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CGMY and Meixner Subordinators are Absolutely Continuous with respect to One Sided Stable Subordinators.

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Abstract

We describe the CGMY and Meixner processes as time changed Brownian motions. The CGMY uses a time change absolutely continuous with respect to the one-sided stable $(Y/2)$ subordinator while the Meixner time change is absolutely continuous with respect to the one sided stable $(1/2)$ subordinator. The required time changes may be generated by simulating the requisite one-sided stable subordinator and throwing away some of the jumps as described in Rosinski (2001).

1 Introduction

Lévy processes are increasingly being used to model the local motion of asset returns, permitting the use of distributions that are both skewed and capable of matching the high levels of kurtosis observed in factors driving equity returns. By way of examples we cite the normal inverse Gaussian process (Barndorff-Nielsen (1998)), the hyperbolic process (Eberlein, Keller and Prause (1998)), and the variance gamma process (Madan, Carr and Chang (1998)). For the valuation of structured equity products the importance of skewness is well recognized and has led to the development of local Lévy processes (See Carr, Geman, Madan and Yor (2004)) that preserve skews in forward implied volatility curves. It is also understood from the steepness of implied volatility curves that tail events have significantly higher prices than those implied by a Gaussian distribution with the consequence that pricing distributions display high levels of excess kurtosis.

On a single asset one may simulate the Lévy process calibrated to the prices of vanilla options to value equity structured products written on a single underlier. Such a simulation (See Rosinski (2001)) may approximate the small jumps using a diffusion process with the large jumps simulated as a compound Poisson process where one uses the normalized large jump Lévy measure as the density of jump magnitudes with the integral of the Lévy measure over the large jumps serving as the jump arrival rate. However, increasingly one sees multiasset structures being traded and this requires a modeling of asset correlations. Given marginal Lévy processes one could accomodate correlations if one can represent the Lévy process as time changed Brownian motion. In this case we correlate the simulated processes by correlating the Brownian motions while preserving the independent time changes for each of the marginal underliers.

It is therefore useful to have representations of Lévy processes as time changed Brownian motions. For some Lévy processes, like the variance gamma process or the normal inverse Gaussian process, these are known by construction of the Lévy process via such a representation. For other Lévy processes, like the $CGMY$ process (Carr, Geman, Madan and Yor (2002), see also Koponen (1995), Boyarchenko and Levendorskii (1999, 2000)) or the *Meixner* process (Schoutens and Teugels (1998) see also Gregelionis (1999), Schoutens (2000), and Pitman and Yor (2003)), the process is defined directly by its Lévy measure and it is not clear a priori whether the processes can be represented as time changed Brownian motions. With a view to enhancing the applicability of these processes, particularly with respect to multiasset structured products, we develop the representations of these processes as time changed Brownian motions.

Section 2 presents for completeness, some preliminary results on Lévy processes that we employ in the subsequent development. In section 3 we develop the $CGMY$ process as a time changed Brownian motion with drift, where the law of the time change is absolutely continuous over finite time intervals with respect to that of the one sided stable $Y/2$ subordinator. The simulation of $CGMY$ as time changed Brownian motion is described in section 3. Section 4 develops the time change for the Meixner process as absolutely continuous with respect to the one-sided stable $1/2$ subordinator. Simulation strategies for the Meixner process based on these representations are described in Section 5. Section 6 reports on the simulation results using chi-squared goodness of fit tests. Section 7 concludes.

2 Preliminary results on Lévy processes

We present three results from the theory of Lévy processes that we make critical use of in our subsequent development. The first result relates the Lévy measure of a process obtained on subordinating a Brownian motion to the Lévy measure of the subordinator. The second result establishes a criterion for the absolute continuity of a subordinator with respect to another subordinator. The third result presents the detailed relationship between the standard presentation of the characteristic function of a two sided jump and one-sided jump stable Lévy process and its Lévy measure. These are presented in three short subsections.

