxdr \le \int_n^{2n} \Phi(r) \int_{|x|=r} |u_1(x)|M(u_2+1)dxdr\,. \end{aligned} \tag{2.20}$$

Since $u_1(x)$, $u_2(x)\to 0$ as $|x|\to\infty$, we deduce that u_1 and u_2 are bounded in $\mathbb{R}^N$. Returning to (2.20) we have

$$\begin{aligned} J_n &\le M(||u_2||_{L^\infty(\mathbb{R}^N)}+1)\sup_{|x|\ge n}|u_1(x)|\cdot\frac{\omega_N}{N}\int_n^{2n}\Phi(r)r^{N-1}dr \\ &\le C\int_0^\infty \Phi(r)r^{N-1}dr\cdot\sup_{|x|\ge n}|u_1(x)|\,. \end{aligned}$$

Since $u_1(x)\to 0$ as $|x|\to\infty$, we have $\sup_{|x|\ge n}|u_1(x)|\to 0$ as $n\to\infty$ which shows that $J_n\to 0$. In particular, this implies $I_n\to 0$ as $n\to\infty$.

Taking $f_n := (u_1\Delta u_2 - u_2\Delta u_1)\psi_n$ we deduce $f_n(x)\to u_1(x)\Delta u_2(x) - u_2(x)\Delta u_1(x)$ as $n\to\infty$. To apply Lebesgue's Dominated Convergence Theorem we need to show that $u_1\Delta u_2 - u_2\Delta u_1\in L^1(\mathbb{R}^N)$. For this purpose it is sufficient to prove that $u_1\Delta u_2\in L^1(\mathbb{R}^N)$. Indeed,

$$\int_{\mathbb{R}^N}|u_1\Delta u_2| \le ||u_1||_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|\Delta u_2| = C\int_{\mathbb{R}^N}|\rho(x)f(u_2)|\,.$$

Thus, using $f(u)\le f(1)(u+1)$ and since u_2 is bounded, the above inequality yields

$$\begin{aligned} \int_{\mathbb{R}^N}|u_1\Delta u_2| &\le C\int_{\mathbb{R}^N}|\rho(x)(u_2+1)| \\ &\le C\int_0^\infty\int_{|x|=r}\Phi(r)dxdr \le C\int_0^\infty \Phi(r)r^{N-1} < +\infty\,. \end{aligned}$$

This shows that $u_1\Delta u_2\in L^1(\mathbb{R}^N)$ and the proof of Step I is completed.

Step II. Let u_1, u_2 be arbitrary solutions of Problem (2.2). For all integer $k\ge 1$, denote $\Omega_k := \{x\in\mathbb{R}^N;\ |x|<k\}$. The Brezis-Oswald theorem implies that the problem

$$\begin{cases} -\Delta v_k = \rho(x)f(v_k) & \text{in } \Omega_k \\ v_k = 0 & \text{on } \partial\Omega_k \end{cases}$$

has a unique solution $v_k \geq 0$. Moreover, by the Maximum Principle, $v_k > 0$ in Ω_k. We define $v_k = 0$ for $|x| > k$. Applying again the Maximum Principle we deduce that $v_k \leq v_{k+1}$ in $\mathbb{R}^N$. Now we prove that $v_k \leq u_1$ in $\mathbb{R}^N$, for all $k \geq 1$. Obviously, this happens outside Ω_k. On the other hand

$$\begin{cases} -\Delta u_1 = \rho(x) f(u_1) & \text{in } \Omega_k \\ u_1 > 0 & \text{on } \partial\Omega_k \end{cases}$$

Arguing by contradiction, we assume that there exists $\overline{x} \in \Omega_k$ such that $v_k(\overline{x}) > u_1(\overline{x})$. Consider the function $h : \Omega_k \to \mathbb{R}$, $h(x) = \ln v_k(x) - \ln u(x)$. Since u_1 is bounded in Ω_k and $\inf_{\partial\Omega_k} u_1 > 0$ we have $\lim\limits_{|x|\to k} h(x) = -\infty$. We deduce that $\max_{\Omega_k} (\ln v_k(x) - \ln u_1(x))$ exists and is positive. Using the same argument as in the case $\ell > 0$ we deduce that $v_k \leq u_1$ in Ω_k, so in $\mathbb{R}^N$. Similarly we obtain $v_k \leq u_2$ in $\mathbb{R}^N$. Hence $v_k \leq \overline{u} := \min\{u_1, u_2\}$. Therefore $v_k \leq v_{k+1} \leq \ldots \leq \overline{u}$. Thus there exists a function u such that $v_k \to u$ pointwise in $\mathbb{R}^N$. Repeating a previous argument we deduce that $u \leq \overline{u}$ is a classical solution of Problem (2.2). Moreover, since $u \geq v_k > 0$ in Ω_k and for all $k \geq 1$, we deduce that $u > 0$ in $\mathbb{R}^N$. This concludes the proof of Step II.

Combining Steps I and II we conclude that $u_1 = u_2$ in $\mathbb{R}^N$. □

2.4 Entire positive solutions of the singular Emden-Fowler equation with nonlinear gradient term

Singular semilinear elliptic problems have been intensively studied in the last decades. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts or in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction-diffusion processes where the unknown $u \geq 0$ is viewed as the density of a reactant (see, e.g., Aris [7]). In this framework a major place is played by the Emden-Fowler singular equation

$$-\Delta u = p(x) u^{-\gamma}, \qquad x \in \Omega, \tag{2.21}$$

where Ω is an open set (bounded or unbounded) in $\mathbb{R}^N$ $(N \geq 3)$, $\gamma > 0$, and $p : \Omega \to (0, \infty)$ is a continuous function. For a comprehensive study of the

Emden-Fowler equation we refer to Dalmasso [34], Edelson [44], Fulks and Maybee [59], Jin [77], Kusano and Swanson [90], Shaker [141], Wong [154] and the references therein. If Ω is bounded, Lazer and McKenna proved in [94] that (2.21) has a unique positive solution if p is a smooth positive function. The existence of entire positive solutions for $\gamma \in (0,1)$ and under certain additional hypotheses has been established in Edelson [44] and in Kusano-Swanson [90]. For instance, Edelson proved the existence of a solution provided that

$$\int_1^\infty r^{N-1+\lambda(N-2)} \max_{|x|=r} p(x) dr < \infty,$$

for some $\lambda \in (0,1)$. This result is generalized for any $\gamma > 0$ via the sub and super solutions method in Shaker [141] or by other methods in Dalmasso [34]. For further results related to singular elliptic equations we also refer to Callegari and Nashman [22, 23], Coclite and Palmieri [30], Crandall, Rabinowitz and Tartar [31], Gomes [63].

The purpose of this chapter is to extend some of these results in the more general framework of singular elliptic equations with nonlinear gradient term. Problems of this type arise in stochastic control theory and have been first studied in Lasry and Lions [93]. The corresponding parabolic equation was considered in Quittner [125]. Elliptic problems with nonlinear gradient term have been also studied in various contexts (see, e.g., Bandle and Giarusso [9], Grenon and Trombetti [66], Mâagli and Zribi [100], Maderna, Pagani and Salsa [102]).

We study the problem

$$\begin{cases} -\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma} \quad \text{in} \quad \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{2.22}$$

where $N \geq 3$, $a > 0$ and $\gamma > 0$. We assume throughout this chapter that p, $q \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$, $p > 0$ and $q \geq 0$ in $\mathbb{R}^N$. Set

$$\Phi(r) = \max_{|x|=r} p(x). \tag{2.23}$$

We impose no growth hypothesis on q but we suppose that p satisfies the following decay condition to zero at infinity:

$$\int_0^\infty r\Phi(r)dr < \infty. \tag{2.24}$$

In particular, potentials $p(x)$ which behave like $|x|^{-\alpha}$ as $|x| \to \infty$, with $\alpha > 2$, satisfy this assumption.

Our main result is the following:

Theorem 5. *Under the above hypotheses, the problem (2.22) has a unique classical solution.*

Proof. We first establish the existence of at least one solution of problem (2.22). For this purpose, for any integer $n \geq 1$, we consider the auxiliary boundary value problem

$$\begin{cases} -\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma} & \text{in} \quad B_n \\ u > 0 \quad \text{in} \quad B_n \\ u = 0, \quad \text{on} \quad \partial B_n, \end{cases} \tag{2.25}$$

where $B_n := \{x \in \mathbb{R}^N;\ |x| < n\}$. We observe that the function $\underline{u} = \varepsilon\varphi_1$ is a subsolution of (2.25), provided that $\varepsilon > 0$ is sufficiently small, where $\varphi_1 > 0$ is the first eigenfunction of $(-\Delta)$ in $H_0^1(B_n)$. In order to find a supersolution of (2.25), we observe that any solution of

$$\begin{cases} -\Delta u = p(x)u^{-\gamma} & \text{in} \quad B_n \\ u > 0 \quad \text{in} \quad B_n \\ u = 0 \quad \text{on} \quad \partial B_n \end{cases} \tag{2.26}$$

is a supersolution of (2.25). But problem (2.26) has a solution, by Theorem 1.1 in Crandall, Rabinowitz and Tartar [31]. Denote by u_n this solution. By standard bootstrap arguments (see Gilbarg and Trudinger [62]), $u_n \in C^2(B_n) \cap C(\overline{B_n})$. Also, by the maximum principle, it follows that $u_n \leq u_{n+1}$ in B_n. Until now we know that there exists $u(x) := \lim_{n\to\infty} u_n(x) \leq +\infty$, for all $x \in \mathbb{R}^N$.

Next, we establish the existence of a positive smooth function v such that $u_n \leq v$ in $\mathbb{R}^N$. Let Φ be defined by (2.23) and set

$$w(r) := K - \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma) d\sigma d\zeta,$$

where

$$K := \int_0^\infty \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)\, d\sigma\, d\zeta \qquad \text{for any } \; r > 0, \tag{2.27}$$

provided that the integral is convergent. Then $-\Delta w = \Phi(r)$ and $\lim_{r\to\infty} w(r) = 0$.

We prove in what follows that $K < +\infty$. An integration by parts yields

$$\int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)\, d\sigma\, d\zeta = (2-N)^{-1} \int_0^r \frac{d}{d\zeta}\zeta^{2-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)\, d\sigma\, d\zeta =$$
$$(N-2)^{-1} \left(-r^{2-N} \int_0^r \sigma^{N-1}\Phi(\sigma)\, d\sigma + \int_0^r \zeta\Phi(\zeta)\, d\zeta \right). \tag{2.28}$$

Next, by L'Hôpital's rule,

$$\lim_{r\to\infty} \left(-r^{2-N} \int_0^r \sigma^{N-1}\Phi(\sigma)\, d\sigma + \int_0^r \zeta\Phi(\zeta)\, d\zeta \right) =$$
$$\lim_{r\to\infty} \frac{-\int_0^r \sigma^{N-1}\Phi(\sigma)\, d\sigma + r^{N-2}\int_0^r \zeta\Phi(\zeta)\, d\zeta}{r^{N-2}} =$$
$$\lim_{r\to\infty} \int_0^r \zeta\Phi(\zeta)\, d\zeta = \int_0^\infty \zeta\Phi(\zeta)\, d\zeta < \infty,$$

by our assumption (2.24). Thus, we obtain that

$$K = (N-2)^{-1} \int_0^\infty \zeta\Phi(\zeta)\, d\zeta < \infty\,.$$

So, by the definition of w, $w(r) < (N-2)^{-1} \int_0^\infty \zeta\Phi(\zeta) d\zeta$, for any $r > 0$.

Set

$$v(r) := [c(2+\gamma)w(r)]^{1/(2+\gamma)} ,$$

where

$$c := [K(2+\gamma)]^{1/(1+\gamma)} .$$

In particular, from $w(r) \to 0$ as $r \to \infty$, we deduce that $v(r) \to 0$ as $r \to \infty$. Since w is a decreasing function, it follows that v decreases, too. Hence

$$\int_0^{v(r)} t^{1+\gamma} dt \le \int_0^{v(0)} t^{1+\gamma} dt = cw(0) = cK = \int_0^{c} t^{1+\gamma} dt.$$

It follows that $v(r) \le c$ for all $r > 0$.

On the other hand,

$$\nabla w = \frac{1}{c}\, v^{1+\gamma} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{c} v^{1+\gamma} \Delta v + \frac{1}{c} \left(v^{1+\gamma}\right)' |\nabla v|^2.$$

Hence

$$\Delta v < c v^{-1-\gamma} \Delta w = -c\, v^{-1-\gamma} \Phi(r) \le -v^{-\gamma} \Phi(r) . \tag{2.29}$$

By (2.26) and (2.29) we obtain that $u_n \le v$ in B_n. Therefore

$$u_1 \le u_2 \le \cdots \le u_n \le u_{n+1} \le \cdots \le v,$$

with v vanishing at infinity. Now, standard bootstrap arguments (see Gilbarg and Trudinger [62]) imply that $u(x) := \lim_{n\to\infty} u_n(x)$ is well defined and smooth in $\mathbb{R}^N$. Moreover, u is a classical solution of problem (2.22).

We justify in what follows the uniqueness of the solution to problem (2.22). Suppose that u and v are arbitrary solutions of (2.22). In order to establish the uniqueness, it is enough to show that $u \le v$ in $\mathbb{R}^N$. Arguing by contradiction, it follows that $\max_{x\in\mathbb{R}^N} (u(x) - v(x)) =: M > 0$. Assume that $u(x_0) - v(x_0) = M$. Then $u(x_0) > v(x_0) > 0$, $\nabla u(x_0) = \nabla v(x_0)$ and $\Delta(u-v)(x_0) \le 0$. But

$$\begin{aligned} \Delta(u-v)(x_0) = \quad & q(x_0) \left[|\nabla u(x_0)|^a - |\nabla v(x_0)|^a \right] + p(x_0) \left(v^{-\gamma}(x_0) - u^{-\gamma}(x_0)\right) = \\ & p(x_0) \left(v^{-\gamma}(x_0) - u^{-\gamma}(x_0)\right) > 0, \end{aligned}$$

which is a contradiction. This implies that $u \le v$, and so $u = v$ in $\mathbb{R}^N$. □

Chapter 3

Semilinear elliptic problems with sign-changing potential and subcritical nonlinearity

Nature and Nature's law lay hid in night: God said,"Let Newton be!," and all was light.

Alexander Pope (1688-1744),
Epitaph on Newton

Abstract. In this chapter we establish existence and multiplicity theorems for a Dirichlet boundary value problem at resonance, which is a nonlinear subcritical perturbation of a linear eigenvalue problem studied by Cuesta. Our framework includes a sign-changing potential and we locate the solutions by using the Mountain Pass lemma and the Saddle Point theorem.

3.1 Subcritical perturbations of resonant linear problems with sign-changing potential

Let Ω be an arbitrary open set in $\mathbb{R}^N$, $N \geq 2$, and assume that $V : \Omega \to \mathbb{R}$ is a variable potential. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x)u \quad \text{in } \Omega\,, \\ u \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

Problems of this type have a long history. If Ω is bounded and $V \equiv 1$, problem (3.1) is related to the Riesz-Fredholm theory of self-adjoint and compact operators (see, e.g., Theorem VI.11 in Brezis [18]). The case of a non-constant potential V has been first considered in the pioneering papers of Bocher [17], Hess and Kato [72], Minakshisundaran and Pleijel [105] and Pleijel [120]. For instance, Minakshisundaran and Pleijel [105], [120] studied the case where Ω is bounded, $V \in L^\infty(\Omega)$, $V \geq 0$ in Ω and $V > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. An important contribution in the study of Problem (3.1) if Ω and V are not necessarily bounded has been given recently by Cuesta [32] (see also Szulkin and Willem [146]) under the assumption that the sign-changing potential V satisfies

$$(H) \qquad V^+ \not\equiv 0 \ \text{ and } \ V \in L^s(\Omega)\,,$$

where $s > N/2$ if $N \geq 2$ and $s = 1$ if $N = 1$. As usually, we have denoted $V^+(x) = \max\{V(x), 0\}$. Obviously, $V = V^+ - V^-$, where $V^-(x) = \max\{-V(x), 0\}$.

In order to study the main properties (isolation, simplicity) of the principal eigenvalue of (3.1), Cuesta [32] proved that the minimization problem

$$\min\left\{\int_\Omega |\nabla u|^2 dx;\ u \in H_0^1(\Omega),\ \int_\Omega V(x)u^2 dx = 1\right\}$$

has a positive solution $\varphi_1 = \varphi_1(\Omega)$, which is an eigenfunction of (3.1) corresponding to the eigenvalue $\lambda_1 := \lambda_1(\Omega) = \int_\Omega |\nabla\varphi_1|^2 dx$.

Our purpose in this chapter is to study the existence of solutions of the perturbed nonlinear boundary value problem

$$\begin{cases} -\Delta u = \lambda_1 V(x) u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

where V satisfies (H) and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $g(x,0) = 0$ and with subcritical growth, that is,

$$|g(x,s)| \leq a_0 \cdot |s|^{r-1} + b_0, \qquad \text{for all } s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \tag{3.3}$$

for some constants a_0, $b_0 > 0$, where $2 \le r < 2^*$. We recall that 2^* denotes the critical Sobolev exponent, that is, $2^* := \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = +\infty$ if $N \in \{1, 2\}$.

Problem (3.2) is resonant at infinity and equations of this type have been first studied by Landesman and Lazer [92] in connection with concrete problems arising in Mechanics.

Set $G(x, s) = \int_0^s g(x, t)dt$. Throughout this chapter we assume that there exist $k,\ m \in L^1(\Omega)$, with $m \ge 0$, such that

$$|G(x, s)| \le k(x), \qquad \text{for all } s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega\,; \tag{3.4}$$

$$\liminf_{s \to 0} \frac{G(x, s)}{s^2} = m(x), \qquad \text{a.e. } x \in \Omega\,. \tag{3.5}$$

The energy functional associated to Problem (3.2) is

$$F(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \lambda_1 V(x) u^2 \right) dx - \int_\Omega G(x, u) dx\,,$$

for all $u \in H_0^1(\Omega)$.

From the variational characterization of λ_1 and using (3.4) we obtain

$$F(u) \ge - \int_\Omega G(x, u(x)) dx \ge -|k|_1 > -\infty\,,$$

for all $u \in H_0^1(\Omega)$ and, consequently, F is bounded from below. Let us consider $u_n = \alpha_n \varphi_1$, where $\alpha_n \to \infty$. Then the estimate $\int_\Omega |\nabla \varphi_1|^2 = \lambda_1 \int_\Omega V(x) \varphi_1^2$ yields $F(u_n) = -\int_\Omega G(x, \alpha_n \varphi_1) dx \le |k|_1 < \infty$. Thus, $\lim_{n\to\infty} F(u_n) < \infty$. Hence the sequence $(u_n)_n \subset H_0^1(\Omega)$ defined by $u_n = \alpha_n \varphi_1$ satisfies $\|u_n\| \to \infty$ and $F(u_n)$ is bounded. In conclusion, if we suppose that (3.4) holds true then the energy functional F is bounded from below and is not coercive.

In the next two theorems, we prove the existence of a solution if $V \in L^\infty(\Omega)$, under the following assumptions on the potential G:

$(G_1)_q$ $\qquad \displaystyle\limsup_{|s| \to \infty} \frac{G(x, s)}{|s|^q} \le b < \infty$ uniformly a.e. $x \in \Omega$, $q > 2$;

$(G_2^+)_\mu$ $\qquad \displaystyle\liminf_{|s| \to \infty} \frac{g(x, s)s - 2G(x, s)}{|s|^\mu} \ge a > 0$ uniformly a.e. $x \in \Omega$;

$$(G_2^-)_\mu \qquad \limsup_{|s|\to\infty} \frac{g(x,s)s - 2G(x,s)}{|s|^\mu} \leq -a < 0 \quad \text{uniformly a.e.} \quad x\in\Omega\,.$$

Theorem 6. *Assume that* G *satisfies conditions* $(G_1)_q$, $(G_2^+)_\mu$ *[or* $(G_2^-)_\mu$*] and* (G_3)

$$\limsup_{s\to 0} \frac{2G(x,s)}{s^2} \leq \alpha < \lambda_1 < \beta \leq \liminf_{|s|\to\infty} \frac{2G(x,s)}{s^2} \quad \textit{uniformly a.e.} \quad x \in \Omega\,,$$

with $\mu > 2N/(q-2)$ *if* $N \geq 3$ *or* $\mu > q-2$ *if* $1 \leq N \leq 2$. *Then Problem (3.2) has at least one solution.*

Theorem 7. *Assume that* $G(x,s)$ *satisfies* $(G_2^-)_\mu$ *[or* $(G_2^+)_\mu$*], for some* $\mu > 0$, *and*

$$(G_4) \qquad \lim_{|s|\to\infty} \frac{G(x,s)}{s^2} = 0 \quad \textit{uniformly a.e.} \quad x \in \Omega\,.$$

Then Problem (3.2) has at least one solution.

The above theorems extend to the anisotropic case $V \not\equiv$ const. some results of Gonçalves and Miyagaki [64] and Ma [99].

3.2 Auxiliary results

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. We start with the following auxiliary result.

Lemma 1. *Let* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function and assume that there exist some constants* a, $b \geq 0$ *such that*

$$|g(x,t)| \leq a + b|t|^{r/s}\,, \qquad \textit{for all } t \in \mathbb{R}\,,\ \textit{a.e. } x \in \Omega\,.$$

Then the application $\varphi(x) \mapsto g(x,\varphi(x))$ *is in* $C(L^r(\Omega), L^s(\Omega))$.

Proof. For any $u \in L^r(\Omega)$ we have

$$\begin{aligned}\int_\Omega |g(x,u(x))|^s dx &\leq \int_\Omega (a + b|u|^{r/s})^s dx \leq 2^s \int_\Omega (a^s + b^s|u|^r)dx \\ &\leq c\int_\Omega (1+|u|^r)dx < \infty\,.\end{aligned}$$

This shows that if $\varphi \in L^r(\Omega)$ then $g(x,\varphi) \in L^s(\Omega)$. Let u_n, $u \in L^r$ be such that $|u_n - u|_r \to 0$. By Theorem IV.9 in Brezis [18], there exist a subsequence $(u_{n_k})_k$ and $h \in L^r$ such that $u_{n_k} \to u$ a.e. in Ω and $|u_{n_k}| \leq h$ a.e. in Ω. By our hypotheses it follows that $g(u_{n_k}) \to g(u)$ a.e. in Ω. Next, we observe that

$$|g(u_{n_k})| \leq a + b|u_{n_k}|^{r/s} \leq a + b|h|^{r/s} \in L^s(\Omega)\,.$$

So, by Lebesgue's Dominated Convergence Theorem,

$$|g(u_{n_k}) - g(u)|_s^s = \int_\Omega |g(u_{n_k}) - g(u)|^s dx \xrightarrow{k} 0\,.$$

This end the proof of the lemma. □

The application $\varphi \mapsto g(x,\varphi(x))$ is the Nemitski operator of the function g.

Proposition 5. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $|g(x,s)| \leq a + b|s|^{r-1}$ for all $(x,s) \in \Omega \times \mathbb{R}$, with $2 \leq r < 2N/(N-2)$ if $N > 2$ or $2 \leq r < \infty$ if $1 \leq N \leq 2$. Denote $G(x,t) = \int_0^t g(x,s)ds$. Let $I : H_0^1(\Omega) \to \mathbb{R}$ be the functional defined by*

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda_1}{2}\int_\Omega V(x)u^2 dx - \int_\Omega G(x,u(x))dx\,,$$

where $V \in L^s(\Omega)$ ($s > N/2$ if $N \geq 2$, $s = 1$ if $N = 1$).

Assume that $(u_n)_n \subset H_0^1(\Omega)$ has a bounded subsequence and $I'(u_n) \to 0$ as $n \to \infty$. Then $(u_n)_n$ has a convergent subsequence.

Proof. We have

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \nabla v dx - \lambda_1 \int_\Omega V(x)uv dx - \int_\Omega g(x,u(x))v(x)dx\,.$$

Denote by

$$\langle a(u), v\rangle = \int_\Omega \nabla u \nabla v dx\,;$$

$$J(u) = \frac{\lambda_1}{2}\int_\Omega V(x)u^2 dx + \int_\Omega G(x,u(x))dx\,.$$

It follows that

$$\langle J'(u), v\rangle = \lambda_1 \int_\Omega V(x)uv dx + \int_\Omega g(x,u(x))v(x)dx$$

and $I^{'}(u) = a(u) - J^{'}(u)$. We prove that a is an isomorphism from $H_0^1(\Omega)$ onto $a(H_0^1(\Omega))$ and $J^{'}$ is a compact operator. This assumption yields

$$u_n = a^{-1}\langle (I^{'}(u_n\rangle) + J^{'}(u_n)) \to \lim_{n\to\infty} a^{-1}\langle (J^{'}(u_n)\rangle) \, .$$

But $J^{'}$ is a compact operator and $(u_n)_n$ is a bounded sequence. This implies that $(J^{'}(u_n))_n$ has a convergent subsequence and, consequently, $(u_n)_n$ has a convergent subsequence. Assume, up to a subsequence, that $(u_n)_n \subset H_0^1(\Omega)$ is bounded. From the compact embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$, we can assume, passing again at a subsequence, that $u_n \to u$ in $L^r(\Omega)$. We have

$$\begin{aligned}
&\|J'(u_n) - J'(u)\| \\
&\le \sup_{\|v\|\le 1} \left| \int_\Omega (g(x,u_n(x)) - g(x,u(x)))\, v(x) dx \right| \\
&+ \sup_{\|v\|\le 1} \lambda_1 \left| \int_\Omega V(x)(u_n - u) v dx \right| \\
&\le \sup_{\|v\|\le 1} \int_\Omega |g(x,u_n(x)) - g(x,u(x))||v(x)| dx \\
&+ \lambda_1 \sup_{\|v\|\le 1} \int_\Omega |V(x)(u_n - u)v| dx \qquad (3.6)\\
&\le \sup_{\|v\|\le 1} \left(\int_\Omega |g(x,u_n) - g(x,u)|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} |v|_r \\
&+ \lambda_1 \sup_{\|v\|\le 1} \int_\Omega |V(x)(u_n - u)v| dx \\
&\le c \sup_{\|v\|\le 1} \left(\int_\Omega |g(x,u_n) - g(x,u)|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \|v\| \\
&+ \lambda_1 |V|_{L^s} \cdot |u_n - u|_\alpha \cdot |v|_\beta \, ,
\end{aligned}$$

where α, $\beta < 2N/(N-2)$ (if $N \ge 2$). Such a choice of α and β is possible due to our choice of s.

By Lemma 1 we obtain $g \in C(L^r, L^{r/(r-1)})$. Next, since $u_n \to u$ in L^r and $u_n \to u$ in L^2, it follows by (3.6) that $J^{'}(u_n) \to J^{'}(u)$ as $n \to \infty$, that is, $J^{'}$ is a compact operator. This completes our proof. □

Let V denote the linear space spanned by φ_1. Set

$$\Gamma := \{\gamma \in C(B, H_0^1(\Omega));\ \gamma(v) = v\, ,\ \text{for all } v \in \partial B\}$$

and denote $B := \{v \in V;\ \|v\| \le R\}$.

Proposition 6. *We have $\gamma(B)\bigcap W \neq \emptyset$, for all $\gamma \in \Gamma$.*

Proof. Let $P : H_0^1(\Omega) \to V$ be the projection of H_0^1 in V. Then P is a linear and continuous operator. If $v \in \partial B$ then $(P \circ \gamma)(v) = P(\gamma(v)) = P(v) = v$ and, consequently, $P \circ \gamma = Id$ on ∂B. We have $P \circ \gamma\ , Id \in C(B, H_0^1)$ and $0 \notin Id(\partial B) = \partial B$. Using a property of the Brouwer topological degree we obtain $\deg(P \circ \gamma, \mathrm{Int}B, 0) = \deg(Id, \mathrm{Int}B, 0)$. But $0 \in \mathrm{Int}\, B$ and it follows that $\deg(Id, \mathrm{Int}\, B, 0) = 1 \neq 0$. So, by the existence property of the Brouwer degree, there exists $v \in \mathrm{Int}\, B$ such that $(P \circ \gamma)(v) = 0$, that is, $P(\gamma(v)) = 0$. Therefore $\gamma(v) \in W$ and this shows that $\gamma(B) \cap W \neq \emptyset$. □

3.3 Cerami's compactness conditions

Let E be a reflexive real Banach space with norm $\|\cdot\|$ and let $I : E \to \mathbb{R}$ be a C^1 functional. We assume that there exists a compact embedding $E \hookrightarrow X$, where X is a real Banach space, and that the following interpolation type inequality holds:

$$(H_1) \qquad \|u\|_X \leq \psi(u)^{1-t}\|u\|^t\,, \qquad \text{for all } u \in E\,,$$

for some $t \in (0,1)$ and some homogeneous function $\psi : E \to \mathbb{R}_+$ of degree one. An example of such a framework is the following: $E = H_0^1(\Omega)$, $X = L^q(\Omega)$, $\psi(u) = |u|_\mu$, where $0 < \mu < q < 2^*$. Then, by the interpolation inequality (see [18, Remarque 2, p. 57]) we have

$$|u|_q \leq |u|_\mu^{1-t}|u|_{2^*}^t\,, \quad \text{where } \frac{1}{q} = \frac{1-t}{\mu} + \frac{t}{2^*}\,.$$

The Sobolev inequality yields $|u|_{2^*} \leq c\|u\|$, for all $u \in H_0^1(\Omega)$. Hence

$$|u|_q \leq k|u|_\mu^{1-t}\|u\|^t\,, \qquad \text{for all } u \in H_0^1(\Omega)$$

and this is a (H_1) type inequality.

We recall below the following Cerami compactness conditions (see [24]).

Definition 1. *a) The functional $I : E \to \mathbb{R}$ is said to satisfy condition (C) at the level $c \in \mathbb{R}$ [denoted $(C)_c$] if any sequence $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|) \cdot \|I'(u_n)\|_{E^*} \to 0$ possesses a convergent subsequence.*

b) The functional $I : E \to \mathbb{R}$ is said to satisfy condition $(\hat{C})$ at the level $c \in \mathbb{R}$ [denoted $(\hat{C})_c$] if any sequence $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|) \cdot \|I^{'}(u_n)\|_{E^*} \to 0$ possesses a bounded subsequence.

We observe that the above conditions are weaker than the usual Palais-Smale condition $(PS)_c$: any sequence $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $\|I^{'}(u_n)\|_{E^*} \to 0$ possesses a convergent subsequence.

Suppose that $I(u) = J(u) - N(u)$, where J is 2-homogeneous and N is not 2-homogeneous at infinity. We recall that J is 2-homogeneous if $J(\tau u) = \tau^2 J(u)$, for all $\tau \in \mathbb{R}$ and for any $u \in E$. We also recall that the functional $N \in C^1(E, \mathbb{R})$ is said to be not 2-homogeneous at infinity if there exist a, $c > 0$ and $\mu > 0$ such that

$$(H_2) \qquad |\langle N^{'}(u), u\rangle - 2N(u)| \geq a\psi(u)^{\mu} - c\,, \qquad \text{for all } u \in E\,.$$

We introduce the following additional hypotheses on the functionals J and N:

$$(H_3) \qquad J(u) \geq k\|u\|^2\,, \qquad \text{for all } u \in E$$

$$(H_4) \qquad |N(u)| \leq b\|u\|_X^q + d\,, \qquad \text{for all } u \in E\,,$$

for some constants k, b, $d > 0$ and $q > 2$.

Theorem 8. *Assume that assumptions (H_1), (H_2), (H_3) and (H_4) are fulfilled, with $qt < 2$. Then the functional I satisfies condition $(\hat{C})_c$, for all $c \in \mathbb{R}$.*

Proof. Let $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I^{'}(u_n)\|_{E^*} \to 0$. We have

$$\begin{aligned} |\langle I^{'}(u), u\rangle - 2I(u)| &= |\langle J^{'}(u) - N^{'}(u), u\rangle - 2J(u) + 2N(u)| \\ &= |\langle J^{'}(u), u\rangle - 2J(u) - (\langle N^{'}(u), u\rangle - 2N(u))|\,. \end{aligned}$$

But J is 2-homogeneous and

$$\frac{J(u + tu) - J(u)}{t} = J(u)\,\frac{(1+t)^2 - 1}{t}\,.$$

This implies $\langle J^{'}(u), u\rangle = 2J(u)$ and

$$|\langle I^{'}(u), u\rangle - 2I(u)| = |\langle N^{'}(u), u\rangle - 2N(u)|\,.$$

From (H_2) we obtain

$$|\langle I^{'}(u), u\rangle - 2I(u)| = |\langle N^{'}(u), u\rangle - 2N(u)| \geq a\psi(u)^{\mu} - c\,.$$

Letting $u = u_n$ in the inequality from above we have:

$$a\psi(u_n)^{\mu} \leq c + \|I^{'}(u_n)\|_{E^*}\|u_n\| + 2|I(u_n)|\,.$$

Thus, by our hypotheses, for some $c_0 > 0$ and all positive integer n, $\psi(u_n) \leq c_0$ and hence, the sequence $\{\psi(u_n)\}$ is bounded. Now, from (H_1) and (H_4) we obtain

$$J(u_n) = I(u_n) + N(u_n) \leq b\|u_n\|_X^q + d_0 \leq b\psi(u_n)^{(1-t)q}\|u_n\|^{qt} + d_0\,.$$

Hence

$$J(u_n) \leq b_0\|u_n\|^{qt} + d_0\,, \qquad \text{for all } n \in \mathbb{N}\,,$$

for some b_0, $d_0 > 0$. Finally, (H_3) implies

$$c\|u_n\|^2 \leq b_0\|u_n\|^{qt} + d_0\,, \qquad \text{for all } n \in \mathbb{N}\,.$$

Since $qt < 2$, we conclude that $(u_n)_n$ is bounded in E. □

Proposition 7. *Assume that $I(u) = J(u) - N(u)$ is as above, where $N^{'} : E \to E^*$ is a compact operator and $J^{'} : E \to E^*$ is an isomorphism from E onto $J^{'}(E)$. Then conditions $(C)_c$ and $(\hat{C})_c$ are equivalent.*

Proof. It is enough to show that $(\hat{C})_c$ implies $(C)_c$. Let $(u_n)_n \subset E$ be a sequence such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I^{'}(u_n)\|_{E^*} \to 0$. From $(\hat{C})_c$ we obtain a bounded subsequence $(u_{n_k})_k$ of $(u_n)_n$. But $N^{'}$ is a compact operator. Then $N^{'}(u_{n_{k_l}}) \xrightarrow{l} f^{'} \in E^*$, where $(u_{n_{k_l}})$ is a subsequence of (u_{n_k}). Since $(u_{n_{k_l}})$ is a bounded sequence and $(1 + \|u_{n_{k_l}}\|)\|I^{'}(u_{n_{k_l}})\|_{E^*} \to 0$, it follows that $\|I^{'}(u_{n_{k_l}})\| \to 0$. Next, using the relation

$$u_{n_{k_l}} = J^{'^{-1}}(I^{'}(u_{n_{k_l}}) + N^{'}(u_{n_{k_l}}))\,,$$

we obtain that $(u_{n_{k_l}})$ is a convergent subsequence of $(u_n)_n$. □

3.4 Proof of main results

PROOF OF THEOREMS 6 AND 7. We will use the following critical point theorems, which are obvious extensions of the Mountain Pass and Rabinowitz Theorems, corresponding to the Cerami compactness condition.

Theorem 9. *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies condition $(C)_c$, for all $c \in \mathbb{R}$ and, for some $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$,*

$$\max\{I(0), I(u_1)\} \leq \hat{\alpha} < \hat{\beta} \leq \inf_{\|u\|=\rho} I(u)\,.$$

Then I has a critical value $\hat{c} \geq \hat{\beta}$, characterized by

$$\hat{c} = \inf_{\gamma\in\Gamma} \max_{0\leq\tau\leq 1} I(\gamma(\tau))\,,$$

where $\Gamma := \{\gamma \in C([0,1], E);\ \gamma(0) = 0, \gamma(1) = u_1\}$.

Theorem 10. *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies condition $(C)_c$, for all $c \in \mathbb{R}$ and, for some $R > 0$ and some $E = V \oplus W$ with $\dim V < \infty$,*

$$\max_{v\in V, \|v\|=R} I(v) \leq \hat{\alpha} < \hat{\beta} \leq \inf_{w\in W} I(w)\,.$$

Then I has a critical value $\hat{c} \geq \hat{\beta}$, characterized by

$$\hat{c} = \inf_{h\in\Gamma} \max_{v\in V, \|v\|\leq R} I(h(v))\,,$$

where $\Gamma = \{h \in C(V \bigcap \bar{B}_R, E);\ h(v) = v$, for all $v \in \partial B_R\}$.

In our arguments we use the following auxiliary result.

Lemma 2. *Assume that G satisfies conditions $(G_1)_q$ and $(G_2^+)_\mu$ [or $(G_2^-)_\mu$], with $\mu > 2N/(q-2)$ if $N \geq 3$ or $\mu > q-2$ if $1 \leq N \leq 2$. Then the functional F satisfies condition $(C)_c$ for all $c \in \mathbb{R}$.*

Proof. Let

$$N(u) = \frac{\lambda_1}{2} \int_\Omega V(x)u^2 dx + \int_\Omega G(x,u)dx \qquad \text{and} \qquad J(u) = \frac{1}{2}\|u\|^2\,.$$

Obviously, J is homogeneous of degree 2 and $J^{'}$ is an isomorphism of $E = H_0^1(\Omega)$ onto $J^{'}(E) \subset H^{-1}(\Omega)$. It is known that $N^{'} : E \to E^*$ is a compact operator. Proposition 7 ensures that conditions $(C)_c$ and $(\hat{C})_c$ are equivalent. So, it suffices to show that $(\hat{C})_c$ holds for all $c \in \mathbb{R}$. Hypothesis (H_3) is trivially satisfied, whereas (H_4) holds true from $(G_1)_q$. Condition $(G_1)_q$ implies that

$$\inf_{|s|>0} \sup_{|t|>|s|} \frac{G(x,t)}{|t|^q} \leq b\,.$$

Therefore there exists $s_0 \neq 0$ such that

$$\sup_{|t|>|s_0|} \frac{G(x,t)}{|t|^q} \leq b \qquad \text{and} \qquad G(x,t) \leq b|t|^q, \quad \text{for all } t \text{ with } |t| > |s_0|\,.$$

The boundedness is provided by the continuity of the application $[-s_0, s_0] \ni t \longmapsto G(x,t)$. It follows that $\int\limits_\Omega G(x,u)dx \leq b|u|_q^q + d$. By the definition of $N(u)$ and since $q > 2$, we deduce that (H_4) holds true if $|u|_q \leq 1$. Indeed, we have $|u|_2 \leq k|u|_q$ because Ω is bounded. Therefore $|u|_2^2 \leq k|u|_q^2 \leq k|u|_q^q$ and finally (H_4) is fulfilled. Hypothesis (H_1) is a direct consequence of the Sobolev inequality. It remains to show that hypothesis (H_2) holds true, that is, the functional N is not 2-homogeneous at infinity. Indeed, using assumption $(G_2^+)_\mu$ (a similar argument works if $(G_2^-)_\mu$ is fulfilled) together with the subcritical condition on g yields

$$\sup_{|s|>0} \inf_{|t|>|s|} \frac{g(x,t)t - 2G(x,t)}{|t|^\mu} \geq a > 0\,.$$

It follows that there exists $s_0 \neq 0$ such that

$$\inf_{|t|>|s_0|} \frac{g(x,t)t - 2G(x,t)}{|t|^\mu} \geq a\,.$$

Hence

$$g(x,t)t - 2G(x,t) \geq a|t|^\mu\,, \qquad \text{for all } |t| > |s_0|\,.$$

The application $t \mapsto g(x,t)t - 2G(x,t)$ is continuous in $[-s_0, s_0]$, therefore it is bounded. We obtain $g(x,t) - 2G(x,t) \geq a_1|t|^\mu - c_1$, for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. We deduce that

$$\begin{aligned} |\langle N^{'}(u), u\rangle - 2N(u)| &= \left| \int\limits_\Omega (g(x,u)u - 2G(x,u))dx \right| \\ &\geq a_1\|u\|_\mu^\mu - c_2\,, \qquad \text{for all } u \in H_0^1(\Omega)\,. \end{aligned}$$

Consequently, the functional N is not 2-homogeneous at infinity.