2.1 Lévy measure of a subordinated Brownian motion

Suppose the Lévy process $X(t)$ is obtained by subordinating Brownian motion with drift (i.e. the process $\theta u + W(u)$, for $(W(u), u \geq 0)$ a Brownian motion) by an independent subordinator $Y(t)$ with Lévy measure $\nu(dy)$. Then applying Sato (1999) theorem 30.1 we get that the Lévy measure of the process $X(t)$ is given by $\mu(dx)$ where

$$\mu(dx) = dx \int_0^\infty \nu(dy) \frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\theta y)^2}{2y}}. \tag{1}$$

2.2 Absolute Continuity Criterion for subordinators

Suppose we have two subordinators $T_A = (T_A(t), t \geq 0), T_B = (T_B(t), t \geq 0)$. The law of the subordinator T_A is absolutely continuous with respect to the subordinator T_B, on finite time intervals, just if there exists a function $f(t)$ such that the Lévy measures $\nu_A(dt), \nu_B(dt)$ for the processes T_A and T_B respectively are related by

$$\nu_A(dt) = f(t)\nu_B(dt) \tag{2}$$

and furthermore, (Sato (1999) Theorem 33.1)

$$\int_0^\infty \nu_B(dt) \left(\sqrt{f(t)} - 1\right)^2 < \infty. \tag{3}$$

2.3 Stable Processes

The Stable Lévy process $\mathcal{S}(\sigma, \alpha, \beta) = (X(t), t \geq 0)$ with parameters (σ, α, β) (For details see DuMouchel (1973, 1975), Bertoin (1996), Samorodnitsky and Taqqu (1998) Nolan (2001), Ito (2004)) has a characteristic function in standard form

$$E[e^{iuX(t)}] = \exp(-t\Psi(u))$$

where the characteristic exponent $\Psi(u)$ is given by

$$\begin{aligned} \Psi(u) &= \sigma^\alpha |u|^\alpha \left(1 - i\beta sign(u) \tan\left(\frac{\pi\alpha}{2}\right)\right), \ \alpha \neq 1 \\ &= \sigma|u| \left(1 + i\beta sign(u) \frac{2}{\pi} \log(|u|)\right), \ \alpha = 1. \end{aligned} \tag{4}$$

The parameters satisfy the restrictions, $\sigma > 0, 0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$. The one sided jump stable processes result when $\beta = 1$ and there are only positive jumps or $\beta = -1$ in which case there are only negative jumps.

The Lévy density of the stable process is of the form

$$k(x) = \frac{c_p}{x^{1+\alpha}} \mathbf{1}_{x>0} + \frac{c_n}{|x|^{1+\alpha}} \mathbf{1}_{x<0} \tag{5}$$

and we have that

$$\beta = \frac{c_p - c_n}{c_p + c_n}. \tag{6}$$

It remains to express σ in terms of the parameters of the Lévy measure. In the one sided case with only positive jumps we have

$$\sigma = \left[\frac{c_p \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(1 - \frac{\alpha}{2}\right)}{2\Gamma(1+\alpha)} \right]^{\frac{1}{\alpha}} \tag{7}$$

and more generally for the two sided jump case we have

$$\sigma = \left[\frac{c_p + c_n}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(1 - \frac{\alpha}{2}\right)}{\Gamma(1+\alpha)} \right]^{\frac{1}{\alpha}}. \tag{8}$$

Conversely, c_p and c_n may be computed in terms of β and σ.

3 CGMY as time changed Brownian motion

We wish to write the $CGMY$ process in the form

$$X_{CGMY}(t) = \theta Y(t) + W(Y(t))$$

for an increasing time change process given by a subordinator $(Y(t), t \geq 0)$ independent of the Brownian motion $(W(s), s \geq 0)$.

The characteristic function of the $CGMY$ process is

$$E\left[\exp\left(iuX_{CGMY}(t)\right)\right] = \left(\phi_{CGMY}(u)\right)^t = \exp\left(tC\Gamma(-Y) \left[\begin{array}{c} (M - iu)^Y - M^Y + \\ (G + iu)^Y - G^Y \end{array} \right] \right)$$

The complex exponentiation is defined via the complex logarithm with a branch cut on the negative real axis with polar coordinate arguments for the complex logarithm restricted to the interval $]-\pi, +\pi]$. The $CGMY$ process is defined as a pure jump Lévy process by its Lévy measure

$$k_{CGMY}(x) = C \left[\frac{\exp(-G|x|)}{|x|^{1+Y}} \mathbf{1}_{x<0} + \frac{\exp\left(-Mx\right)}{x^{1+Y}} \mathbf{1}_{x>0} \right].$$

On the other hand we have, in all generality, by conditioning on the time change that

$$\begin{aligned} E\left[e^{iu(\theta Y(t) + W(Y(t))}\right] &= E\left[\exp\left(iu\theta Y(t) - \frac{Y(t)}{2}u^2\right)\right] \\ &= E\left[\exp\left(-\left(\frac{u^2}{2} - iu\theta\right) Y(t)\right)\right] \end{aligned}$$

Take $u(\lambda)$ to be any solution of

$$\lambda = \left(\frac{u^2}{2} - iu\theta \right);$$

Then we have the Laplace transform of the time change subordinator as

$$E[e^{-\lambda Y(t)}] = \exp\left(tC\Gamma(-Y) \left[(M - iu(\lambda))^Y - M^Y + (G + iu(\lambda))^Y - G^Y \right] \right)$$

The solutions for u are:

$$u = i\theta \pm \sqrt{2\lambda - \theta^2}$$

where we suppose that $\theta^2 < 2\lambda$.

We shall see that a good choice for θ , for sufficiently large λ, is

$$\theta = \frac{G - M}{2}$$

and in this case

$$\begin{aligned} M - iu &= \frac{G + M}{2} + i\sqrt{2\lambda - \left(\frac{G - M}{2} \right)^2} \\ G + iu &= \frac{G + M}{2} - i\sqrt{2\lambda - \left(\frac{G - M}{2} \right)^2}. \end{aligned}$$

It follows that the Laplace transform of the subordinator is

$$\begin{aligned} E[e^{-\lambda Y(t)}] &= \exp\left(tC\Gamma(-Y) \left[2r^Y \cos(\eta Y) - M^Y - G^Y \right] \right) \\ r &= \sqrt{2\lambda + GM} \\ \eta &= \arctan\left(\frac{\sqrt{2\lambda - \left(\frac{G-M}{2}\right)^2}}{\left(\frac{G+M}{2}\right)} \right) \end{aligned}$$

In the special case of $G = M$ we have

$$E[e^{-\lambda Y(t)}] = \exp\left(2tC\Gamma(-Y) \left[\left(2\lambda + M^2\right)^{Y/2} \cos\left(Y \arctan\left(\frac{\sqrt{2\lambda}}{M} \right) \right) - M^Y \right] \right)$$

3.1 The explicit time change for CGMY

We shall show that the time change subordinator $Y(t)$ associated with the $CGMY$ process is absolutely continuous with respect to the one-sided stable

$Y/2$ subordinator and in particular that its Lévy measure $\nu(dy)$ takes the form

$$\begin{aligned}
\nu(dy) &= \frac{K}{y^{1+\frac{Y}{2}}} f(y)dy \\
f(y) &= e^{-\frac{(B^2-A^2)y}{2}} E\left[e^{-\frac{B^2 y}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}} \right] \\
B &= \frac{G+M}{2} \\
K &= \left[\frac{C\Gamma\left(\frac{Y}{4}\right)\Gamma\left(1-\frac{Y}{4}\right)}{2\Gamma(1+\frac{Y}{2})} \right]
\end{aligned} \tag{9}$$

where $\gamma_{\frac{Y}{2}}, \gamma_{\frac{1}{2}}$ are two independent gamma variates with unit scale parameters and shape parameters $Y/2, 1/2$ respectively. Further we explicitly evaluate the expectation in equation (9) in terms of the Hermite functions as follows.

$$E\left[e^{-\frac{B^2 y}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}} \right] = \frac{\Gamma\left(\frac{Y}{2}+\frac{1}{2}\right)}{\Gamma(Y)\Gamma(\frac{1}{2})} 2^Y \left(\frac{B^2 y}{2}\right)^{\frac{Y}{2}} I\left(Y, B^2 y, \frac{B^2 y}{2}\right)$$

where

$$I(\nu, a, \lambda) = \int_0^\infty x^{\nu-1} e^{-ax-\lambda x^2} dx = (2\lambda)^{-\nu/2}\Gamma(\nu) h_{-\nu}\left(\frac{a}{\sqrt{2\lambda}}\right)$$

and $h_{-\nu}(z)$ is the Hermite function with parameter $-\nu$ (see e.g Lebedev (1972), p 290-291).