Finally, when $N \geq 3$, we observe that condition $\mu > N(q-2)/2$ is equivalent with $\mu > 2^*(q-2)/2^* - 2$. From $1/q = (1-t)/\mu + t/2^*$ we obtain $(1-t)/\mu = (2^* - qt)/(2^* q)$. Hence $(2^* - qt)/q < (1-t)(2^* - 2)/(q-2)$ and, consequently, $(q - 2^*)(2 - tq) < 0$. But $q < 2^*$ and this implies $2 > tq$. Similarly, when $1 \leq N \leq 2$, we choose some $2^{**} > 2$ sufficiently large so that $\mu > 2^{**}(q-2)/(2^{**} - 2)$ and $t \in (0,1)$ be as above. The proof of Lemma is complete in view of Theorem 8. □

Our next step is to show that condition (G_3) implies the geometry of the Mountain Pass theorem for the functional F.

Lemma 3. *Assume that G satisfies the hypotheses*

$(G_1)_q$ $$\limsup_{|s|\to\infty} \frac{G(x,s)}{|s|^q} \leq b < \infty \quad \textit{uniformly a.e. } x \in \Omega$$

(G_3)
$$\limsup_{s\to 0} \frac{2G(x,s)}{s^2} \leq \alpha < \lambda_1 < \beta \leq \liminf_{|s|\to\infty} \frac{2G(x,s)}{|s|^2} \qquad \textit{uniformly a.e.} \quad x \in \Omega\,.$$

Then there exists ρ, $\gamma > 0$ such that $F(u) \geq \gamma$ if $|u| = \rho$. Moreover, there exists $\varphi_1 \in H_0^1(\Omega)$ such that $F(t\varphi_1) \to -\infty$ as $t \to \infty$.

Proof. In view of our hypotheses and the subcritical growth condition, we obtain
$$\liminf_{|s|\to\infty} \frac{2G(x,s)}{s^2} \geq \beta \text{ is equivalent with } \sup_{s\neq 0} \inf_{|t|>|s|} \frac{2G(x,t)}{t^2} \geq \beta\,.$$

There exists $s_0 \neq 0$ such that $\inf_{|t|>|s_0|} \frac{2G(x,t)}{t^2} \geq \beta$ and therefore $\frac{2G(x,t)}{t^2} \geq \beta$, for all $|t| > |s_0|$ or $G(x,t) \geq \frac{1}{2}\beta t^2$, provided $|t| > |s_0|$. We choose t_0 such that $|t_0| \leq |s_0|$ and $G(x,t_0) < \frac{1}{2}\beta|t_0|^2$. Fix $\varepsilon > 0$. There exists $B(\varepsilon, t_0)$ such that $G(x,t_0) \geq \frac{1}{2}(\beta - \varepsilon)|t_0|^2 - B(\varepsilon,t_0)$. Denote $B(\varepsilon) = \sup_{|t_0|\leq|s_0|} B(\varepsilon,t_0)$. We obtain for any given $\varepsilon > 0$ there exists $B = B(\varepsilon)$ such that
$$G(x,s) \geq \frac{1}{2}\,(\beta - \varepsilon)|s|^2 - B\,, \qquad \text{for all } s \in \mathbb{R}\,,\ \text{a.e.} \quad x \in \Omega\,. \tag{3.7}$$

Fix arbitrarily $\varepsilon > 0$. In the same way, using the second inequality of (G_3) and $(G_1)_q$ it follows that there exists $A = A(\varepsilon) > 0$ such that
$$2G(x,t) \leq (\alpha + \varepsilon)t^2 + 2(b + A(\varepsilon))|t|^q\,, \qquad \text{for all } t \in \mathbb{R}\,,\ \text{a.e.} \quad x \in \Omega\,. \tag{3.8}$$

We now choose $\varepsilon > 0$ so that $\alpha + \varepsilon < \lambda_1$ and we use (3.8) together with the Poincaré inequality to obtain the first assertion of the lemma.

Set $H(x,s) = \lambda_1 V(x) s^2/2 + G(x,s)$. Then H satisfies

$$(H_1)_q \qquad \limsup_{|s|\to\infty} \frac{H(x,s)}{|s|^q} \le b < \infty\,, \qquad \text{uniformly a.e.} \quad x \in \Omega$$

(H_3)

$$\limsup_{s\to 0} \frac{2H(x,s)}{s^2} \le \alpha < \lambda_1 < \beta \le \liminf_{|s|\to\infty} \frac{2H(x,s)}{s^2}\,, \qquad \text{uniformly a.e.} \quad x\in\Omega\,.$$

In the same way, for any given $\varepsilon > 0$ there exists $A = A(\varepsilon) > 0$ and $B = B(\varepsilon)$ such that

$$\begin{aligned} \frac{1}{2}(\beta - \varepsilon)s^2 - B \quad &\le H(x,s) \\ &\le \frac{1}{2}(\alpha+\varepsilon)s^2 + A|s|^q\,, \qquad \text{for all } s \in \mathbb{R}\,,\ \text{a.e. } x \in \Omega\,. \end{aligned} \tag{3.9}$$

We have

$$\begin{aligned} F(u) \quad &= \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u)dx \ge \frac{1}{2}\|u\|^2 - \frac{1}{2}(\alpha+\varepsilon)|u|_2^2 - A|u|_q^q \\ &\ge \frac{1}{2}\left(1 - \frac{\varepsilon+\alpha}{\lambda_1}\right)\|u\|^2 - Ak\|u\|^q\,. \end{aligned}$$

We can assume without loss of generality that $q > 2$. Thus, the above estimate yields $F(u) \ge \gamma$ for some $\gamma > 0$, as long as $\rho > 0$ is small, thus proving the first assertion of the lemma.

On the other hand, choosing now $\varepsilon > 0$ so that $\beta - \varepsilon > \lambda_1$ and using (3.9), we obtain

$$F(u) \le \frac{1}{2}\|u\|^2 - \frac{\beta-\varepsilon}{2}\,|u|_2^2 + B|\Omega|\,.$$

We consider φ_1 be the λ_1-eigenfunction with $\|\varphi_1\| = 1$. It follows that

$$F(t\varphi_1) \le \frac{1}{2}\left(1 - \frac{\beta-\varepsilon}{\lambda_1}\right)t^2 + B|\Omega| \to -\infty \qquad \text{as } t \to \infty.$$

This proves the second assertion of our lemma. □

Lemma 4. *Assume that $G(x,s)$ satisfies the conditions $(G_2^-)_\mu$ (for some $\mu > 0$) and*

$$(G_4) \qquad \lim_{|s|\to\infty} \frac{G(x,s)}{s^2} = 0\,, \qquad \textit{uniformly a.e.} \quad x \in \Omega\,.$$

Then there exists a subspace W *of* $H_0^1(\Omega)$ *such that* $H_0^1(\Omega) = V \oplus W$ *and*
(i) $F(v) \to -\infty$*, as* $\|v\| \to \infty$*,* $v \in V$*;*
(ii) $F(w) \to \infty$*, as* $\|w\| \to \infty$*,* $w \in W$*.*

Proof. (i) The condition $(G_2^-)_\mu$ is equivalent with the fact that there exists $s_0 \neq 0$ such that

$$g(x,s)s - 2G(x,s) \le -a|s|^\mu\,, \quad \text{for all } |s| \ge |s_0| = R_1\,,\ \text{a.e. } x \in \Omega\,.$$

Integrating the identity

$$\frac{d}{ds}\frac{G(x,s)}{|s|^2} = \frac{g(x,s)s^2 - 2|s|G(x,s)}{s^4} = \frac{g(x,s)|s| - 2G(x,s)}{|s|^3}$$

over an interval $[t,T] \subset [R,\infty)$ and using the above inequality we find

$$\frac{G(x,T)}{T^2} - \frac{G(x,t)}{t^2} \le -a\int_t^T s^{\mu-3}ds = \frac{a}{2-\mu}\left(\frac{1}{T^{2-\mu}} - \frac{1}{t^{2-\mu}}\right).$$

Since we can assume that $\mu < 2$ and using the above relation, we obtain

$$G(x,t) \ge \hat{a}t^\mu\,, \qquad \text{for all } t \ge R_1\,,\ \text{where } \hat{a} = \frac{a}{2-\mu} > 0\,.$$

Similarly, we show that

$$G(x,t) \ge \hat{a}|t|^\mu\,, \qquad \text{for } |t| \ge R_1\,.$$

Consequently, $\lim_{|t|\to\infty} G(x,t) = \infty$. Now, letting $v = t\varphi_1 \in V$ and using the variational characterization of λ_1, we have

$$F(v) \ge -\int_\Omega G(x,v)dx \to -\infty\,, \qquad \text{as } \|v\| = |t|\|\varphi_1\| \to \infty\,.$$

This result is a consequence of the Lebesgue's Dominated Convergence Theorem.

(ii) Let $V = \mathrm{Sp}\,(\varphi_1)$ and $W \subset H_0^1(\Omega)$ be a closed complementary subspace to V. Since λ_1 is an eigenvalue of Problem (3.1), it follows that there exists $d > 0$ such that

$$\inf_{0\neq w\in W}\frac{\int_\Omega |\nabla w|^2dx}{\int_\Omega V(x)w^2dx} \ge \lambda_1 + d\,.$$

Therefore

$$\|w\|^2 \geq (\lambda_1 + d)|w|_2^2\,, \qquad \text{for all } w \in W\,.$$

Let $0 < \varepsilon < d$. From (G_4) we deduce that there exists $\delta = \delta(\varepsilon) > 0$ such that for all s satisfying $|s| > \delta$ we have $2G(x,s)/s^2 \leq \varepsilon$, a.e. $x \in \Omega$. In conclusion

$$G(x,s) - \frac{1}{2}\varepsilon s^2 \leq M\,, \qquad \text{for all } s \in \mathbb{R}\,,$$

where

$$M := \sup_{|s|\leq\delta} \left(G(x,s) - \frac{1}{2}\,\varepsilon\, s^2 \right) < \infty\,.$$

Therefore

$$\begin{aligned} F(w) \;&\geq \frac{1}{2}\|w\|^2 - \frac{\lambda_1}{2}|w|_2^2 - \frac{1}{2}\varepsilon|w|_2^2 - M \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda_1 + \varepsilon}{\lambda_1 + d}\right)\|w\|^2 - M = N\|w\|^2 - M\,, \qquad \text{for all } w \in W\,. \end{aligned}$$

It follows that $F(w) \to \infty$ as $\|w\| \to \infty$, for all $w \in W$, which completes the proof of the lemma. □

Proof of Theorem 6. In view of Lemmas 2 and 3, we may apply the Mountain Pass theorem with $u_1 = t_1\varphi_1$, $t_1 > 0$ being such that $F(t_1\varphi_1) \leq 0$ (this is possible from Lemma 3). Since $F(u) \geq \gamma$ if $\|u\| = \rho$, we have

$$\max\{F(0), F(u_1)\} = 0 = \hat{\alpha} < \inf_{\|u\|=\rho} F(u) = \hat{\beta}\,.$$

It follows that the energy functional F has a critical value $\hat{c} \geq \hat{\beta} > 0$ and, hence, Problem (3.2) has a nontrivial solution $u \in H_0^1(\Omega)$. □

Proof of Theorem 7. In view of Lemmas 2 and 4, we may apply the Saddle Point theorem with $\hat{\beta} := \inf_{w\in W} F(w)$ and $R > 0$ being such that $\sup_{\|v\|=R} F(v) := \hat{\alpha} < \hat{\beta}$, for all $v \in V$ (this is possible because $F(v) \to -\infty$ as $\|v\| \to \infty$). It follows that F has a critical value $\hat{c} \geq \hat{\beta}$, which is a weak solution of Problem (3.2). □

Chapter 4

Boundary value problems in Sobolev spaces with variable exponent

If I have seen further it is by standing on the shoulders of giants.

Sir Isaac Newton (1642-1727),
Letter to Robert Hooke, 1675

Abstract. In the first part of this chapter we consider a class of nonlinear Dirichlet problems involving the $p(x)$–Laplace operator. Our framework is based on the theory of Sobolev spaces with variable exponent and we establish the existence of a weak solution in such a space. Next, we study the boundary value problem $-\text{div}((|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u) = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$. We focus on the cases when $f_{\pm}(x,u) = \pm(-\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u)$, where $m(x) := \max\{p_1(x), p_2(x)\} < q(x) < \frac{N\cdot m(x)}{N-m(x)}$ for any $x \in \overline{\Omega}$. In the first case we show the existence of infinitely many weak solutions for any $\lambda > 0$. In the second case we prove that if λ is large enough then there exists a nontrivial weak solution. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with a $\mathbb{Z}_2$-symmetric version for even functionals of the Mountain Pass Lemma and some adequate variational methods.

4.1 Basic properties of Sobolev spaces with variable exponent

In this section we recall the main properties of Lebesgue and Sobolev spaces with variable exponent. We point out that these functional spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [116], who proved various results (including Hölder's inequality) in a discrete framework. Orlicz also considered the variable exponent function space $L^{p(x)}$ on the real line, and proved the Hölder inequality in this setting, too. Next, Orlicz abandoned the study of variable exponent spaces, to concentrate on the theory of the function spaces that now bear his name. The first systematic study of spaces with variable exponent (called *modular spaces*) is due to Nakano [109]. In the appendix of this book, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers [109, p. 284]. Despite their broad interest, these spaces have not reached the same main-stream position as Orlicz spaces. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Polish mathematicians. We refer to the book by Musielak [107] for a nice presentation of modular function spaces. This book, although not dealing specifically with the spaces that interest us, is still specific enough to contain several interesting results regarding variable exponent spaces. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably Sharapudinov. These investigations originated in a paper by Tsenov [149]. The question raised by Tsenov and solved by Sharapudinov [142] is the minimization of $\int_a^b |u(x) - v(x)|^{p(x)}dx$, where u is a fixed function and v varies over a finite dimensional subspace of $L^{p(x)}([a, b])$. Sharapudinov also introduces the Luxemburg norm for the Lebesgue space and shows that this space is reflexive if the exponent satisfies $1 < p^- \leq p^+ < \infty$. In the 80's Zhikov started a new line of investigation, that was to become intimately related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions. These notions have been widely applied in various fields, including electrorheological fluids (sometimes referred to as "smart fluids"), which are particular fluids of high technological interest whose apparent viscosity changes reversibly in response to an electric field. The electrorheological fluids have been intensively studied from the 1940's to the present. The first major discovery on electrorheological fluids is due to Willis M. Winslow [153]. He noticed that such fluids' (for instance lithium polymetachrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For a general account of the underlying physics we refer to Halsey [69], while for some technical applications we refer to Pfeiffer, Mavroidis, Bar-Cohen and Doljin [119]. We just remember that any device which currently depends upon hydraulics, hydrodynamics or hydrostatics can benefit from electrorheological fluids' properties. Consequently, electrorheological fluids are most promising in aircraft and aerospace applications. For more information on properties and the application of these fluids we refer to Acerbi and Mingione [2], Diening [38], Halsey [69] and Rabinowitz [126].

We recall in what follows the main properties of Sobolev spaces with variable exponent.

Let Ω be a bounded open set in $\mathbb{R}^N$.

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue and Sobolev spaces

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On these spaces we define, respectively, the following norms

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\} \qquad \text{(called Luxemburg norm)}$$

and

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}\,.$$

Variable exponent Lebesgue and Sobolev spaces resemble classical Lebesgue and Sobolev spaces in many respects: they are Banach spaces [84, Theorem 2.5], the Hölder inequality holds [84, Theorem 2.1], they are reflexive if and only if $1 < p^- \le p^+ < \infty$ [84, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [84, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [84, Theorem 2.8]: if $0 < |\Omega| < \infty$ and p_1, $p_2 \in C_+(\overline{\Omega})$ are variable exponent so that $p_1(x) \le p_2(x)$ in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\;dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) |u|_{p(x)} |v|_{p'(x)} \tag{4.1}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)}$: $L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\;dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)} \tag{4.2}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)} \tag{4.3}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \tag{4.4}$$

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [45].

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. On this space we can use the equivalent norm $\|u\| = |\nabla u|_{p(x)}$. The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. The dual of this space is denoted by $W_0^{-1,p'(x)}(\Omega)$. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact, while $W_0^{1,p(x)}(\Omega)$ is continuously embedded into $L^{p^*(x)}(\Omega)$, where $p^\star(x)$ denotes the critical Sobolev exponent, that is, $p^\star(x) = Np(x)/(N - p(x))$, provided that $p(x) < N$ for all $x \in \overline{\Omega}$.

Remark 1. *If* $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$, *then* $m(x) \in C_+(\overline{\Omega})$, *where* $m(x) = \max\{p_1(x), p_2(x)\}$, *for any* $x \in \overline{\Omega}$. *On the other hand, since* $p_1(x)$, $p_2(x) \leq m(x)$ *for any* $x \in \overline{\Omega}$, *it follows that* $W_0^{1,m(x)}(\Omega)$ *is continuously embedded in* $W_0^{1,p_i(x)}(\Omega)$ *for* $i \in \{1, 2\}$.

We refer to Diening [38], Edmunds and Rákosník [46, 47], Fan and Han [52], Fan, Shen and Zhao [53], Fan, Zhang and Zhao [55], Fan and Zhao [56], Kováčik and Rákosník [84] and Ruzicka [140] for further properties and applications of variable exponent Lebesgue–Sobolev spaces.

4.2 A nonlinear eigenvalue problem

The Mountain Pass Theorem is due to Ambrosetti and Rabinowitz [5] and is one of the most powerful tools in Nonlinear Analysis for proving the existence of critical points of energy functionals. One of the simplest versions of the Mountain Pass Theorem asserts that if a continuously differential functional has two local minima, then (under some natural assumptions) such a function has a third critical point. This fact is elementary for functions of one real variable. However, even for functions on the plane the proof of such a theorem requires deep topological ideas. The Mountain Pass Theorem has numerous generalizations and has been applied in the treatment of various classes of boundary value problems. We refer to the recent monograph by Jabri [75] for an excellent survey of some of the most interesting applications of this abstract result. We do not intend to insist on the wide spectrum of applications of the Mountain Pass Theorem. We remark only that this theorem has been applied in the last few years in very concrete situations. For instance, in Lewin [95] it is considered a neutral molecule that possesses two distinct stable positions for its nuclei, and it is looked for a mountain pass point between the two minima in the non-relativistic Schrödinger framework.

As showed in Ambrosetti and Rabinowitz [5], one of the simplest applications of the Mountain Pass Theorem implies the existence of solutions for the

Dirichlet problem

$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2 < p < 2N/(N-2)$ if $N \geq 3$ and $p \in (2, \infty)$ if $N = 1$ or $N = 2$.

This equation is called the *Kazdan–Warner equation* and the existence results are related not only to the values of p, but also to the geometry of Ω. For instance, problem (4.5) has no solution if $p \geq 2N/(N-2)$ and if Ω is a *starshaped domain* with respect to a certain point (the proof uses the *Pohozaev identity*, which is obtained after multiplication in (4.5) with $x \cdot \nabla u$ and integration by parts). If Ω is **not** starshaped, Kazdan and Warner proved in [82] that problem (4.5) has a solution for **any** $p > 2$, where Ω is an **annulus** in $\mathbb{R}^N$.

Under the same assumptions on the subcritical exponent p, similar arguments show that the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution for any $\lambda < \lambda_1$, where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. The proof of this result relies on the fact that the operator $(-\Delta - \lambda I)$ is coercive if $\lambda < \lambda_1$. Moreover, by multiplication with φ_1 and integration on Ω we deduce that there is no solution if $\lambda \geq \lambda_1$, where φ_1 stands for the first eigenfunction of the Laplace operator. We refer to Precup [122] for interesting localization results of solutions to problems of the above type, as well as for a lower bound of all nontrivial solutions.