3.2 Determining the time change for CGMY

For an explicit evaluation of the time change we begin by writing the $CGMY$ Lévy density in the form

$$k_{CGMY}(x) = C\frac{e^{Ax-B|x|}}{x^{1+Y}}, \text{ where: } A = \frac{G-M}{2};\ B = \frac{G+M}{2}$$

Henceforth, when we encounter a Lévy measure $\mu(dx)$ that is absolutely continuous with respect to Lebesgue measure we shall denote its density by $\mu(x)$. We now employ the result (1) and seek to find a Lévy measure of a subordinator satisfying

$$\begin{aligned}
C\frac{e^{Ax-B|x|}}{|x|^{1+Y}} &= \int_0^\infty \nu(dy)\frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\theta y)^2}{2y}} \\
&= \int_0^\infty \nu(dy)\frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}-\frac{\theta^2 y}{2}+\theta x}
\end{aligned}$$

We set $\theta = A$ and observe that the right choice for θ is $(G-M)/2$ as remarked earlier, and identify $\nu(dy)$ such that

$$C\frac{e^{-B|x|}}{|x|^{1+Y}} = \int_0^\infty \nu(dy)\frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}-\frac{\theta^2 y}{2}} \tag{10}$$

We now recognize that the Lévy measure for the $CGMY$ is (taking $C = \frac{\Gamma(\frac{Y}{2})\Gamma(1-\frac{Y}{2})}{\Gamma(1+\frac{Y}{2})}$, now), that of the symmetric stable Y Lévy process with Lévy measure tilted as

$$k_{CGMY}(x) = e^{Ax-B|x|}k_{Stable(Y)}(x).$$

We also know that

$$X_{Stable(Y)}(t) = B_{Y^0(t)}$$

where $Y^0(t)$ is the one sided stable $Y/2$ subordinator, independent of the Brownian motion (B_u) .

We now write

$$X_{CGMY}(t) = \theta Y^{(1)}(t) + W_{Y^{(1)}(t)}$$

and we seek to relate the Lévy measures $\nu^{(1)}$ and $\nu^{(0)}$ of the processes $Y^{(1)}$ and $Y^{(0)}$.

From the result (1) we may write

$$\begin{aligned} \mu_0(x) &= \int_0^\infty \nu^{(0)}(dy)\frac{e^{-\frac{x^2}{2y}}}{\sqrt{2\pi y}} \\ \mu_1(x) &= \int_0^\infty \nu^{(1)}(dy)\frac{e^{-\frac{(x-\theta y)^2}{2y}}}{\sqrt{2\pi y}} \end{aligned}$$

Hence we must have that

$$\int_0^\infty \nu^{(1)}(dy)\frac{e^{-\frac{(x-\theta y)^2}{2y}}}{\sqrt{y}} = e^{Ax-B|x|}\int_0^\infty \nu^{(0)}(dy)\frac{e^{-\frac{x^2}{2y}}}{\sqrt{y}}$$

Taking $\theta = A$, we get:

$$\int_0^\infty \nu^{(1)}(dy)\frac{e^{-\frac{x^2}{2y}-\frac{A^2y}{2}}}{\sqrt{y}} = e^{-B|x|}\int_0^\infty \nu^{(0)}(dy)\frac{e^{-\frac{x^2}{2y}}}{\sqrt{y}}$$

We now use the well known fact that

$$e^{-B|x|} = \int_0^\infty du\frac{B}{\sqrt{2\pi u^3}}e^{-\frac{B^2}{2u}-\frac{x^2}{2}u}$$

to write

$$\int_0^\infty \nu^{(1)}(dy)\frac{e^{-\frac{x^2}{2y}-\frac{A^2y}{2}}}{\sqrt{y}} = \int_0^\infty du\frac{B}{\sqrt{2\pi u^3}}e^{-\frac{B^2}{2u}}\int_0^\infty \nu^{(0)}(dy)\frac{e^{-\frac{x^2}{2}\left(\frac{1}{y}+u\right)}}{\sqrt{y}}$$