The main purpose of the first part of this chapter is to study a related problem, but for a more general differential operator, the so-called $p(x)$–Laplace operator. This degenerate differential operator is defined by $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ (where $p(x)$ is a certain function whose properties will be stated in what follows) and that generalizes the celebrated p–Laplace operator, defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a constant. The $p(x)$–Laplace operator possesses more complicated nonlinearity than the p–Laplacian, for example, it is inhomogeneous. We only recall that Δ_p describes a variety of phenomena in the nature. For instance, the equation governing the motion of a fluid involves the p–Laplace operator. More exactly, the shear stress $\vec{\tau}$ and the velocity gradient ∇u of the fluid are related in the manner that $\vec{\tau}(x) = r(x)|\nabla u|^{p-2}\nabla u$, where $p = 2$ (resp., $p < 2$ or $p > 2$) if the fluid is Newtonian (resp., pseudoplastic or dilatant). Other applications of the p–Laplacian also appear in the study of flow through porous media ($p = 3/2$), Nonlinear Elasticity ($p \geq 2$), or Glaciology ($1 < p \leq 4/3$).

Assume that Ω is a smooth bounded open set in $\mathbb{R}^N$ ($N \geq 2$), λ is a real parameter and $p \in C_+(\overline{\Omega})$.

Consider the boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda u^{p(x)-1} + u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega\,, \end{cases} \tag{4.6}$$

where $p \in C_+(\overline{\Omega})$ such that $p^+ < N$, and q is a real number.

Definition 2. *Let λ be a real number. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a solution of Problem (4.6) if $u \geq 0$, $u \not\equiv 0$ in Ω and*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \lambda \int_\Omega u^{p(x)-1} v dx + \int_\Omega u^{q-1} v dx, \qquad \forall v \in W_0^{1,p(x)}(\Omega)\,.$$

A crucial role in the statement of our result will be played by the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega\,. \end{cases} \tag{4.7}$$

It follows easily that if (u, λ) is a solution of (4.7) then

$$\lambda = \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}$$

and hence $\lambda > 0$. Let Λ denote the set of eigenvalues of (4.7), that is,

$$\Lambda = \Lambda_{p(x)} = \{\lambda \in \mathbb{R};\ \lambda \text{ is an eigenvalue of Problem (4.7)}\}.$$

In Garcia Azorero and Peral Alonso [60] it is showed that if the function $p(x)$ is a constant $p > 1$ (we refer to Brezis [18] for the linear case $p(x) \equiv 2$), then Problem (4.7) has a sequence of eigenvalues, $\sup \Lambda = +\infty$ and $\inf \Lambda = \lambda_1 = \lambda_{1,p} > 0$, where $\lambda_{1,p}$ is the first eigenvalue of $(-\Delta_p)$ in $W_0^{1,p}(\Omega)$ and

$$\lambda_1 = \lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega \setminus \{0\})} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}.$$

In Fan, Zhang and Zhao [55] it is showed that for general functions $p(x)$ the set Λ is infinite and $\sup \Lambda = +\infty$. Moreover, it may arise that $\inf \Lambda = 0$. Set

$$\lambda^* = \lambda^*_{p(x)} = \inf \Lambda.$$

In Fan, Zhang and Zhao [55] it is argued that if $N = 1$ then $\lambda^* > 0$ if and only if the function $p(x)$ is monotone. In arbitrary dimension, $\lambda^* = 0$ provided that there exist an open set $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) <$ (or $>$) $p(x)$ for all $x \in \partial U$.

Theorem 11. *Assume that $\lambda < \lambda^*$ and $p^+ < q < Np^-/(N - p^-)$. Then Problem (4.6) has at least a solution.*

We cannot expect that Problem (4.6) has a solution for any $\lambda \geq \lambda^*$. Indeed, consider the simplest case $p(x) \equiv 2$, take $\lambda \geq \lambda_1$ and multiply the equation in (4.6) by $\varphi_1 > 0$. Integrating on Ω we find

$$(\lambda - \lambda_1) \int_\Omega u \varphi_1 dx + \int_\Omega u^{q-1} \varphi_1 dx = 0$$

which yields a contradiction.

The proof of the above result relies on the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [5]. We refer to Dincă, Jebelean and Mawhin [39, 40] for variants of Theorem 11 corresponding to the p–Laplace operator.

4.3 Proof of Theorem 11

Our hypothesis $\lambda < \lambda^*$ implies that there exists $C_0 > 0$ such that

$$\int_\Omega (|\nabla v|^{p(x)} - \lambda |v|^{p(x)})dx \geq C_0 \int_\Omega |\nabla v|^{p(x)} dx \qquad \text{for all } v \in W_0^{1,p(x)}(\Omega)\,. \tag{4.8}$$

Set

$$g(u) = \begin{cases} u^{q-1}, & \text{if } u \geq 0\,, \\ 0, & \text{if } u < 0 \end{cases}$$

and $G(u) = \int_0^u g(t)dt$. Define the energy functional associated to Problem (4.6) by

$$J(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} - \lambda |u|^{p(x)}\right) dx - \int_\Omega G(u)dx \qquad \text{for all } u \in W_0^{1,p(x)}\,.$$

Observe that

$$|G(u)| \leq C\,|u|^q$$

and, by our hypotheses on $p(x)$ and q, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow L^q(\Omega)$, which implies that J is well defined on $W_0^{1,p(x)}(\Omega)$.

A straightforward computation shows that J is of class C^1 and, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$J'(u)(v) = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v - \lambda\,|u|^{p(x)-2} uv)dx - \int_\Omega g(u)v dx\,.$$

We prove in what follows that J satisfies the hypotheses of the Mountain Pass Theorem.

Firstly, let us observe that we may write, for every $u \in \mathbb{R}$,

$$|g(u)| \leq |u|^{q-1}\,.$$

Thus, for all $u \in \mathbb{R}$,

$$|G(u)| \leq \frac{1}{q}\,|u|^q\,. \tag{4.9}$$

Next, by (4.8) and (4.9),

$$\begin{aligned} J(u) &\geq \frac{C_0}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - C \int_\Omega |u|^q dx \\ &= C_1 \int_\Omega |\nabla u|^{p(x)} dx - C_2\,\|u\|_{L^q}^q dx \end{aligned}\,, \tag{4.10}$$

for every $u \in W_0^{1,p(x)}(\Omega)$, where C_1 and C_2 are positive constants. So, by relation (4.3) and using the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^q(\Omega)$ combined with the assumption $p^+ < q$ we find, for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = |\nabla u|_{p(x)} = R$ sufficiently small,

$$J(u) \geq C_1 \, |\nabla u|_{p(x)}^{p^+} - C_3 \, |\nabla u|_{p(x)}^q \geq c_0 > 0 \, .$$

For the second geometric assumption of the Mountain Pass Theorem, we choose $u_0 \in W_0^{1,p(x)}(\Omega)$ such that $u_0 > 0$ in Ω. Since $p^+ < q$, it follows that if $t > 0$ is large enough then

$$J(tu_0) = \int_\Omega \frac{t^{p(x)}}{p(x)} \left(|\nabla u_0|^{p(x)} - \lambda |u_0|^{p(x)} \right) dx - \frac{t^q}{q} \int_\Omega u_0^q dx < 0 \, .$$

Verification of the Palais-Smale condition. Let (u_n) be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\sup_n |J(u_n)| < +\infty \tag{4.11}$$

$$\|J'(u_n)\|_{W^{-1,p'(x)}} \to 0 \qquad \text{as } n \to \infty \, . \tag{4.12}$$

We first prove that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Remark that (4.12) implies that, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$\int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla v - \lambda \, |u_n|^{p(x)-2} \, u_n v) dx = \int_\Omega g(u_n) v dx + o(1) \, \|v\| \, . \tag{4.13}$$

Choosing $v = u_n$ in (4.13) we find

$$\int_\Omega \left(|\nabla u_n|^{p(x)} - \lambda \, |u_n|^{p(x)} \right) dx = \int_\Omega g(u_n) u_n dx + o(1) \, \|u_n\| \, . \tag{4.14}$$

Relation (4.11) implies that there exists $M > 0$ such that, for any $n \geq 1$,

$$\left| \int_\Omega \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} - \lambda \, |u_n|^{p(x)} \right) dx - \int_\Omega G(u_n) dx \right| \leq M \, . \tag{4.15}$$

But a simple computation yields

$$\int_\Omega g(u_n) u_n dx = q \int_\Omega G(u_n) dx \, . \tag{4.16}$$

Combining (4.14), (4.15) and (4.16) and using our assumption $p^+ < q$ we find

$$\int_\Omega G(u_n)dx = O(1) + o(1)\,\|u_n\|\,. \tag{4.17}$$

Thus, by (4.14) and (4.17),

$$\int_\Omega |\nabla u_n|^{p(x)}dx = O(1) + o(1)\,\|u_n\|\,,$$

which means that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$.

It remains to prove that (u_n) is relatively compact. We first remark that (4.13) may be rewritten as

$$\int_\Omega |\nabla u_n|^{p(x)-2}\,\nabla u_n \cdot \nabla v dx = \int_\Omega h(x,u_n)v dx + o(1)\,\|v\|\,, \tag{4.18}$$

for every $v \in W_0^{1,p(x)}(\Omega)$, where

$$h(x,u) = g(u) + \lambda\,|u|^{p(x)-2}\,u\,,$$

where $\lambda < \lambda^*$ is fixed. Obviously, h is continuous and, since $q < Np(x)/(N - p(x))$ for all $x \in \overline{\Omega}$, there exists $C > 0$ such that

$$|h(x,u)| \le C\,\left(1 + |u|^{(Np(x)-N+p(x))/(N-p(x))}\right) \qquad \text{for all } x \in \overline{\Omega} \text{ and } u \in \mathbb{R}\,. \tag{4.19}$$

Moreover

$$h(x,u) = o\left(|u|^{Np(x)/(N-p(x))}\right) \qquad \text{as } |u| \to \infty, \text{ uniformly for } x \in \overline{\Omega}\,. \tag{4.20}$$

Define $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ by $Au = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. Then A is invertible and $A^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is a continuous operator. Thus, by (4.18), it suffices to show that $h(x,u_n)$ is relatively compact in $W^{-1,p'(x)}(\Omega)$. By continuous embeddings for Sobolev spaces with variable exponent, this will be achieved by proving that a subsequence of $h(x,u_n)$ is convergent in

$$(L^{Np(x)/(N-p(x))}(\Omega)\,)^\star = L^{Np(x)/(Np(x)-N+p(x))}(\Omega)\,.$$

Since (u_n) is bounded in $W_0^{1,p(x)}(\Omega) \subset L^{Np(x)/(N-p(x))}(\Omega)$ we can suppose that, up to a subsequence,

$$u_n \to u \in L^{Np(x)/(N-p(x))}(\Omega) \qquad \text{a.e. in } \Omega\,.$$

Moreover, by Egorov's Theorem, for each $\delta > 0$, there exists a subset A of Ω with $|A| < \delta$ and such that

$$u_n \to u \qquad \text{uniformly in } \Omega \setminus A\,.$$

So, it is sufficient to show that

$$\int_A |h(u_n) - h(u)|^{Np(x)/(Np(x)-N+p(x))}\, dx \leq \eta\,,$$

for any fixed $\eta > 0$. But, by (4.19),

$$\int_A |h(u)|^{Np(x)/(Np(x)-N+p(x))} dx \leq C \int_A (1 + |u|^{Np(x)/(N-p(x))}) dx\,,$$

which can be made arbitrarily small if we choose a sufficiently small $\delta > 0$.

We have, by (4.20),

$$\int_A |h(u_n) - h(u)|^{Np(x)/(Np(x)-N+p(x))} dx \leq \varepsilon \int_A |u_n - u|^{Np(x)/(N-p(x))} dx + C_\varepsilon\, |A|\,,$$

which can be also made arbitrarily small, by continuous embeddings for Sobolev spaces with variable exponent combined with the boundedness of (u_n) in $W_0^{1,p(x)}(\Omega)$. Hence, J satisfies the Palais-Smale condition. Thus, by the Mountain Pass Theorem, the boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2} u + g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$. It remains to show that $u \geq 0$. Indeed, multiplying the equation by u^- and integrating we find

$$\int_\Omega |\nabla u^-|^{p(x)} dx - \lambda \int_\Omega (u^-)^{p(x)} dx = 0\,.$$

Thus, since $\lambda < \lambda^*$, we deduce that $u^- = 0$ in Ω or, equivalently, $u \geq 0$ in Ω. □

A careful analysis of the above proof shows that the existence result stated in Theorem 11 remains valid if u^{q-1} is replaced by the more general nonlinearity $f(x,u)$, where $f(x,u) : \overline{\Omega} \to \mathbb{R}$ is a continuous functions satisfying

$$|f(x,u)| \leq C(|u| + |u|^{q-1}), \qquad \forall x \in \Omega,\ \forall u \in \mathbb{R}$$

with $p^- < q < Np^+/(N-p^+)$ if $N \geq 3$ and $q \in (p^-, \infty)$ if $N = 1$ or $N = 2$,

$$\lim_{\varepsilon \searrow 0} \sup \left\{ \left| \frac{f(x,t)}{t} \right| ;\ (x,t) \in \overline{\Omega} \times (-\varepsilon, \varepsilon) \right\} = 0 \qquad \text{uniformly for } x \in \overline{\Omega}$$

and

$$0 \leq \mu F(x,u) \leq u f(x,u) \qquad \text{for } 0 < u \text{ large and some } \mu > p^+,$$

where $F(x,u) = \int_0^u f(x,t)dt$.

The following result shows that Theorem 11 still remains valid if the right hand-side is affected by a small perturbation. Consider the boundary value problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u + |u|^{q-2}u + a(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4.21}$$

where $a \in L^\infty(\Omega)$, $p \in C_+(\overline{\Omega})$ such that $p^+ < N$, and q is a real number.

Corollary 1. *Assume that $\lambda < \lambda^*$ and $p^+ < q < Np^-/(N-p^-)$. There exists $\delta > 0$ such that if $\|a\|_{L^\infty} < \delta$ then Problem (4.21) has at least a solution.*

Proof. For any $u \in W_0^{1,p(x)}(\Omega)$ define the energy functional

$$E(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} - \lambda |u|^{p(x)} \right) dx - \frac{1}{q} \int_\Omega |u|^q dx - \int_\Omega a(x) u dx\,.$$

We have already seen that if $a = 0$ then Problem (4.21) has a nontrivial and nonnegative solution. If $\|a\|_{L^\infty}$ is sufficiently small then the verification of the Palais-Smale condition, as well as of the two geometric assumptions can be made following the same ideas as in the proof of Theorem 11. Thus, by the Mountain Pass Theorem, the functional E has a nontrivial critical point $u \in W_0^{1,p(x)}(\Omega)$, which is a solution of Problem (4.21). However, we are not able to decide if this solution is nonnegative. This result remains true if $a \geq 0$, as we can see easily after multiplication with u^- and integration. □

4.4 A nonlinear eigenvalue problem with two variable exponents

We study in what follows the boundary value problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u)=f(x,u), & \text{for} \quad x\in\Omega \\ u=0, & \text{for} \quad x\in\partial\Omega \end{cases} \tag{4.22}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary and $1<p_i(x)$, $p_i(x)\in C(\overline{\Omega})$ for $i\in\{1,2\}$. We are looking for nontrivial weak solutions of Problem (4.22) in the generalized Sobolev space $W^{1,m(x)}(\Omega)$, where $m(x)=\max\{p_1(x),p_2(x)\}$ for any $x\in\overline{\Omega}$. We point out that problems of type (4.22) were intensively studied in the past decades. We refer to Chabrowski and Fu [25], Fan and Zhang [54], Fan, Zhang and Zhao [55] for some interesting results.

We study Problem (4.22) if $f(x,t)=\pm(-\lambda|t|^{m(x)-2}t+|t|^{q(x)-2}t)$, where

$$m(x):=\max\{p_1(x),p_2(x)\}<q(x)<\begin{cases} \dfrac{N\cdot m(x)}{N-m(x)} & \text{if } m(x)<N \\ +\infty & \text{if } m(x)\geq N\,, \end{cases}$$

for any $x\in\overline{\Omega}$ and all $\lambda>0$.

We first consider the problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u)=-\lambda|u|^{m(x)-2}u+|u|^{q(x)-2}u, & x\in\Omega \\ u=0, & x\in\partial\Omega. \end{cases} \tag{4.23}$$

We say that $u\in W_0^{1,m(x)}(\Omega)$ is a *weak solution* of problem (4.23) if

$$\int_\Omega(|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u\nabla v\,dx \quad +\lambda\int_\Omega|u|^{m(x)-2}uv\,dx - \int_\Omega|u|^{q(x)-2}uv\,dx=0,$$

for all $v\in W_0^{1,m(x)}(\Omega)$.

We prove

Theorem 12. *For every $\lambda > 0$ problem (4.23) has infinitely many weak solutions, provided that $2 \leq p_i^-$ for $i \in \{1, 2\}$, $m^+ < q^-$ and $q^+ < \frac{N \cdot m^-}{N - m^-}$.*

Next, we study the problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = \lambda |u|^{m(x)-2}u - |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{4.24}$$

We say that $u \in W_0^{1,m(x)}(\Omega)$ is a *weak solution* of problem (4.24) if

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v \, dx \quad -\lambda \int_\Omega |u|^{m(x)-2} uv \, dx + \int_\Omega |u|^{q(x)-2} uv \, dx = 0,$$

for all $v \in W_0^{1,m(x)}(\Omega)$.

We prove

Theorem 13. *There exists $\lambda^\star > 0$ such that for any $\lambda \geq \lambda^\star$ problem (4.24) has a nontrivial weak solution, provided that $m^+ < q^-$ and $q^+ < \frac{N \cdot m^-}{N - m^-}$.*

There are strong similarities but also differences between problems (4.23) and (4.24). We first observe that the signs are reversed in the right hand-sides. Next, Problem (4.23) admits infinitely many solutions for **any** $\lambda > 0$. In contrast, Problem (4.24) admits **at least** one solution, provided λ is **sufficiently large**.

4.5 Proof of Theorem 12

The key argument in the proof of Theorem 12 is the following $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass Lemma (see Theorem 9.12 in Rabinowitz [126]).

Theorem 14. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition (that is, any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I^{'}(x_n) \to 0$ in $X^\star$ has a convergent subsequence) and $I(0) = 0$. Suppose that*

(I1) There exist two constants ρ, $a > 0$ such that $I(x) \geq a$ if $\|x\| = \rho$.

(I2) For each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1;\ I(x) \geq 0\}$ is bounded.

Then I has an unbounded sequence of critical values.

Let E denote the generalized Sobolev space $W_0^{1,m(x)}(\Omega)$.

The energy functional corresponding to problem (4.23) is defined by $J_\lambda : E \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx \quad +\lambda \int_\Omega \frac{1}{m(x)}|u|^{m(x)}\,dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx.$$

A simple calculation based on Remark 1, relations (4.2) and (4.3) and the compact embedding of E into $L^{s(x)}(\Omega)$ for all $s \in C_+(\overline{\Omega})$ with $s(x) < m^\star(x)$ on $\overline{\Omega}$ shows that J_λ is well-defined on E and $J_\lambda \in C^1(E,\mathbb{R})$ with the derivative given by

$$\langle J_\lambda'(u), v\rangle = \int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v\,dx \quad +\lambda \int_\Omega |u|^{m(x)-2}uv\,dx - \int_\Omega |u|^{q(x)-2}uv\,dx,$$

for any $u,\ v \in E$. Thus the weak solutions of (4.23) are exactly the critical points of J_λ.

Lemma 5. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\|_{m(x)} = \eta$.*

Proof. We first point out that since $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$ then

$$|\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)} \geq |\nabla u(x)|^{m(x)}, \quad \forall x \in \overline{\Omega}. \tag{4.25}$$

On the other hand, we have

$$|u(x)|^{q^-} + |u(x)|^{q^+} \geq |u(x)|^{q(x)}, \quad \forall x \in \overline{\Omega}. \tag{4.26}$$

Using (4.25) and (4.26) we deduce that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{\max\{p_1^+, p_2^+\}} \cdot \int_\Omega |\nabla u|^{m(x)}\,dx - \frac{1}{q^-} \cdot \left(\int_\Omega |u|^{q^-}\,dx + \int_\Omega |u|^{q^+}\,dx\right) \\ &\geq \frac{1}{m^+} \cdot \int_\Omega |\nabla u|^{m(x)}\,dx - \frac{1}{q^-} \cdot \left(\int_\Omega |u|^{q^-}\,dx + \int_\Omega |u|^{q^+}\,dx\right), \end{aligned} \tag{4.27}$$

for any $u \in E$.