By uniqueness of Laplace transforms we get that for every function $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$\int_0^\infty \nu^{(1)}(dy)\frac{e^{-\frac{A^2y}{2}}}{\sqrt{y}}f\left(\frac{1}{y}\right) = \int_0^\infty du\frac{B}{\sqrt{2\pi u^3}}e^{-\frac{B^2}{2u}}\int_0^\infty \nu^{(0)}(dy)\frac{1}{\sqrt{y}}f\left(\frac{1}{y}+u\right)$$

or equivalently that, for every function $g : \mathbb{R}_+ \to \mathbb{R}_+$

$$\begin{aligned}
\int_0^\infty \nu^{(1)}(dy)\frac{e^{-\frac{A^2 y}{2}}}{\sqrt{y}}g(y) &= \int_0^\infty du\frac{B}{\sqrt{2\pi u^3}}e^{-\frac{B^2}{2u}}\int_0^\infty \nu^{(0)}(dy)\frac{1}{\sqrt{y}}g\left(\frac{y}{1+uy}\right) \\
&= \int_0^\infty du\frac{B}{\sqrt{2\pi u^3}}e^{-\frac{B^2}{2u}}\int_0^{\frac{1}{u}} d\left(\frac{s}{1-us}\right)\frac{\nu^{(0)}(\frac{s}{1-us})}{\sqrt{\frac{s}{1-us}}}g(s) \\
&= \int_0^\infty du\frac{B}{\sqrt{2\pi u^3}}e^{-\frac{B^2}{2u}}\int_0^{\frac{1}{u}} \frac{ds}{(1-su)^2}\frac{\nu^{(0)}(\frac{s}{1-us})}{\sqrt{\frac{s}{1-us}}}g(s)
\end{aligned}$$

Hence it is the case that

$$\begin{aligned}
\nu^{(1)}(y)e^{-\frac{A^2 y}{2}} &= \int_0^{\frac{1}{y}} \frac{duBe^{-\frac{B^2}{2u}}\nu^{(0)}(\frac{y}{1-uy})}{\sqrt{2\pi(u(1-uy))^3}} \\
&= \sqrt{y}\int_0^1 \frac{dvBe^{-\frac{B^2 y}{2v}}\nu^{(0)}(\frac{y}{1-v})}{\sqrt{2\pi(v(1-v))^3}}
\end{aligned}$$

In particular we have

$$\nu^{(1)}(y) = \sqrt{y}\int_0^1 \frac{dvBe^{-\frac{y}{2}\left(\frac{B^2}{v}-A^2\right)}\nu^{(0)}(\frac{y}{1-v})}{\sqrt{2\pi(v(1-v))^3}}$$

We now introduce the explicit form of $\nu_0(y)$ for our case where it is the Lévy density of the one-sided stable $Y/2$ subordinator,

$$\nu_0(y) = \frac{K}{y^{\left(\frac{Y}{2}+1\right)}}.$$

This gives the representation

$$\begin{aligned}
\nu_1(y) &= \frac{K}{y^{\frac{Y+1}{2}}}\int_0^1 \frac{dvBe^{-\frac{y}{2}\left(\frac{B^2}{v}-A^2\right)}(1-v)^{\left(\frac{Y}{2}+1\right)}}{\sqrt{2\pi(v(1-v))^3}} \\
&= \frac{K}{y^{\frac{Y+1}{2}}}\int_1^\infty \frac{dw}{w^2}\frac{Be^{-\frac{y}{2}\left(B^2 w-A^2\right)}}{\sqrt{2\pi(\frac{1}{w}(1-\frac{1}{w}))^3}}\left(1-\frac{1}{w}\right)^{\left(\frac{Y}{2}+1\right)} \\
&= \frac{K}{y^{\frac{Y+1}{2}}}\int_1^\infty \frac{dw}{\sqrt{2\pi w}}Be^{-\frac{y}{2}\left(B^2 w-A^2\right)}\left(\frac{w-1}{w}\right)^{\frac{Y-1}{2}} \\
&= \frac{KBe^{-\frac{y}{2}\left(B^2-A^2\right)}}{y^{\frac{Y+1}{2}}}\int_0^\infty \frac{dh}{\sqrt{2\pi}}e^{-\frac{yB^2 h}{2}}\frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y}{2}}}
\end{aligned}$$

3.2.1 Absolute Continuity relations

This subsection investigates the absolute continuity relation in general between two subordinated processes and the absolute continuity of the subordinators as processes. It is easy to show that the laws of the $CGMY$ process and the symmetric stable Y process are locally equivalent, i.e. for each t,their laws, as restricted to their past $\sigma - fields\ \mathcal{F}_t$ up to time t, are equivalent (from now on, as a slight abuse of language, we shall say of 2 such processes, that they are equivalent). Now that we have identified these processes as subordinated processes, we look for the equivalence in law of the subordinators. Indeed we first observe that if the subordinators are equivalent then the subordinated processes will be equivalent but the converse may not be true.