Since $m^+ < q^- \leq q^+ < m^\star(x)$ for any $x \in \overline{\Omega}$ and E is continuously embedded in $L^{q^-}(\Omega)$ and in $L^{q^+}(\Omega)$ it follows that there exist two positive constants C_1 and C_2 such that

$$\|u\|_{m(x)} \geq C_1 \cdot |u|_{q^+}, \quad \|u\|_{m(x)} \geq C_2 \cdot |u|_{q^-}, \quad \forall u \in E. \tag{4.28}$$

Assume that $u \in E$ and $\|u\|_{m(x)} < 1$. Thus, by (4.3),

$$\int_\Omega |\nabla u|^{m(x)}\, dx \geq \|u\|^{m^+}_{m(x)}. \tag{4.29}$$

Relations (4.27), (4.28) and (4.29) yield

$$\begin{aligned} J_\lambda(u) \quad &\geq \quad \frac{1}{m^+} \cdot \|u\|^{m^+}_{m(x)} - \frac{1}{q^-} \cdot \left[\left(\frac{1}{C_1} \cdot \|u\|_{m(x)} \right)^{q^+} + \left(\frac{1}{C_2} \cdot \|u\|_{m(x)} \right)^{q^-} \right] \\ &= \quad \left(\beta - \gamma \cdot \|u\|^{q^+ - m^+}_{m(x)} - \delta \cdot \|u\|^{q^- - m^+}_{m(x)} \right) \cdot \|u\|^{m^+}_{m(x)} \end{aligned}$$

for any $u \in E$ with $\|u\|_{m(x)} < 1$, where β, γ and δ are positive constants.

We remark that the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = \beta - \gamma \cdot t^{q^+ - m^+} - \delta \cdot t^{q^- - m^+}$$

is positive in a neighborhood of the origin. We conclude that Lemma 5 holds true. □

Lemma 6. *Let E_1 be a finite dimensional subspace of E. Then the set $S = \{u \in E_1;\ J_\lambda(u) \geq 0\}$ is bounded.*

Proof. In order to prove Lemma 6, we first show that

$$\int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)}\, dx \leq K_1 \cdot (\|u\|^{p_1^-}_{m(x)} + \|u\|^{p_1^+}_{m(x)}), \quad \forall u \in E \tag{4.30}$$

where K_1 is a positive constant.

Indeed, using relations (4.2) and (4.3) we have

$$\int_\Omega |\nabla u|^{p_1(x)}\, dx \leq |\nabla u|^{p_1^-}_{p_1(x)} + |\nabla u|^{p_1^+}_{p_1(x)} = \|u\|^{p_1^-}_{p_1(x)} + \|u\|^{p_1^+}_{p_1(x)}, \quad \forall u \in E. \tag{4.31}$$

On the other hand, Remark 1 implies that there exists a positive constant K_0 such that

$$\|u\|_{p_1(x)} \leq K_0 \cdot \|u\|_{m(x)}, \quad \forall u \in E. \tag{4.32}$$

Inequalities (4.31) and (4.32) yield

$$\int_\Omega |\nabla u|^{p_1(x)}\,dx \leq (K_0\cdot\|u\|_{m(x)})^{p_1^-}+(K_0\cdot\|u\|_{m(x)})^{p_1^+},\quad \forall u\in E$$

and thus (4.30) holds true.

With similar arguments we deduce that there exists a positive constant K_2 such that

$$\int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx \leq K_2\cdot(\|u\|_{m(x)}^{p_2^-}+\|u\|_{m(x)}^{p_2^+}),\quad \forall u\in E. \tag{4.33}$$

Using again (4.2) and (4.3) we have

$$\int_\Omega |u|^{m(x)}\,dx \leq |u|_{m(x)}^{m^-}+|u|_{m(x)}^{m^+},\quad \forall u\in E.$$

Since E is continuously embedded in $L^{m(x)}(\Omega)$, there exists of a positive constant $\overline{K}$ such that

$$|u|_{m(x)} \leq \overline{K}\cdot\|u\|_{m(x)},\quad \forall u\in E.$$

The last two inequalities show that for each $\lambda>0$ there exists a positive constant $K_3(\lambda)$ such that

$$\lambda\cdot\int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)}\,dx \leq K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-}+\|u\|_{m(x)}^{m^+}),\quad \forall u\in E. \tag{4.34}$$

By inequalities (4.30), (4.33) and (4.34) we get

$$\begin{aligned} J_\lambda(u) &\leq K_1\cdot\left(\|u\|_{m(x)}^{p_1^-}+\|u\|_{m(x)}^{p_1^+}\right)+K_2\cdot\left(\|u\|_{m(x)}^{p_2^-}+\|u\|_{m(x)}^{p_2^+}\right)\\ &+K_3(\lambda)\cdot\left(\|u\|_{m(x)}^{m^-}+\|u\|_{m(x)}^{m^+}\right)-\frac{1}{q^+}\int_\Omega |u|^{q(x)}\,dx, \end{aligned}$$

for all $u\in E$.

Let $u\in E$ be arbitrary but fixed. We define

$$\Omega_< = \{x\in\Omega;\ |u(x)|<1\},\quad \Omega_\geq = \Omega\setminus\Omega_<.$$

Therefore

$$\begin{aligned} J_\lambda(u) &\le K_1\cdot(\|u\|^{p_1^-}_{m(x)}+\|u\|^{p_1^+}_{m(x)})+K_2\cdot(\|u\|^{p_2^-}_{m(x)}+\|u\|^{p_2^+}_{m(x)}) \\ &+K_3(\lambda)\cdot(\|u\|^{m^-}_{m(x)}+\|u\|^{m^+}_{m(x)})-\frac{1}{q^+}\int_\Omega |u|^{q(x)}\,dx \\ &\le K_1\cdot(\|u\|^{p_1^-}_{m(x)}+\|u\|^{p_1^+}_{m(x)})+K_2\cdot(\|u\|^{p_2^-}_{m(x)}+\|u\|^{p_2^+}_{m(x)}) \\ &+K_3(\lambda)\cdot(\|u\|^{m^-}_{m(x)}+\|u\|^{m^+}_{m(x)})-\frac{1}{q^+}\int_{\Omega_\ge} |u|^{q(x)}\,dx \\ &\le K_1\cdot(\|u\|^{p_1^-}_{m(x)}+\|u\|^{p_1^+}_{m(x)})+K_2\cdot(\|u\|^{p_2^-}_{m(x)}+\|u\|^{p_2^+}_{m(x)}) \\ &+K_3(\lambda)\cdot(\|u\|^{m^-}_{m(x)}+\|u\|^{m^+}_{m(x)})-\frac{1}{q^+}\int_{\Omega_\ge} |u|^{q^-}\,dx \\ &\le K_1\cdot(\|u\|^{p_1^-}_{m(x)}+\|u\|^{p_1^+}_{m(x)})+K_2\cdot(\|u\|^{p_2^-}_{m(x)}+\|u\|^{p_2^+}_{m(x)}) \\ &+K_3(\lambda)\cdot(\|u\|^{m^-}_{m(x)}+\|u\|^{m^+}_{m(x)})-\frac{1}{q^+}\int_\Omega |u|^{q^-}\,dx \\ &+\frac{1}{q^+}\int_{\Omega_<} |u|^{q^-}\,dx. \end{aligned}$$

But there exists a positive constant K_4 such that, for all $u\in E$,

$$\frac{1}{q^+}\int_{\Omega_<}|u|^{q^-}\le K_4.$$

Hence

$$\begin{aligned} J_\lambda(u) &\le K_1\cdot\left(\|u\|^{p_1^-}_{m(x)}+\|u\|^{p_1^+}_{m(x)}\right)+K_2\cdot\left(\|u\|^{p_2^-}_{m(x)}+\|u\|^{p_2^+}_{m(x)}\right) \\ &+K_3(\lambda)\cdot\left(\|u\|^{m^-}_{m(x)}+\|u\|^{m^+}_{m(x)}\right)-\frac{1}{q^+}\int_\Omega |u|^{q^-}\,dx+K_4, \end{aligned}$$

for all $u\in E$. The functional $|\cdot|_{q^-}:E\to\mathbb{R}$ defined by

$$|u|_{q^-}=\left(\int_\Omega |u|^{q^-}\,dx\right)^{1/q^-}$$

is a norm in E. In the finite dimensional subspace E_1 the norms $|\cdot|_{q^-}$ and $\|\cdot\|_{m(x)}$ are equivalent, so there exists a positive constant $K=K(E_1)$ such that

$$\|u\|_{m(x)}\le K\cdot|u|_{q^-},\quad \forall u\in E_1.$$

As a consequence we have that there exists a positive constant K_5 such that

$$\begin{aligned} J_\lambda(u) &\le K_1\cdot(\|u\|^{p_1^-}_{m(x)}+\|u\|^{p_1^+}_{m(x)})+K_2\cdot(\|u\|^{p_2^-}_{m(x)}+\|u\|^{p_2^+}_{m(x)}) \\ &+K_3(\lambda)\cdot(\|u\|^{m^-}_{m(x)}+\|u\|^{m^+}_{m(x)})-K_5\cdot\|u\|^{q^-}_{m(x)}+K_4, \end{aligned}$$

for all $u \in E_1$. Hence

$$K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) +$$
$$K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - K_5 \cdot \|u\|_{m(x)}^{q^-} + K_4 \geq 0, \quad \forall u \in S$$

and since $q^- > m^+$ we conclude that S is bounded in E. The proof of Lemma 6 is complete. □

Lemma 7. *Assume that $\{u_n\} \subset E$ is a sequence which satisfies the properties:*

$$|J_\lambda(u_n)| < M \tag{4.35}$$

$$J_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty \tag{4.36}$$

where M is a positive constant. Then $\{u_n\}$ possesses a convergent subsequence.

Proof. First, we show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually at a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|_{m(x)} \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\|_{m(x)} > 1$ for any integer n.

By (4.36) we deduce that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|J_\lambda'(u_n)\| \leq 1.$$

On the other hand, for any $n > N_1$ fixed, the application

$$E \ni v \to \langle J_\lambda'(u_n), v\rangle$$

is linear and continuous. The above information yields

$$|\langle J_\lambda'(u_n), v\rangle| \leq \|J_\lambda'(u_n)\| \cdot \|v\|_{m(x)} \leq \|v\|_{m(x)}, \quad \forall v \in E,\ n > N_1.$$

Setting $v = u_n$ we have

$$-\|u_n\|_{m(x)} \leq \int_\Omega |\nabla u_n|^{p_1(x)}\,dx + \int_\Omega |\nabla u_n|^{p_2(x)}\,dx + \lambda \int_\Omega |u_n|^{m(x)}\,dx$$
$$- \int_\Omega |u_n|^{q(x)}\,dx \leq \|u_n\|_{m(x)},$$

for all $n > N_1$. We obtain

$$\begin{aligned} &-\|u_n\|_{m(x)} - \int_\Omega |\nabla u_n|^{p_1(x)}\,dx - \int_\Omega |\nabla u_n|^{p_2(x)}\,dx \\ &-\lambda \int_\Omega |u_n|^{m(x)}\,dx \le -\int_\Omega |u_n|^{q(x)}\,dx\,, \end{aligned} \tag{4.37}$$

for any $n > N_1$.

Assuming that $\|u_n\|_{m(x)} > 1$, relations (4.35), (4.37) and (4.2) imply

$$\begin{aligned} M > J_\lambda(u_n) &\geq \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \int_\Omega (|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)})\,dx \\ &+ \lambda \cdot \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \int_\Omega |u_n|^{m(x)}\,dx - \frac{1}{q^-} \cdot \|u_n\|_{m(x)} \\ &\geq \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \int_\Omega |\nabla u_n|^{m(x)}\,dx - \frac{1}{q^-}\|u_n\|_{m(x)} \\ &\geq \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \|u_n\|_{m(x)}^{m^-} - \frac{1}{q^-}\|u_n\|_{m(x)}. \end{aligned}$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E, there exist a subsequence, again denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. Since E is compactly embedded in $L^{m(x)}(\Omega)$ and in $L^{q(x)}(\Omega)$ it follows that $\{u_n\}$ converges strongly to u_0 in $L^{m(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. The above information and relation (4.36) imply

$$\langle J_\lambda'(u_n) - J_\lambda'(u_0), u_n - u_0\rangle \to 0 \quad \text{as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned} &\int_\Omega (|\nabla u_n|^{p_1(x)-2}\nabla u_n + |\nabla u_n|^{p_2(x)-2}\nabla u_n) \cdot (\nabla u_n - \nabla u_0)dx \\ &- \int_\Omega (|\nabla u_0|^{p_1(x)-2}\nabla u_0 + |\nabla u_0|^{p_2(x)-2}\nabla u_0) \cdot (\nabla u_n - \nabla u_0)dx \\ &= \langle J_\lambda'(u_n) - J_\lambda'(u_0), u_n - u_0\rangle \\ &-\lambda \cdot \int_\Omega (|u_n|^{m(x)-1}u_n - |u_0|^{m(x)-1}u_0)(u_n - u_0)\,dx \\ &+ \int_\Omega (|u_n|^{q(x)-1}u_n - |u_0|^{q(x)-1}u_0)(u_n - u_0)\,dx\,. \end{aligned} \tag{4.38}$$

Using the fact that $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}(\Omega)$ and inequality (4.1) we have

$$\left|\int_\Omega (|u_n|^{q(x)-1}u_n - |u_0|^{q(x)-1}u_0)(u_n-u_0)\,dx\right| \leq$$
$$\left|\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u_0)\,dx\right| +$$
$$\left|\int_\Omega |u_0|^{q(x)-2}u_0(u_n-u_0)\,dx\right| \leq$$
$$C_3\cdot ||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot|u_n-u_0|_{q(x)} + C_4\cdot ||u_0|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot|u_n-u_0|_{q(x)}\,,$$

where C_3 and C_4 are positive constants. Since $|u_n-u_0|_{q(x)}\to 0$ as $n\to\infty$ we deduce that

$$\lim_{n\to\infty}\int_\Omega (|u_n|^{q(x)-1}u_n - |u_0|^{q(x)-1}u_0)(u_n-u_0)\,dx = 0. \tag{4.39}$$

With similar arguments we obtain

$$\lim_{n\to\infty}\int_\Omega (|u_n|^{m(x)-1}u_n - |u_0|^{m(x)-1}u_0)(u_n-u_0)\,dx = 0. \tag{4.40}$$

By (4.38), (4.39) and (4.40) we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega (|\nabla u_n|^{p_1(x)-2}\nabla u_n &+ |\nabla u_n|^{p_2(x)-2}\nabla u_n - |\nabla u_0|^{p_1(x)-2}\nabla u_0\\ &- |\nabla u_0|^{p_2(x)-2}\nabla u_0)\cdot(\nabla u_n-\nabla u_0)\,dx = 0.\end{aligned} \tag{4.41}$$

Next, we apply the following elementary inequality (see [37, Lemma 4.10])

$$(|\xi|^{r-2}\xi - |\psi|^{r-2}\psi)\cdot(\xi-\psi)\geq C\,|\xi-\psi|^r,\quad \forall r\geq 2,\ \xi,\psi\in\mathbb{R}^N. \tag{4.42}$$

Relations (4.41) and (4.42) yield

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n-\nabla u_0|^{p_1(x)}\,dx + \int_\Omega |\nabla u_n-\nabla u_0|^{p_2(x)}\,dx = 0$$

or using relation (4.25) we get

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n-\nabla u_0|^{m(x)}\,dx = 0.$$

That fact and relation (4.4) imply $\|u_n-u_0\|_{m(x)}\to 0$ as $n\to\infty$. The proof of Lemma 7 is complete. □

Proof of Theorem 12 completed. It is clear that the functional J_λ is even and verifies $J_\lambda(0) = 0$. Lemma 7 implies that J_λ satisfies the Palais-Smale condition. On the other hand, Lemmas 5 and 6 show that conditions (I1) and (I2) are satisfied. Applying Theorem 14 to the functional J_λ we conclude that equation (4.23) has infinitely many weak solutions in E. The proof of Theorem 12 is complete. □

4.6 Proof of Theorem 13

Define the energy functional associated to Problem (4.24) by $I_\lambda : E \to \mathbb{R}$,

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx - \lambda\int_\Omega \frac{1}{m(x)}|u|^{m(x)}\,dx \\ &+ \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx. \end{aligned}$$

The same arguments as those used in the case of functional J_λ show that I_λ is well-defined on E and $I_\lambda \in C^1(E,\mathbb{R})$ with the derivative given by

$$\begin{aligned} \langle I_\lambda'(u), v\rangle &= \int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\nabla v\,dx - \lambda\int_\Omega |u|^{m(x)-2}uv\,dx \\ &+ \int_\Omega |u|^{q(x)-2}uv\,dx, \end{aligned}$$

for any u, $v \in E$. We obtain that the weak solutions of (4.24) are the critical points of I_λ.

This time our idea is to show that I_λ possesses a nontrivial global minimum point in E. With that end in view we start by proving two auxiliary results.

Lemma 8. *The functional I_λ is coercive on E.*

Proof. In order to prove Lemma 8 we first show that for any a, $b > 0$ and $0 < k < l$ the following inequality holds

$$a \cdot t^k - b \cdot t^l \le a \cdot \left(\frac{a}{b}\right)^{k/(l-k)}, \quad \forall\, t \ge 0. \tag{4.43}$$

Indeed, since the function

$$[0,\infty) \ni t \to t^\theta$$

is increasing for any $\theta > 0$ it follows that

$$a - b \cdot t^{l-k} < 0, \quad \forall\, t > \left(\frac{a}{b}\right)^{1/(l-k)},$$

and

$$t^k \cdot (a - b \cdot t^{l-k}) \le a \cdot t^k < a \cdot \left(\frac{a}{b}\right)^{k/(l-k)}, \quad \forall\, t \in \left[0, \left(\frac{a}{b}\right)^{1/(l-k)}\right].$$

The above two inequalities show that (4.43) holds true.

Using (4.43) we deduce that for any $x \in \Omega$ and $u \in E$ we have

$$\frac{\lambda}{m^-}|u(x)|^{m(x)} - \frac{1}{q^+}|u(x)|^{q(x)} \le \frac{\lambda}{m^-}\left[\frac{\lambda \cdot q^+}{m^-}\right]^{m(x)/(q(x)-m(x))} \le$$
$$\frac{\lambda}{m^-}\left[\left(\frac{\lambda \cdot q^+}{m^-}\right)^{m^+/(q^- - m^+)} + \left(\frac{\lambda \cdot q^+}{m^-}\right)^{m^-/(q^+ - m^-)}\right] = \mathcal{C},$$

where $\mathcal{C}$ is a positive constant independent of u and x. Integrating the above inequality over Ω we obtain

$$\frac{\lambda}{m^-}\int_\Omega |u|^{m(x)}\,dx - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,dx \le \mathcal{D} \tag{4.44}$$

where $\mathcal{D}$ is a positive constant independent of u.

Using inequalities (4.25) and (4.44) we obtain that for any $u \in E$ with $\|u\|_{m(x)} > 1$ we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{m^+}\int_\Omega |\nabla u|^{m(x)}\,dx - \frac{\lambda}{m^-}\int_\Omega |u|^{m(x)}\,dx + \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,dx \\ &\ge \frac{1}{m^+}\|u\|_{m(x)}^{m^-} - \left(\frac{\lambda}{m^-}\int_\Omega |u|^{m(x)}\,dx - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,dx\right) \\ &\ge \frac{1}{m^+}\|u\|_{m(x)}^{m^-} - \mathcal{D}. \end{aligned}$$

Thus I_λ is coercive and the proof of Lemma 8 is complete. □

Lemma 9. *The functional I_λ is weakly lower semicontinuous.*

Proof. In a first instance we prove that the functionals $\Lambda_i : E \to \mathbb{R}$,

$$\Lambda_i(u) = \int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}\,dx, \quad \forall\, i \in \{1, 2\}$$

are convex. Indeed, since the function

$$[0, \infty) \ni t \to t^\theta$$

is convex for any $\theta > 1$, we deduce that for each $x \in \Omega$ fixed it holds that

$$\left|\frac{\xi+\psi}{2}\right|^{p_i(x)} \leq \left|\frac{|\xi|+|\psi|}{2}\right|^{p_i(x)} \leq \frac{1}{2}|\xi|^{p_i(x)} + \frac{1}{2}|\psi|^{p_i(x)}, \quad \forall \xi, \psi \in \mathbb{R}^N,\ i \in \{1,2\}.$$

Using the above inequality we deduce that

$$\left|\frac{\nabla u+\nabla v}{2}\right|^{p_i(x)} \leq \frac{1}{2}|\nabla u|^{p_i(x)} + \frac{1}{2}|\nabla v|^{p_i(x)}, \quad \forall u, v \in E,\ x \in \Omega,\ i \in \{1,2\}.$$

Multiplying with $\frac{1}{p_i(x)}$ and integrating over Ω we obtain

$$\Lambda_i\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda_i(u) + \frac{1}{2}\Lambda_i(v), \quad \forall u, v \in E,\ i \in \{1,2\}.$$

Thus Λ_1 and Λ_2 are convex. It follows that $\Lambda_1 + \Lambda_2$ is convex.

Next, we show that the functional $\Lambda_1 + \Lambda_2$ is weakly lower semicontinuous on E. Taking into account that $\Lambda_1 + \Lambda_2$ is convex, by Corollary III.8 in Brezis [18] it is enough to show that $\Lambda_1 + \Lambda_2$ is strongly lower semicontinuous on E. We fix $u \in E$ and $\epsilon > 0$. Let $v \in E$ be arbitrary. Since $\Lambda_1 + \Lambda_2$ is convex and inequality (4.1) holds true we have

$$\begin{aligned}
\Lambda_1(v) + \Lambda_2(v) &\geq \Lambda_1(u) + \Lambda_2(u) + \langle \Lambda_1'(u) + \Lambda_2'(u), v - u \rangle \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \int_\Omega |\nabla u|^{p_1(x)-1} |\nabla(v-u)|\, dx \\
&- \int_\Omega |\nabla u|^{p_2(x)-1} |\nabla(v-u)|\, dx \\
&\geq \Lambda_1(u) + \Lambda_2(u) - D_1 \cdot ||\nabla u|^{p_1(x)-1}|_{\frac{p_1(x)}{p_1(x)-1}} \cdot |\nabla(u-v)|_{p_1(x)} \\
&- D_2 \cdot ||\nabla u|^{p_2(x)-1}|_{\frac{p_2(x)}{p_2(x)-1}} \cdot |\nabla(u-v)|_{p_2(x)} \\
&\geq \Lambda_1(u) + \Lambda_2(u) - D_3 \cdot \|u - v\|_{m(x)} \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \epsilon
\end{aligned}$$

for all $v \in E$ with

$$\|u - v\|_{m(x)} < \frac{\epsilon}{\left||\nabla u|^{p_1(x)-1}\right|_{\frac{p_1(x)}{p_1(x)-1}} + \left||\nabla u|^{p_2(x)-1}\right|_{\frac{p_2(x)}{p_2(x)-1}}},$$

where D_1, D_2 and D_3 are positive constants. It follows that $\Lambda_1 + \Lambda_2$ is strongly lower semicontinuous and since it is convex we obtain that $\Lambda_1 + \Lambda_2$ is weakly lower semicontinuous.

Finally, we remark that if $\{u_n\} \subset E$ is a sequence which converges weakly to u in E then $\{u_n\}$ converges strongly to u in $L^{m(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. Thus, I_λ is weakly lower semicontinuous. The proof of Lemma 9 is complete. □

Proof of Theorem 13. By Lemmas 8 and 9 we deduce that I_λ is coercive and weakly lower semicontinuous on E. Then Theorem 1.2 in Struwe [144] implies that there exists $u_\lambda \in E$ a global minimizer of I_λ and thus a weak solution of problem (4.24).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$ we deduce that there exists $u_0 \in C_0^\infty(\Omega) \subset E$ such that $u_0(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \le u_0(x) \le t_0$ in $\Omega \setminus \Omega_1$. We have

$$\begin{aligned} I_\lambda(u_0) &= \int_\Omega \frac{1}{p_1(x)}|\nabla u_0|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u_0|^{p_2(x)}\,dx \\ &\quad -\lambda \int_\Omega \frac{1}{m(x)}|u_0|^{m(x)}\,dx + \int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}\,dx \\ &\le L - \frac{\lambda}{m^+}\int_{\Omega_1}|u_0|^{m(x)}\,dx \\ &\le L - \frac{\lambda}{m^+}\cdot t_0^{m^-}\cdot|\Omega_1|\,, \end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $I_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $I_\lambda(u_\lambda) < 0$ for any $\lambda \ge \lambda^\star$ and thus u_λ is a nontrivial weak solution of problem (4.24) for λ large enough. The proof of Theorem 13 is complete. □

Chapter 5

Two multivalued versions of the nonlinear Schrödinger equation on the whole space

I don't like it, and I'm sorry I
ever had anything to do with it.

Erwin Schrödinger talking about
Quantum Physics

Abstract. We first establish the existence of an entire solution for a class of stationary Schrödinger equations with subcritical discontinuous nonlinearity and lower bounded potential that blows-up at infinity. In the second part of this chapter we prove the existence of an entire solution for a class of stationary Schrödinger systems with subcritical discontinuous nonlinearities and lower bounded potentials that blow-up at infinity. The abstract framework are related to Lebesgue–Sobolev spaces with variable exponent. The proofs are based on the critical point theory in the sense of Clarke and we apply Chang's version of the Mountain Pass Lemma without the Palais–Smale condition for locally Lipschitz functionals. Our results generalize in a nonsmooth framework a theorem of Rabinowitz [127] on the existence of ground-state solutions of the nonlinear Schrödinger equation.

5.1 General results on the stationary Schrödinger equation

In 1923, L. de Broglie recovers Bohr's formula for hydrogen atom by associating to each particle a wave of some frequency and identifying the stationary states of the electron to the stationary character of the wave. Independently and the same year, Schrödinger proposes to express the Bohr's quatification conditions as an eigenvalue problem. The Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear form of Schrödinger's equation is

$$\Delta\psi + \frac{8\pi^2 m}{\hbar^2}\,(E(x) - V(x))\,\psi = 0\,,$$

where ψ is the Schrödinger wave function, m is the mass, $\hbar$ denotes Planck's constant, E is the energy, and V stands for the potential energy. The structure of the nonlinear Schrödinger equation is much more complicated. This equation is a prototypical dispersive nonlinear partial differential equation that has been central for almost four decades now to a variety of areas in Mathematical Physics. The relevant fields of application may vary from optics and propagation of the electric field in optical fibers (see Hasegawa and Kodama [71], Malomed [103]), to the self-focusing and collapse of Langmuir waves in plasma physics (see Zakharov [156]) and the behaviour of deep water waves and freak waves (the so-called rogue waves) in the ocean (see Benjamin and Feir [14], Onorato, Osborne, Serio and Bertone [114]). The nonlinear Schrödinger equation also describes various phenomena arising in: self-channelling of a high-power ultra-short laser in matter, in the theory of Heisenberg ferromagnets and magnons, in dissipative quantum mechanics, in condensed matter theory, in plasma physics (e.g., the Kurihara superfluid film equation). We refer to Ablowitz, Prinari and Trubatch [1], Grosse and Martin [67], Sulem [145] for a modern overview, including applications.

Schrödinger gives later the now classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He also develops a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proves the equivalence between his wave mechanics and

Heisenberg's matrix, and introduces the time dependent Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\,\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \qquad \text{in } \mathbb{R}^N\ (N \geq 2)\,, \tag{5.1}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$. In physical problems, a cubic nonlinearity corresponding to $p = 3$ is common; in this case (5.1) is called the Gross-Pitaevskii equation. In the study of Eq. (5.1), Oh [112] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some real number a) introduced in Kato [79]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov-Schmidt type reduction, Oh [112] proved the existence of a standing wave solution of Problem (5.1), that is, a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x)\,. \tag{5.2}$$

Note that substituting the ansatz (5.2) into (5.1) leads to

$$-\frac{\hbar^2}{2}\,\Delta u + (V(x) - E)\,u = |u|^{p-1}u\,.$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2\,(V_\hbar(x) - E)\,u = |u|^{p-1}u \qquad \text{in } \mathbb{R}^N\,, \tag{5.3}$$

where $V_\hbar(x) = V(\hbar x)$.

If for some $\xi \in \mathbb{R}^N \setminus \{0\}$, $V(x + s\xi) = V(x)$ for all $s \in \mathbb{R}$, equation (5.1) is invariant under the Galilean transformation

$$\psi(x,t) \longmapsto \psi(x - \xi t, t)\exp\left(i\xi\cdot x/\hbar - \frac{1}{2}i|\xi|^2 t/\hbar\right)\psi(x - \xi t, t)\,.$$

Thus, in this case, standing waves reproduce solitary waves travelling in the direction of ξ. In other words, Schrödinger discovered that the standing waves are scalar waves rather than vector electromagnetic waves. This is an important difference, vector electromagnetic waves are mathematical waves which describe a direction (vector) of force, whereas the wave Motions of Space are scalar waves which are simply described by their wave-amplitude. The importance of this discovery was pointed out by Albert Einstein, who wrote:

"The Schrodinger method, which has in a certain sense the character of a field theory, does indeed deduce the existence of only discrete states, in surprising agreement with empirical facts. It does so on the basis of differential equations applying a kind of resonance argument". (*On Quantum Physics*, 1954).

In a celebrated paper, Rabinowitz [126] proved that Equation (5.3) has a ground-state solution (mountain-pass solution) for $\hbar > 0$ small, under the assumption that $\inf_{x\in\mathbb{R}^N} V(x) > E$. After making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + a(x)u = f(x,u) \qquad \text{in } \mathbb{R}^N, \tag{5.4}$$

under suitable conditions on a and assuming that f is smooth, superlinear and has a subcritical growth. A related equation has been considered in Lions [97], where it is studied the problem

$$\begin{cases} -\Delta u + u = a(x)u^{p-1} \\ u > 0 \quad \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N)\,, \end{cases}$$

where $2 < p < 2N/(N-2)$ and $a(x) \geq a_\infty := \lim_{|x|\to\infty} a(x)$. The complementary case has been studied in Tintarev [148].

Our purpose in this chapter is to study two multivalued versions of Equation (5.4). We first consider a more general class of differential operators, the so-called $p(x)$–Laplace operators. This degenerate quasilinear operator is defined by $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ (where $p(x)$ is a certain function whose properties will be stated in what follows) and it generalizes the celebrated p–Laplace operator $\Delta_p u := \text{div}\,(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a constant. The $p(x)$–Laplace operator possesses more complicated nonlinearity than the p–Laplacian, for example, it is inhomogeneous. In the last part of this chapter we establish the existence of a weak solution for a class of nonlinear Schrödinger systems.

5.2 Entire solutions of a multivalued Schrödinger equation in Sobolev spaces with variable exponent

The analysis we develop throughout this chapter is carried out in terms of Clarke's critical point theory for locally Lipschitz functionals and in generalized Sobolev spaces. That is why we recall in this section some basic facts related to Clarke's generalized gradient (see Clarke [28, 29] for more details) and Lebesgue-Sobolev spaces with variable exponent.

Let E be a real Banach space and assume that $I : E \to \mathbb{R}$ is a locally Lipschitz functional. Then the Clarke generalized gradient is defined by

$$\partial I(u) = \{\xi \in E^*;\, I^0(u,v) \geq \langle \xi, v\rangle\,,\ \text{for all } v \in E\}\,,$$

where $I^0(u,v)$ stands for the directional derivative of I at u in the direction v, that is,

$$I^0(u,v) = \limsup_{\substack{w\to u\\ \lambda\searrow 0}} \frac{I(w+\lambda v) - I(w)}{\lambda}\,.$$

For any function $h(x,\cdot) \in L^\infty_{\text{loc}}(\mathbb{R})$ we denote by $\underline{h}$ (resp., $\overline{h}$) the lower (resp., upper) limit of h in its second variable, that is,

$$\underline{h}(x,t) := \lim_{\varepsilon\searrow 0}\ \operatorname{essinf}\,\{h(x,s);\ |t-s|<\varepsilon\}\,;$$

$$\overline{h}(x,t) = \lim_{\varepsilon\searrow 0}\ \operatorname{esssup}\,\{h(x,s);\ |t-s|<\varepsilon\}\,.$$

Let $a \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ be a variable potential such that, for some $a_0 > 0$,

$$a(x) \geq a_0 \qquad \text{a.e. } x \in \mathbb{R}^N \qquad \text{and} \qquad \operatorname{ess}\lim_{|x|\to\infty} a(x) = +\infty\,. \tag{5.5}$$

Let $p : \mathbb{R}^N \to \mathbb{R}$ ($N \geq 2$) be a continuous function. Set $p^+ := \sup_{x\in\mathbb{R}^N} p(x)$ and $p^- := \inf_{x\in\mathbb{R}^N} p(x)$. We assume throughout this section that p^+ is finite.

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that, for some $C > 0$, $q \in \mathbb{R}$ with $p^+ < q+1 \leq Np^-/(N-p^-)$ if $p^- < N$ and $p^+ < q+1 < +\infty$ if $p^- \geq N$, and $\mu > p^+$, we have

$$|f(x,t)| \leq C(|t| + |t|^q) \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}\,; \tag{5.6}$$

$$\lim_{\varepsilon \searrow 0} \ \text{esssup} \left\{ \left| \frac{f(x,t)}{t^{p^+-1}} \right| ; \ (x,t) \in \mathbb{R}^N \times (-\varepsilon, \varepsilon) \right\} = 0 \, ; \tag{5.7}$$

$$0 \leq \mu F(x,t) \leq s\underline{f}(x,t) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0, +\infty) \, . \tag{5.8}$$

Our hypothesis $q \leq Np^-/(N-p^-)$ enables us to allow an almost critical behaviour on f. We also point out that we do not assume that the nonlinearity f is continuous.

Let E denote the set of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $[a(x)]^{1/p(x)} \, u \in L^{p(x)}(\mathbb{R}^N)$ and $|\nabla u| \in L^{p(x)}(\mathbb{R}^N)$. Then E is a Banach space if it is endowed with the norm

$$\|u\|_E := \left| [a(x)]^{1/p(x)} \, u \right|_{p(x)} + |\nabla u|_{p(x)} \, .$$

We remark that E is continuously embedded in $W^{1,p(x)}(\mathbb{R}^N)$. In the case $p(x) \equiv 2$ and if the potential $a(x)$ fulfills more general hypotheses than (5.5), then the embedding $E \subset L^{q+1}(\mathbb{R}^N)$ is compact, whenever $2 \leq q < (N+2)/(N-2)$ (see, e.g., Bartsch, Liu and Weth [12] and Bartsch, Pankov and Wang [13]). We do not know if this compact embedding still holds true in our "variable exponent" framework and under assumption (5.5).

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between E^* and E.

Set $F(x,t) := \int_0^t f(x,s) ds$ and

$$\Psi(u) := \int_{\mathbb{R}^N} F(x, u(x)) dx \, .$$

We observe that Ψ is locally Lipschitz on E. This follows by (5.6), Hölder's inequality and the continuous embedding $E \subset L^{q+1}(\mathbb{R}^N)$. Indeed, for all u, $v \in E$,

$$|\Psi(u) - \Psi(v)| \leq C \, \|u - v\|_E \, ,$$

where $C = C(\|u\|_E, \|v\|_E) > 0$ depends only on $\max\{\|u\|_E, \|v\|_E\}$.

In the first part of this chapter we are concerned with the problem

$$\begin{cases} -\text{div} \, (|\nabla u|^{p(x)-2} \nabla u) + a(x)|u|^{p(x)-2} u \in [\underline{f}(x,u), \overline{f}(x,u)] & \text{in } \mathbb{R}^N \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N \, . \end{cases} \tag{5.9}$$

We notice that the semilinear anisotropic case corresponding to $p(x) \equiv 2$ has been analyzed in Gazzola and Rădulescu [61].

We refer to Bertone–do Ó [16] and Kristály [86] for the study (by means of other methods) of certain classes of Schrödinger type equations which involve discontinuous nonlinearities.

Definition 3. *We say that $u \in E$ is a solution of Problem (5.9) if $u \geq 0$, $u \not\equiv 0$, and $0 \in \partial I(u)$, where*

$$I(u) := \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)} \right) dx - \int_{\mathbb{R}^N} F(x, u^+) dx \,, \qquad \forall \, u \in E \,.$$

The mapping $I : E \to \mathbb{R}$ is called the energy functional associated to Problem (5.9). Our previous remarks show that I is locally Lipschitz on the Banach space E.

The above definition may be reformulated, equivalently, in terms of hemivariational inequalities. More precisely, $u \in E$ is a solution of (5.9) if $u \geq 0$, $u \not\equiv 0$ in $\mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2} uv \right) dx + \int_{\mathbb{R}^N} (-F)^0(x, u; v) dx \geq 0,$$

for all $v \in E$.

Our main result is the following

Theorem 15. *Assume that hypotheses (5.5)–(5.8) are fulfilled. Then Problem (5.9) has at least one solution.*

Proof. We first claim that there exist positive constants C_1 and C_2 such that

$$f(x,t) \geq C_1 t^{\mu - 1} - C_2 \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0, +\infty) \,. \tag{5.10}$$

Indeed, by the definition of $\underline{f}$ we deduce that

$$\underline{f}(x,t) \leq f(x,t) \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0, +\infty) \,. \tag{5.11}$$

Set $\underline{F}(x,t) := \int_0^t \underline{f}(x,s) ds$. Thus, by our assumption (5.8),

$$0 \leq \mu \underline{F}(x,t) \leq t \underline{f}(x,t) \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0, +\infty) \,. \tag{5.12}$$

Next, by (5.12), there exist positive constants R and K_1 such that

$$\underline{F}(x,t) \geq K_1 t^{\mu} \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times [R, +\infty) \,. \tag{5.13}$$

Our claim (5.10) follows now directly by relations (5.11), (5.12) and (5.13).

Next, we observe that

$$\partial I(u) = -\mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u) + a(x)|u|^{p(x)-2}u - \partial\Psi(u^+) \qquad \text{in } E^*\,.$$

So, by [58, Theorem 2.2] and [106, Theorem 3], we have

$$\partial\Psi(u) \subset [\underline{f}(x,u(x)), \overline{f}(x,u(x))] \qquad \text{a.e. } x \in \mathbb{R}^N\,,$$

in the sense that if $w \in \partial\Psi(u)$ then

$$\underline{f}(x,u(x)) \le w(x) \le \overline{f}(x,u(x)) \qquad \text{a.e. } x \in \mathbb{R}^N\,. \tag{5.14}$$

This means that if u_0 is a critical point of I, then there exists $w \in \partial\Psi(u_0)$ such that

$$-\mathrm{div}\,(|\nabla u_0|^{p(x)-2}\nabla u_0) + a(x)|u_0|^{p(x)-2}u_0 = w \qquad \text{in } E^*\,.$$

This argument shows that, for proving Theorem 15, it is enough to show that the energy functional I has at least a nontrivial critical point $u_0 \in E$, $u_0 \ge 0$. We prove the existence of a solution of Problem (5.9) by arguing that the hypotheses of Chang's version of the Mountain Pass Lemma for locally Lipschitz functionals (see Chang [26]) are fulfilled. More precisely, we check the following geometric assumptions:

$$I(0) = 0 \text{ and there exists } v \in E \text{ such that } I(v) \le 0\,; \tag{5.15}$$

$$\text{there exist } \beta,\, \rho > 0 \quad \text{such that} \quad I \ge \beta \quad \text{on} \quad \{u \in E;\ \|u\|_E = \rho\}\,. \tag{5.16}$$

Verification of (5.15). Fix $w \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ such that $w \ge 0$ in $\mathbb{R}^N$. In particular, we have

$$\int_{\mathbb{R}^N} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right) dx < +\infty\,.$$

So, by (5.10),

$$\begin{aligned} I(tw) &= \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right) dx - \Psi(tw) \\ &\le \frac{t^{p^+}}{p_-} \int_{\mathbb{R}^N} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right) dx + C_2 t \int_{\mathbb{R}^N} w dx \\ &\quad -C_1' t^\mu \int_{\mathbb{R}^N} w^\mu dx\,, \qquad \text{for all } t > 0\,. \end{aligned}$$

Since, by hypothesis, $1 < p^+ < \mu$, we deduce that $I(tw) < 0$ for $t > 0$ large enough.