Indeed, consider two subordinators

$$T_A(t),\ T_B(t)$$

such that the relation (2) between their Lévy measures holds for some function $f(t)$ for $t > 0$.

We suppose the absolute continuity of T_A with respect to T_B or the condition (3).

We also define the subordinated processes

$$\begin{aligned} X_A(t) &= \beta_{T_A(t)} \\ X_B(t) &= \beta_{T_B(t)} \end{aligned}$$

where (β_u) is a Brownian motion assumed to be independent of either T_A or T_B.

We have from the result (1) that at the level of Lévy measures μ_A, μ_B for X_A, X_B

$$\begin{aligned} \mu_A(x) &= \int_0^\infty \nu_A(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \\ \mu_B(x) &= \int_0^\infty \nu_B(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \end{aligned}$$

The following then holds as a consequence of (3), for every functional $F \geq 0$:

$$E\left[F\left(T_A(s), s \leq t\right)\right] = E\left[F(T_B(s), s \leq t)\phi\left(T_B(s), s \leq t\right)\right]$$

where

$$\phi\left(T_B(s), s \leq t\right) = \left(\frac{dP_{T_A}}{dP_{T_B}}\right)_t$$

As a consequence we deduce that, for every $G \geq 0$:

$$E\left[G\left(X_A(s), s \leq t\right)\right] = E\left[G\left(X_B(s), s \leq t\right)\phi(T_B(s), s \leq t)\right]$$

Consequently we may write

$$E\left[G\left(X_A(s), s \leq t\right)\right] = E\left[G\left(X_B(s), s \leq t\right)\psi(X_B(s), s \leq t)\right]$$

where

$$\psi(X_B(s), s \le t) = E\left[\phi(T_B(s), s \le t) | (X_B(s), s \le t)\right]$$

This implies that we should have

$$\mu_A(dx) = g(x)\mu_B(dx)$$

with

$$\int_{-\infty}^{\infty} \left(\sqrt{g(x)} - 1\right)^2 \mu_B(dx) < \infty \tag{11}$$

We want to show that (3) implies (11).

Now we have explicitly that

$$\begin{aligned} g(x) &= \frac{\int \nu_A(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}}}{\int \nu_B(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}}} \\ &= \frac{\int \nu_B(dt) f(t) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}}}{\int \nu_B(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}}} \end{aligned}$$

Let

$$\gamma^{(x)}(dt) = \frac{\nu_B(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}}}{\int \nu_B(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}}}$$

and note that

$$g(x) = \int \gamma^{(x)}(dt) f(t)$$

We then have

$$\sqrt{g(x)} - 1 = \left(\int \gamma^{(x)}(dt) f(t)\right)^{\frac{1}{2}} - 1$$

and

$$\int (\sqrt{g(x)} - 1)^2 \mu_B(dx) = \int (\sqrt{g(x)} - 1)^2 \left(\int \nu_B(dt) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx\right)$$

Observe that

$$
\begin{aligned}
& (\sqrt{g(x)}-1)^2\mu_B(x) \\
= & \left(\left(\int \gamma^{(x)}(dt)f(t)\right)^{\frac{1}{2}}-1\right)^2 \int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \\
= & \left(\left(\frac{\int \nu_B(dt)f(t)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}}{\int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}}\right)^{\frac{1}{2}}-1\right)^2 \int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \\
= & \int \nu_B(dt)f(t)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}+\int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \\
& -2\left(\int \nu_B(dt)f(t)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}}\left(\int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}}
\end{aligned}
$$

We wish to show that the integral over x of the right hand side is smaller than

$$\int \nu_B(dt)f(t)+\int \nu_B(dt)-2\int \nu_B(dt)\sqrt{f(t)}$$

and this follows provided

$$\int dx\left(\int \nu_B(dt)f(t)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}}\left(\int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}} \geq \int \nu_B(dt)\sqrt{f(t)}$$

For this consider

$$
\begin{aligned}
& \int \nu_B(dt)\sqrt{f(t)} = \int \nu_B(dt)\sqrt{f(t)}\int dx\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \\
= & \int dx\int \nu_B(dt)\sqrt{f(t)}\left(\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}}\left(\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}} \\
\leq & \int dx\left(\int \nu_B(dt)f(t)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}}\left(\int \nu_B(dt)\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used Cauchy-Schwarz for fixed x.