VERIFICATION OF (5.16). Our hypotheses (5.6) and (5.7) imply that, for any $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t| + C_\varepsilon|t|^q \qquad \text{a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}\,. \tag{5.17}$$

By (5.17) and Sobolev embeddings in variable exponent spaces we have, for any $u \in E$,

$$\begin{aligned}\Psi(u) \quad &\leq \varepsilon \int_{\mathbb{R}^N} \frac{1}{p(x)}|u|^{p(x)}dx + \frac{A_\varepsilon}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} \frac{1}{p(x)}|u|^{p(x)}dx + C_4 \|u\|^{q+1}_{L^{q+1}(\mathbb{R}^N)},\end{aligned}$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus, by our hypotheses,

$$\begin{aligned}I(u) \quad &= \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right) dx - \Psi(u^+) \\ &\geq \frac{1}{p^+} \int_{\mathbb{R}^N} \left[|\nabla u|^{p(x)} + (a_0 - \varepsilon)|u|^{p(x)}\right] dx - C_4 \|u\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \geq \beta > 0\,,\end{aligned}$$

for $\|u\|_E = \rho$, with ρ, ε and β are small enough positive constants. □

Denote

$$\mathcal{P} := \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } I(\gamma(1)) \leq 0\}$$

and

$$c := \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} I(\gamma(t))\,.$$

Set

$$\lambda_I(u) := \min_{\zeta \in \partial I(u)} \|\zeta\|_{E^*}\,.$$

We are now in position to apply Chang's version of the Mountain Pass Lemma for locally Lipschitz functionals (see Chang [26]). So, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \qquad \text{and} \qquad \lambda_I(u_n) \to 0\,. \tag{5.18}$$

Moreover, since $I(|u|) \leq I(u)$ for all $u \in E$, we can assume without loss of generality that $u_n \geq 0$ for every $n \geq 1$. So, for all positive integer n, there exists $\{w_n\} \in \partial\Psi(u_n) \subset E^*$ such that, for any $v \in E$,

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)-2}\nabla u_n \nabla v + a(x)u_n^{p(x)-1}v\right) dx - \langle w_n, v\rangle \to 0 \qquad \text{as } n \to \infty\,. \tag{5.19}$$

Note that for all $u \in E$, $u \geq 0$, the definition of Ψ and our hypotheses yield

$$\Psi(u) \leq \frac{1}{\mu} \int_{\mathbb{R}^N} u(x)\underline{f}(x, u(x))dx\,.$$

Therefore, by (5.14), for every $u \in E$, $u \geq 0$, and for any $w \in \partial\Psi(u)$,

$$\Psi(u) \leq \frac{1}{\mu} \int_{\mathbb{R}^N} u(x)w(x)dx\,.$$

Hence

$$\begin{aligned}
I(u_n) \;&\geq \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right) dx \\
&+\frac{1}{\mu} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)} - w_n u_n\right) dx + \frac{1}{\mu} \int_{\mathbb{R}^N} w_n u_n dx - \Psi(u_n) \\
&\geq \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right) dx \\
&+\frac{1}{\mu} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)} - w_n u_n\right) dx \\
&= \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right) dx \\
&+\frac{1}{\mu}\langle -\Delta_{p(x)}u_n + au_n - w_n, u_n\rangle \\
&= \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right) dx + o(1)\|u_n\|_E\,.
\end{aligned}$$

This relation and (5.18) show that the Palais-Smale sequence $\{u_n\}$ is bounded in E. It follows that $\{u_n\}$ converges weakly (up to a subsequence) in E and strongly in $L^{p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ to some $u_0 \geq 0$. Taking into account that $w_n \in \partial\Psi(u_n)$ for all m, that $u_n \rightharpoonup u_0$ in E and that there exists $w_0 \in E^*$ such that $w_n \rightharpoonup w_0$ in E^* (up to a subsequence), we infer that $w_0 \in \partial\Psi(u_0)$. This follows from the fact that the map $u \longmapsto F(x,u)$ is compact from E into L^1. Moreover, if we take $\varphi \in C_c^\infty(\mathbb{R}^N)$ and let $\omega := \operatorname{supp}\varphi$, then by (5.19) we get

$$\int_\omega \left(|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla\varphi + a(x)u_0^{p(x)-1}\varphi - w_0\varphi\right) dx = 0\,.$$

So, by relation (4) p.104 in Chang [26] and by the definition of $(-F)^0$, we deduce that

$$\int_\omega \langle(|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla\varphi + a(x)u_0^{p(x)-1}\varphi\rangle)dx + \int_\omega (-F)^0(x, u_0; \varphi)dx \geq 0\,.$$

By density, this hemivariational inequality holds for all $\varphi \in E$ and this means that u_0 solves Problem (5.9).

It remains to prove that $u_0 \not\equiv 0$. If w_n is as in (5.19), then by (5.14) (recall that $u_n \geq 0$) and (5.18) (for large m) we deduce that

$$\begin{aligned} \frac{c}{2} &\leq I(u_n) - \frac{1}{p^-} \langle -\Delta_{p(x)} u_n + a u_n - w_n, u_n \rangle \\ &= \frac{1}{p^-} \langle w_n, u_n \rangle - \int_{\mathbb{R}^N} F(x, u_n) dx \leq \frac{1}{p^-} \int_{\mathbb{R}^N} u_n \overline{f}(x, u_n) dx \,. \end{aligned} \tag{5.20}$$

Now, taking into account its definition, one deduces that $\overline{f}$ verifies (5.17), too. So, by (5.20), we obtain

$$0 < \frac{c}{2} \leq \frac{1}{p^-} \int_{\mathbb{R}^N} (\varepsilon u_n^2 + A_\varepsilon u_n^{q+1}) dx = \frac{\varepsilon}{p^-} \|u_n\|^2_{L^2(\mathbb{R}^N)} + \frac{A_\varepsilon}{p^-} \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \,.$$

In particular, this shows that $\{u_n\}$ does not converge strongly to 0 in $L^{q+1}(\mathbb{R}^N)$. It remains to argue that $u_0 \not\equiv 0$. Since both $\|u_n\|_{L^{p^-}(\mathbb{R}^N)}$ and $\|\nabla u_n\|_{L^{p^-}(\mathbb{R}^N)}$ are bounded, it follows by Lemma I.1 in Lions [98] that the sequence $\{u_n\}$ "does not vanish" in $L^{p^-}(\mathbb{R}^N)$. Thus, there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ and $C > 0$ such that, for some $R > 0$,

$$\int_{z_n + B_R} u_n^{p^-} dx \geq C \,. \tag{5.21}$$

We claim that the sequence $\{z_n\}$ is bounded in $\mathbb{R}^N$. Indeed, if not, up to a subsequence, it follows by (5.5) that

$$\int_{\mathbb{R}^N} a(x) u_n^{p^-} dx \to +\infty \qquad \text{as } n \to \infty \,,$$

which contradicts our assumption $I(u_n) = c + o(1)$. Therefore, by (5.21), there exists an open bounded set $D \subset \mathbb{R}^N$ such that

$$\int_D u_n^{p^-} dx \geq C > 0 \,.$$

In particular, this relation implies that $u_0 \not\equiv 0$ and our proof is concluded. □

5.3 Entire solutions of Schrödinger elliptic systems with sign–changing potential and discontinuous nonlinearity

Consider the following class of coupled elliptic systems in $\mathbb{R}^N$ ($N \geq 3$):

$$\begin{cases} -\Delta u_1 + a(x)u_1 = f(x, u_1, u_2) & \text{in } \mathbb{R}^N \\ -\Delta u_2 + b(x)u_2 = g(x, u_1, u_2) & \text{in } \mathbb{R}^N\,. \end{cases} \tag{5.22}$$

We point out that coupled nonlinear Schrödinger systems describe some physical phenomena such as the propagation in birefringent optical fibers or Kerr-like photorefractive media in optics. Another motivation to the study of coupled Schrödinger systems arises from the Hartree-Fock theory for the double condensate, that is a binary mixture of Bose-Einstein condensates in two different hyperfine states, cf. Esry *et all.* [50]. System (5.22) is also important for industrial applications in fiber communications systems (see Hasegawa and Kodama [71]) and all-optical switching devices (see Islam [74]).

Throughout this section we assume that a, $b \in L^\infty_{\rm loc}(\mathbb{R}^N)$ and there exist $\underline{a}\,,\ \underline{b} > 0$ such that

$$a(x) \geq \underline{a} \qquad \text{and} \qquad b(x) \geq \underline{b} \ \text{ a.e. in } \mathbb{R}^N, \tag{5.23}$$

and $\text{esslim}_{|x|\to\infty} a(x) = \text{esslim}_{|x|\to\infty} b(x) = +\infty$. Our aim in this section is to study the existence of solutions to the above problem in the case when f, g are not continuous functions. Our goal is to show how variational methods can be used to find existence results for stationary nonsmooth Schrödinger systems.

Throughout this section we assume that $f(x,\cdot,\cdot)$, $g(x,\cdot,\cdot) \in L^\infty_{\rm loc}(\mathbb{R}^2)$. Denote:

$$\underline{f}(x, t_1, t_2) = \lim_{\delta\to 0} \text{essinf}\{f(x, s_1, s_2)\,;\, |t_i - s_i| \leq \delta\,;\, i = 1, 2\}$$

$$\overline{f}(x, t_1, t_2) = \lim_{\delta\to 0} \text{esssup}\{f(x, s_1, s_2)\,;\, |t_i - s_i| \leq \delta\,;\, i = 1, 2\}$$

$$\underline{g}(x, t_1, t_2) = \lim_{\delta\to 0} \text{essinf}\{g(x, s_1, s_2)\,;\, |t_i - s_i| \leq \delta\,;\, i = 1, 2\}$$

$$\overline{g}(x, t_1, t_2) = \lim_{\delta\to 0} \text{esssup}\{g(x, s_1, s_2)\,;\, |t_i - s_i| \leq \delta\,;\, i = 1, 2\}\,.$$

Under these conditions we reformulate Problem (5.22) as follows:

$$\begin{cases} -\Delta u_1 + a(x)u_1 \in [\underline{f}(x,u_1(x),u_2(x)), \overline{f}(x,u_1(x),u_2(x))] & \text{a.e. } x \in \mathbb{R}^N \\ -\Delta u_2 + b(x)u_2 \in [\underline{g}(x,u_1(x),u_2(x)), \overline{g}(x,u_1(x),u_2(x))] & \text{a.e. } x \in \mathbb{R}^N \,. \end{cases} \tag{5.24}$$

Let $H^1 = H(\mathbb{R}^N, \mathbb{R}^2)$ denote the Sobolev space of all $U = (u_1, u_2) \in (L^2(\mathbb{R}^N))^2$ with weak derivatives $\dfrac{\partial u_1}{\partial x_j}, \dfrac{\partial u_2}{\partial x_j}$ $(j = 1, \ldots, N)$ also in $L^2(\mathbb{R}^N)$, endowed with the usual norm

$$\|U\|^2_{H_1} = \int_{\mathbb{R}^N} (|\nabla U|^2 + |U|^2)\, dx = \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2 + u_1^2 + u_2^2)\, dx \,.$$

Given the functions $a,\, b : \mathbb{R}^N \to \mathbb{R}$ as above, define the subspace

$$E = \{U = (u_1, u_2) \in H^1\,;\, \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\, dx < +\infty\} \,.$$

Then the space E endowed with the norm

$$\|U\|^2_E = \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\, dx$$

becomes a Hilbert space.

Since $a(x) \geq \underline{a} > 0$, $b(x) \geq \underline{b} > 0$, we have the continuous embeddings $H^1 \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ for all $2 \leq q \leq 2^* = 2N/(N-2)$.

We assume throughout the section that $f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are nontrivial measurable functions satisfying the following hypotheses:

$$\begin{cases} |f(x,t)| \leq C(|t| + |t|^p) \text{ for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2 \\ |g(x,t)| \leq C(|t| + |t|^p) \text{ for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2 \,, \end{cases} \tag{5.25}$$

where $p < 2^*$;

$$\begin{cases} \lim\limits_{\delta \to 0} \operatorname{esssup} \left\{ \dfrac{|f(x,t)|}{|t|};\, (x,t) \in \mathbb{R}^N \times (-\delta, +\delta)^2 \right\} = 0 \\ \lim\limits_{\delta \to 0} \operatorname{esssup} \left\{ \dfrac{|g(x,t)|}{|t|};\, (x,t) \in \mathbb{R}^N \times (-\delta, +\delta)^2 \right\} = 0; \end{cases} \tag{5.26}$$

f and g are chosen so that the mapping $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,t_1,t_2) := \int_0^{t_1} f(x,\tau,t_2)\,d\tau + \int_0^{t_2} g(x,0,\tau)\,d\tau$ satisfies

$$\begin{cases} F(x,t_1,t_2) = \int\limits_0^{t_2} g(x,t_1,\tau)\,d\tau + \int\limits_0^{t_1} f(x,\tau,0)\,d\tau \\ \text{and } F(x,t_1,t_2)=0 \text{ if and only if } t_1=t_2=0; \end{cases} \tag{5.27}$$

there exists $\mu > 2$ such that for any $x \in \mathbb{R}^N$

$$0 \le \mu F(x,t_1,t_2) \le \begin{cases} t_1\underline{f}(x,t_1,t_2) + t_2\underline{g}(x,t_1,t_2);\ t_1,\ t_2 \ge 0 \\ t_1\underline{f}(x,t_1,t_2) + t_2\overline{g}(x,t_1,t_2);\ t_1 \ge 0,\ t_2 \le 0 \\ t_1\overline{f}(x,t_1,t_2) + t_2\overline{g}(x,t_1,t_2);\ t_1,\ t_2 \le 0 \\ t_1\overline{f}(x,t_1,t_2) + t_2\underline{g}(x,t_1,t_2);\ t_1 \le 0,\ t_2 \ge 0\,. \end{cases} \tag{5.28}$$

Definition 4. *A function $U=(u_1,u_2) \in E$ is called solution of the problem (5.24) if there exists a function $W=(w_1,w_2) \in L^2(\mathbb{R}^N,\mathbb{R}^2)$ such that*

(i) $\underline{f}(x,u_1(x),u_2(x)) \le w_1(x) \le \overline{f}(x,u_1(x),u_2(x))$ *a.e.* $x \in \mathbb{R}^N$*;*
$\underline{g}(x,u_1(x),u_2(x)) \le w_2(x) \le \overline{g}(x,u_1(x),u_2(x))$ *a.e.* $x \in \mathbb{R}^N$*;*

(ii) $\int\limits_{\mathbb{R}^N} (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x)u_1v_1 + b(x)u_2v_2)\,dx = \int\limits_{\mathbb{R}^N} (w_1v_1 + w_2v_2)\,dx,$
for all $(v_1,v_2) \in E$*.*

Our main result is the following.

Theorem 16. *Assume that conditions (5.25)-(5.28) are fulfilled. Then Problem (5.24) has at least a nontrivial solution in E.*

5.4 Auxiliary results

Let Ω be an arbitrary domain in $\mathbb{R}^N$. Set

$$E_\Omega = \left\{ U=(u_1,u_2) \in H^1(\Omega;\mathbb{R}^2)\,;\ \int\limits_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + au_1^2 + bu_2^2\right) < +\infty \right\}$$

which is endowed with the norm

$$\|U\|_{E_\Omega}^2 = \int\limits_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\,dx\,.$$

Then E_Ω becomes a Hilbert space.

Lemma 10. *The functional* $\Psi_\Omega : E_\Omega \to \mathbb{R}$ *defined by* $\Psi_\Omega(U) = \int\limits_\Omega F(x,U)\,dx$ *is locally Lipschitz on* E_Ω*.*

Proof. We first observe that

$$\begin{aligned} F(x,U) &= F(x,u_1,u_2) = \int\limits_0^{u_1} f(x,\tau,u_2)\,d\tau + \int\limits_0^{u_2} g(x,0,\tau)\,d\tau \\ &= \int\limits_0^{u_2} g(x,u_1,\tau)\,d\tau + \int\limits_0^{u_1} f(x,\tau,0)\,d\tau \end{aligned}$$

is a Carathéodory functional which is locally Lipschitz with respect to the second variable. Indeed, by (5.25),

$$\begin{aligned} |F(x,t_1,t) - F(x,s_1,t)| &= \left|\int\limits_{s_1}^{t_1} f(x,\tau,t)\,d\tau\right| \le \left|\int\limits_{s_1}^{t_1} C(|\tau,t| + |\tau,t|^p)\,d\tau\right| \\ &\le k(t_1,s_1,t)\,|t_1 - s_1|\,. \end{aligned}$$

Similarly,

$$|F(x,t,t_2) - F(x,t,s_2)| \le k(t_2,s_2,t)|t_2 - s_2|\,.$$

Therefore

$$\begin{aligned} |F(x,t_1,t_2) - F(x,s_1,s_2)| &\le |F(x,t_1,t_2) - F(x,s_1,t_2)| \\ &\quad + |F(x,t_1,s_2) - F(x,s_1,s_2)| \\ &\le k(V)\,|(t_2,s_2) - (t_1,s_1)|\,, \end{aligned}$$

where V is a neighbourhood of (t_1,t_2), (s_1,s_2).

Set

$$\chi_1(x) = \max\{u_1(x), v_1(x)\}\,, \quad \chi_2(x) = \max\{u_2(x), v_2(x)\}\,, \qquad \text{for all } x \in \Omega\,.$$

It is obvious that if $U = (u_1, u_2)$, $V = (v_1, v_2)$ belong to E_Ω, then $(\chi_1, \chi_2) \in E_\Omega$. So, by Hölder's inequality and the continuous embedding $E_\Omega \subset L^p(\Omega; \mathbb{R}^2)$,

$$|\Psi_\Omega(U) - \Psi_\Omega(V)| \leq C(\|\chi_1, \chi_2\|_{E_\Omega}) \|U - V\|_{E_\Omega} ,$$

which concludes the proof. □

The following result is a generalization of Lemma 6 in Mironescu and Rădulescu [106].

Lemma 11. *Let Ω be an arbitrary domain in $\mathbb{R}^N$ and let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function such that $f(x, .) \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Then $\underline{f}$ and $\overline{f}$ are Borel functions.*

Proof. Since the requirement is local we may suppose that f is bounded by M and it is nonnegative. Denote

$$f_{m,n}(x, t_1, t_2) = \left(\int_{t_1-\frac{1}{n}}^{t_1+\frac{1}{n}} \int_{t_2-\frac{1}{n}}^{t_2+\frac{1}{n}} |f(x, s_1, s_2)|^m \, ds_1 ds_2 \right)^{\frac{1}{m}} .$$

Since $\overline{f}(x, t_1, t_2) = \lim_{\delta \to 0} \text{esssup}\{f(x, s_1, s_2) \,;\, |t_i - s_i| \leq \delta \,;\, i = 1, 2\}$ we deduce that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}^*$ such that for $|t_i - s_i| < \frac{1}{n}$ $(i = 1, 2)$ we have $|\text{esssup} f(x, s_1, s_2) - \overline{f}(x, t_1, t_2)| < \varepsilon$ or, equivalently,

$$\overline{f}(x, t_1, t_2) - \varepsilon < \text{esssup} f(x, s_1, s_2) < \overline{f}(x, t_1, t_2) + \varepsilon . \tag{5.29}$$

By the second inequality in (5.29) we obtain

$$f(x, s_1, s_2) \leq \overline{f}(x, t_1, t_2) + \varepsilon \quad \text{a.e. } x \in \Omega \quad \text{for } |t_i - s_i| < \frac{1}{n} \ (i = 1, 2)$$

which yields

$$f_{m,n}(x, t_1, t_2) \leq (\overline{f}(x, t_1, t_2) + \varepsilon) \left(\sqrt{4/n^2} \right)^{\frac{1}{m}} . \tag{5.30}$$

Let

$$A = \left\{ (s_1, s_2) \in \mathbb{R}^2 \ ; \ |t_i - s_i| < \frac{1}{n} \ (i = 1, 2) \,;\ \overline{f}(x, t_1, t_2) - \varepsilon \leq f(x, s_1, s_2) \right\} .$$

By the first inequality in (5.29) and the definition of the essential supremum we obtain that $|A|>0$ and

$$f_{m,n} \leq \left(\int\int_A (f(x,s_1,s_2))^m \, ds_1 \, ds_2 \right)^{\frac{1}{m}} \geq (\overline{f}(x,s_1,s_2)-\varepsilon)\, |A|^{1/m} \,. \quad (5.31)$$

Since (5.30) and (5.31) imply

$$\overline{f}(x,t_1,t_2) = \lim_{n\to\infty} \lim_{m\to\infty} f_{m,n}(x,t_1,t_2)\,,$$

it suffices to prove that $f_{m,n}$ is a Borel function. Let

$$\mathcal{M} = \{ f : \Omega \times \mathbb{R}^2 \to \mathbb{R};\ |f| \leq M \text{ and } f \text{ is a Borel function}\}$$

$$\mathcal{N} = \{ f \in \mathcal{M};\ f_{m,n} \text{ is a Borel function}\}\,.$$

Cf. Berberian [15, p.178], $\mathcal{M}$ is the smallest set of functions having the following properties:

(i) $\{ f \in C(\Omega \times \mathbb{R}^2; \mathbb{R});\ |f| \leq M\} \subset \mathcal{M}$;

(ii) $f^{(k)} \in \mathcal{M}$ and $f^{(k)} \xrightarrow{k} f$ imply $f \in \mathcal{M}$.

Since $\mathcal{N}$ contains obviously the continuous functions and (ii) is also true for $\mathcal{N}$ then, by the Lebesgue Dominated Convergence Theorem, we obtain that $\mathcal{M} = \mathcal{N}$. For $\underline{f}$ we note that $\underline{f} = -(-\overline{f})$ and the proof of Lemma 11 is complete. □

Let us now assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain. By the continuous embedding $L^{p+1}(\Omega;\mathbb{R}^2) \hookrightarrow L^2(\Omega;\mathbb{R}^2)$, we may define the locally Lipchitz functional $\Psi_\Omega : L^{p+1}(\Omega;\mathbb{R}^2) \to \mathbb{R}$ by $\Psi_\Omega(U) = \int_\Omega F(x,U)\,dx$.

Lemma 12. *Under the above assumptions and for any* $U \in L^{p+1}(\Omega;\mathbb{R}^2)$, *we have*

$$\partial\Psi_\Omega(U)(x) \subset [\underline{f}(x,U(x)), \overline{f}(x,U(x))] \times [\underline{g}(x,U(x)), \overline{g}(x,U(x))] \qquad a.e.\ x \in \Omega\,,$$

in the sense that if $W = (w_1,w_2) \in \partial\Psi_\Omega(U) \subset L^{p+1}(\Omega;\mathbb{R}^2)$ *then*

$$\underline{f}(x,U(x)) \leq w_1(x) \leq \overline{f}(x,U(x)) \qquad a.e.\ x \in \Omega \quad (5.32)$$

$$\underline{g}(x,U(x)) \leq w_2(x) \leq \overline{g}(x,U(x)) \qquad a.e.\ x \in \Omega\,. \quad (5.33)$$

Proof. By the definition of the Clarke gradient we have

$$\int_\Omega (w_1 v_1 + w_2 v_2)\,dx \le \Psi^0_\Omega(U,V) \qquad \text{for all } V=(v_1,v_2)\in L^{p+1}(\Omega;\mathbb{R}^2)\,.$$

Choose $V=(v,0)$ such that $v\in L^{p+1}(\Omega)$, $v\ge 0$ a.e. in Ω. Thus, by Lemma 11,

$$\begin{aligned}\int_\Omega w_1 v &\le \limsup_{\substack{(h_1,h_2)\to U\\ \lambda\searrow 0}} \frac{\int_\Omega \left(\int_{h_1(x)}^{h_1(x)+\lambda v(x)} f(x,\tau,h_2(x))d\tau\right)dx}{\lambda}\\ &\le \int_\Omega \left(\limsup_{\substack{(h_1,h_2)\to U\\ \lambda\searrow 0}} \frac{1}{\lambda}\int_{h_1(x)}^{h_1(x)+\lambda v(x)} f(x,\tau,h_2(x))d\tau\right)dx\\ &\le \int_\Omega \overline{f}(x,u_1(x),u_2(x))v(x)\,dx\,.\end{aligned} \tag{5.34}$$

Analogously we obtain

$$\int_\Omega \underline{f}(x,u_1(x),u_2(x))v(x)\,dx \le \int_\Omega w_1 v\,dx \qquad \text{for all } v\ge 0 \text{ in } \Omega. \tag{5.35}$$

Arguing by contradiction, suppose that (5.32) is false. Then there exist $\varepsilon>0$, a set $A\subset\Omega$ with $|A|>0$ and w_1 as above such that

$$w_1(x) > \overline{f}(x,U(x)) + \varepsilon \qquad \text{in } A\,. \tag{5.36}$$

Taking $v=\mathbf{1}_A$ in (5.34) we obtain

$$\int_\Omega w_1 v\,dx = \int_A w_1\,dx \le \int_A \overline{f}(x,U(x))\,dx\,,$$

which contradicts (5.36). Proceeding in the same way we obtain the corresponding result for g in (5.33). □

By Lemma 12, Lemma 2.1 in Chang [26] and the embedding $E_\Omega \hookrightarrow L^{p+1}(\Omega,\mathbb{R}^2)$ we obtain also that for $\Psi_\Omega : E_\Omega\to\mathbb{R}$, $\Psi_\Omega(U)=\int_\Omega F(x,U)\,dx$ we have

$$\partial\Psi_\Omega(U)(x)\subset[\underline{f}(x,U(x)),\overline{f}(x,U(x))]\times[\underline{g}(x,U(x)),\overline{g}(x,U(x))] \qquad \text{a.e. } x\in\Omega\,.$$

Let $V \in E_\Omega$. Then $\tilde{V} \in E$, where $\tilde{V} : \mathbb{R}^N \to \mathbb{R}^2$ is defined by

$$\tilde{V} = \begin{cases} V(x) & x \in \Omega \\ 0 & \text{otherwise}\,. \end{cases}$$

For $W \in E^*$ we consider $W_\Omega \in E^*_\Omega$ such that $\langle W_\Omega, V\rangle = \langle W, \tilde{V}\rangle$ for all V in E_Ω. Set $\Psi : E \to \mathbb{R}$, $\Psi(U) = \int\limits_{\mathbb{R}^N} F(x, U)$.

Lemma 13. *Let $W \in \partial\Psi(U)$, where $U \in E$. Then $W_\Omega \in \partial\Psi_\Omega(U)$, in the sense that $W_\Omega \in \partial\Psi_\Omega(U|_\Omega)$.*

Proof. By the definition of the Clarke gradient we deduce that $\langle W, \tilde{V}\rangle \leq \Psi^0(U, \tilde{V})$ for all V in E_Ω

$$\begin{aligned} \Psi^0(U, \tilde{V}) &= \limsup_{\substack{H \to U,\, H \in E \\ \lambda \to 0}} \frac{\Psi(H + \lambda\tilde{V}) - \Psi(H)}{\lambda} \\ &= \limsup_{\substack{H \to U,\, H \in E \\ \lambda \to 0}} \frac{\int\limits_{\mathbb{R}^N} (F(x, H + \lambda\tilde{V}) - F(x, H))\, dx}{\lambda} \\ &= \limsup_{\substack{H \to U,\, H \in E \\ \lambda \to 0}} \frac{\int\limits_{\Omega} (F(x, H + \lambda\tilde{V}) - F(x, H)) dx}{\lambda} \\ &= \limsup_{\substack{H \to U,\, H \in E_\Omega \\ \lambda \to 0}} \frac{\int\limits_{\Omega} (F(x, H + \lambda\tilde{V}) - F(x, H)) dx}{\lambda} = \Psi^0_\Omega(U, V)\,. \end{aligned}$$

Hence $\langle W_\Omega, V\rangle \leq \Psi^0_\Omega(U, V)$ which implies $W_\Omega \in \partial\Psi^0_\Omega(U)$. □

By Lemmas 12 and 13 we obtain that for any $W \in \partial\Psi(U)$ (with $U \in E$), W_Ω satisfies (5.32) and (5.33). We also observe that for Ω_1, $\Omega_2 \subset \mathbb{R}^N$ we have $W_{\Omega_1}|_{\Omega_1 \cap \Omega_2} = W_{\Omega_2}|_{\Omega_1 \cap \Omega_2}$.

Let $W_0 : \mathbb{R}^N \to \mathbb{R}$, where $W_0(x) = W_\Omega(x)$ if $x \in \Omega$. Then W_0 is well defined and

$$W_0(x) \in [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))] \qquad \text{a.e. } x \in \mathbb{R}^N$$

and, for all $\varphi \in C^\infty_c(\mathbb{R}^N, \mathbb{R}^2)$, $\langle W, \varphi\rangle = \int\limits_{\mathbb{R}^N} W_0\varphi$. By density of $C^\infty_c(\mathbb{R}^N, \mathbb{R}^2)$ in

E we deduce that $\langle W, V\rangle = \int\limits_{\mathbb{R}^N} W_0 V\,dx$ for all V in E. Hence, for a.e. $x \in \mathbb{R}^N$,

$$W(x) = W_0(x) \in [\underline{f}(x,U(x)), \overline{f}(x,U(x))] \times [\underline{g}(x,U(x)), \overline{g}(x,U(x))]\,. \quad (5.37)$$

5.5 Proof of Theorem 16

Define the energy functional $I : E \to \mathbb{R}$ by

$$\begin{aligned} I(U) &= \frac{1}{2}\int\limits_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right)\,dx - \int\limits_{\mathbb{R}^N} F(x,U)\,dx \\ &= \frac{1}{2}\|U\|_E^2 - \Psi(U)\,. \end{aligned} \quad (5.38)$$

The existence of solutions to problem (5.24) will be justified by a nonsmooth variant of the Mountain-Pass Theorem (see Chang [26]) applied to the functional I, even if the Palais-Smale condition is not fulfilled. More precisely, we check the following geometric hypotheses:

$$I(0) = 0 \quad \text{and there exists } V \in E \quad \text{such that } I(V) \le 0; \quad (5.39)$$

$$\text{there exist } \beta, \rho > 0 \quad \text{such that } I \ge \beta \quad \text{on } \{U \in E;\ \|U\|_E = \rho\}. \quad (5.40)$$

Verification of (5.39). It is obvious that $I(0) = 0$. For the second assertion we need the following lemma.

Lemma 14. *There exist two positive constants C_1 and C_2 such that*

$$f(x,s,0) \ge C_1 s^{\mu-1} - C_2 \quad \textit{for a.e. } x \in \mathbb{R}^N;\ s \in [0,+\infty)\,.$$

Proof. We first observe that (5.28) implies

$$0 \le \mu F(x,s,0) \le \begin{cases} s\underline{f}(x,s,0), & \text{if } s \in [0,+\infty) \\ s\overline{f}(x,s,0), & \text{if } s \in (-\infty,0]\,, \end{cases}$$

which places us in the conditions of Lemma 5 in Mironescu and Rădulescu [106].

VERIFICATION OF (5.39) CONTINUED. Choose $v \in C_c^\infty(\mathbb{R}^N) - \{0\}$ so that $v \geq 0$ in $\mathbb{R}^N$. We have $\int_{\mathbb{R}^N} (|\nabla v|^2 + a(x)v^2)dx < \infty$, hence $t(v,0) \in E$ for all $t \in \mathbb{R}$. Thus by Lemma 14 we obtain

$$
\begin{aligned}
I(t(v,0)) &= \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + a(x)v^2)\,dx - \int_{\mathbb{R}^N}\int_0^{tv} f(x,\tau,0)\,d\tau \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + a(x)v^2)\,dx - \int_{\mathbb{R}^N}\int_0^{tv} (C_1\tau^{\mu-1} - C_2)\,d\tau \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + a(x)v^2)\,dx + C_2 t\int_{\mathbb{R}^N} v\,dx - C_1' t^\mu \int_{\mathbb{R}^N} v^\mu\,dx < 0
\end{aligned}
$$

for $t > 0$ large enough.

VERIFICATION OF (5.40). We observe that (5.26), (5.27) and (5.28) imply that, for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that

$$
\begin{aligned}
|f(x,s)| &\leq \varepsilon|s| + A_\varepsilon|s|^p \\
|g(x,s)| &\leq \varepsilon|s| + A_\varepsilon|s|^p
\end{aligned}
\qquad \text{for a.e. } (x,s) \in \mathbb{R}^N \times \mathbb{R}^2 . \tag{5.41}
$$

By (5.41) and Sobolev's embedding theorem we have, for any $U \in E$,

$$
\begin{aligned}
|\Psi(U)| &= |\Psi(u_1,u_2)| \leq \int_{\mathbb{R}^N}\int_0^{|u_1|} |f(x,\tau,u_2)|d\tau + \int_{\mathbb{R}^N}\int_0^{u_2} |g(x,0,\tau)|\,d\tau \\
&\leq \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2}|(u_1,u_2)|^2 + \frac{A_\varepsilon}{p+1}|(u_1,u_2|^{p+1}\right)\,dx \\
&+ \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2}|u_2|^2 + \frac{A_\varepsilon}{p+1}|u_2|^{p+1}\right)\,dx \\
&\leq \varepsilon\|U\|_{L^2}^2 + \frac{2A_\varepsilon}{p+1}\|U\|_{L^{p+1}}^{p+1} \leq \varepsilon C_3\|U\|_E^2 + C_4\|U\|_E^{p+1} ,
\end{aligned}
$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus

$$
I(U) = \frac{1}{2}\|U\|_E^2 - \Psi(U) \geq \frac{1}{2}\|U\|_E^2 - \varepsilon C_3\|U\|_E^2 - C_4\|U\|_E^{p+1} \geq \beta > 0 ,
$$

for $\|U\|_E = \rho$, with ρ, ε and β sufficiently small positive constants.

Denote

$$
\mathcal{P} = \{\gamma \in C([0,1],E);\ \gamma(0) = 0,\ \gamma(1) \neq 0 \quad \text{and } I(\gamma(1)) \leq 0\}
$$

and

$$c = \inf_{\gamma \in \mathcal{P}} \max_{t\in[0,1]} I(\gamma(t))\,.$$

Set

$$\lambda_I(U) = \min_{\xi \in \partial I(U)} \|\xi\|_{E^*}\,.$$

Thus, by the nonsmooth version of the Mountain Pass Lemma (see Chang [26]), there exists a sequence $\{U_M\} \subset E$ such that

$$I(U_m) \to c \qquad \text{and} \qquad \lambda_I(U_m) \to 0\,. \tag{5.42}$$

So, there exists a sequence $\{W_m\} \subset \partial\Psi(U_m)$, $W_m = (w_m^1, w_m^2)$, such that

$$(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + a(x)u_m^2 - w_m^2) \to 0 \quad \text{in } E^*\,. \tag{5.43}$$

Note that, by (5.28),

$$\Psi(U) \le \frac{1}{\mu}\left(\int\limits_{u_1\ge 0} u_1\underline{f}(x,U) + \int\limits_{u_1\le 0} u_1\overline{f}(x,U) + \int\limits_{u_2\ge 0} u_1\underline{g}(x,U) + \int\limits_{u_2\le 0} u_2\overline{g}(x,U)\right).$$

Therefore, by (5.37),

$$\Psi(U) \le \frac{1}{\mu}\int\limits_{\mathbb{R}^N} U(x)W(x)\,dx = \frac{1}{\mu}\int\limits_{\mathbb{R}^N} (u_1w_1 + u_2w_2)\,dx\,,$$

for every $U \in E$ and $W \in \partial\Psi(U)$. Hence, if $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E^* and E, we have

$$\begin{aligned}
I(U_m) \quad &= \frac{\mu-2}{2\mu}\int\limits_{\mathbb{R}^N} (|\nabla u_m^1|^2 + |\nabla u_m|^2 + a(x)|u_m|^1 + b(x)|u_m|^2)\,dx \\
&+\frac{1}{\mu}\langle(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m\rangle \\
&+\frac{1}{\mu}\langle W_m, U_m\rangle - \Psi(U_m) \\
&\ge \frac{\mu-2}{2\mu}\int\limits_{\mathbb{R}^N} (|\nabla u_m^1|^2 + |\nabla u_m^2|^2 + a(x)|u_m^1|^2 + b(x)|u_m^2|^2)\,dx \\
&+\frac{1}{\mu}\langle(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m\rangle \\
&\ge \frac{\mu-2}{2\mu}\|U_m\|_E^2 - o(1)\|U_m\|_E\,.
\end{aligned}$$

This relation in conjunction with (5.42) implies that the Palais-Smale sequence $\{U_m\}$ is bounded in E. Thus, it converges weakly (up to a subsequence) in E and strongly in $L^2_{\rm loc}(\mathbb{R}^N)$ to some U. Taking into account that $W_m \in \partial\Psi(U_m)$ and $U_m \rightharpoonup U$ in E, we deduce from (5.43) that there exists $W \in E^*$ such that $W_m \rightharpoonup W$ in E^* (up to a subsequence). Since the mapping $U \longmapsto F(x,U)$ ia compact from E to L^1, it follows that $W \in \partial\Psi(U)$. Therefore

$$W(x) \in [\underline{f}(x,U(x)), \overline{f}(x,U(x))] \times [\underline{g}(x,U(x)), \overline{g}(x,U(x))] \quad \text{a.e. } x \in \mathbb{R}^N$$

and

$$(-\Delta u^1_m + a(x)u^1_m - w^1_m, -\Delta u^2_m + b(x)u^2_m - w^2_m) = 0 \Longleftrightarrow$$
$$\int_{\mathbb{R}^N} (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x)u_1v_1 + b(x)u_2v_2)dx = \int_{\mathbb{R}^N} (w_1v_1 + w_2v_2)dx\,,$$

for all $(v_1, v_2) \in E$. These last two relations show that U is a solution of the problem (5.24).

It remains to prove that $U \not\equiv 0$. If $\{W_m\}$ is as in (5.43), then by (5.28), (5.37), (5.42) and for large m

$$\begin{aligned}\frac{c}{2} &\leq I(U_m) - \frac{1}{2}\langle(-\Delta u^1_m + a(x)u^1_m - w^1_m, -\Delta u^2_m + b(x)u^2_m - w^2_m), U_m\rangle \\ &= \frac{1}{2}\langle W_m, U_m\rangle - \int_{\mathbb{R}^N} F(x,U_m)\,dx \\ &\leq \frac{1}{2}\left(\int_{u_1\geq 0} u_1\underline{f}(x,U) + \int_{u_1\leq 0} u_1\overline{f}(x,U) + \int_{u_2\geq 0} u_1\underline{g}(x,U) + \int_{u_2\leq 0} u_2\overline{g}(x,U)\right).\end{aligned} \tag{5.44}$$

Now, taking into account the definition of $\overline{f}$, $\underline{f}$, $\overline{g}$, $\underline{g}$, we deduce that these functions verify (5.39), too. So, by (5.44),

$$\frac{c}{2} \leq \int_{\mathbb{R}^N} \left(\varepsilon|U_m|^2 + A_\varepsilon|u_m|^{p+1}\right)dx = \varepsilon\|U_m\|^2_{L^2} + A_\varepsilon\|U_m\|^{p+1}_{L^{p+1}}\,.$$

Thus, $\{U_m\}$ does not converge strongly to 0 in $L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$. Next, since $\{U_m\}$ is bounded in $E \subset L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$, it follows that $\{U_m\}$ and $\{\nabla U_m\}$ are bounded in $L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$. So, by [98, Lemma I.1], there exist a sequence $\{z_m\}$ and positive numbers C, R such that, for all $m \geq 1$,

$$\int_{z_m+B_R} \left[(u^1_m)^2 + (u^2_m)\right]dx \geq C\,. \tag{5.45}$$

The next step consists in showing that $\{z_m\}$ is bounded. Arguing by contradiction and using (5.23) we obtain, up to a subsequence,

$$\int_{\mathbb{R}^N} \left[a(x)(u_m^1)^2 + b(x)(u_m^2)^2\right] dx \to +\infty \quad \text{as } m \to \infty\,.$$

But this relation contradicts our assumption $I(U_m) \to c$. So, by (5.45), there exists an open bounded set $D \subset \mathbb{R}^N$ such that

$$\int_D \left[a(x)(u_m^1)^2 + b(x)(u_m^2)^2\right] dx \geq C\,.$$

This relation implies that $U \not\equiv 0$, which concludes our proof. □

Bibliography

We believe that the human mind is a "meteor" in the same way as the rainbow – a natural phenomenon; and that Hilbert realizing the "spectral decomposition" of linear operators, Perrin analyzing the blue color of the sky, Monet, Debussy and Proust recreating, for our wonder, the scintillation of the light on the see, all worked for the same aim, which will also be that of the future: the knowledge of the whole Universe.

Roger Godement

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