Hence we have

$$\int(\sqrt{g(x)}-1)^2\mu_B(dx) \leq \int_0^\infty \left(\sqrt{f(t)}-1\right)^2\nu_B(dt)$$

The result does not go in the other direction as we may take

$$
\begin{aligned}
\nu_A(dt) &= \varepsilon_a(dt) \\
\nu_B(dt) &= \varepsilon_b(dt)
\end{aligned}
$$

for $a \neq b$. These are not equivalent subordinators but in this case

$$\begin{aligned} \mu_A(x) &= \frac{e^{-\frac{x^2}{2a}}}{\sqrt{2\pi a}} \\ \mu_B(x) &= \frac{e^{-\frac{x^2}{2b}}}{\sqrt{2\pi b}} \end{aligned}$$

two Lévy densities, which in fact are probability densities, so that the corresponding Lévy processes which are indeed Compound Poisson, are (locally) equivalent.

3.2.2 Absolute Continuity of the subordinators for CGMY and Stable Y/2

We now establish precisely the absolute continuity relationship between the subordinator associated with the CGMY process, and the one sided stable $Y/2$ subordinator.

We note that

$$\begin{aligned} \nu_{CGMY}(dy) &= f(y)\nu_0(dy) \\ f(y) &= e^{-\frac{y}{2}(B^2-A^2)}\left(B\sqrt{y}\right)\int_0^\infty \frac{dh}{\sqrt{2\pi}} e^{-\frac{B^2 y}{2}h} \frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y}{2}}} \end{aligned}$$

We first check that as $B \to 0$ for $A = 0$ we get the expected result that $f(y) \to 1$.

For this we let $z = B\sqrt{y}$ and make the change of variable

$$k = z^2 h$$

to get

$$\begin{aligned} f(y) &= e^{-\frac{z^2}{2}} \int_0^\infty \frac{dk}{\sqrt{2\pi z}} e^{-\frac{k}{2}} \frac{\left(\frac{k}{z^2}\right)^{\frac{Y-1}{2}}}{\left(1+\frac{k}{z^2}\right)^{\frac{Y}{2}}} \\ &= e^{-\frac{z2}{2}} \int_0^\infty \frac{dk}{\sqrt{2\pi k}\frac{z}{\sqrt{k}}} e^{-\frac{k}{2}} \frac{\left(\frac{k}{z^2}\right)^{\frac{Y-1}{2}}}{\left(1+\frac{k}{z^2}\right)^{\frac{Y}{2}}} \\ &\to \int_0^\infty \frac{dk}{\sqrt{2\pi k}} e^{-\frac{k}{2}}, \text{ as } z \to 0 \\ &= 2\int_0^\infty \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ &= 1 \end{aligned}$$

For the equivalence of the two subordinators we must check that

$$\int_0^\infty \frac{dy}{y^{\frac{Y}{2}+1}} \left(\sqrt{f(y)} - 1\right)^2 < \infty.$$

We break up this quantity in 2 parts dealing with the integral near 0 and ∞ separately. First consider the integral over $[1,\infty)$. Here we write

$$\int_1^\infty \nu_0(dy)f(y) = \int_1^\infty \frac{dy}{y^{\frac{Y}{2}+1}} e^{-\frac{y}{2}(B^2-A^2)} \left(B\sqrt{y}\right) \int_0^\infty \frac{dh}{\sqrt{2\pi}} e^{-\frac{B^2 y}{2}h} \frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y+2}{2}}}$$

and check that

$$\left(B\sqrt{y}\right) \int_0^\infty \frac{dh}{\sqrt{2\pi}} e^{-\frac{B^2 y}{2}h} \frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y+2}{2}}}$$

is bounded in y.

Write again $B\sqrt{y} = z$, make the change of variable $k = z^2 h$ and observe that

$$\begin{aligned}
\left(B\sqrt{y}\right) \int_0^\infty \frac{dh}{\sqrt{2\pi}} e^{-\frac{B^2 y}{2}h} \frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y+2}{2}}} &= \int_0^\infty \frac{dk}{\sqrt{2\pi k}} e^{-\frac{k}{2}} \frac{\left(\frac{k}{z^2}\right)^{\frac{Y}{2}}}{\left(1+\frac{k}{z^2}\right)^{\frac{Y}{2}}} \\
&\leq \int_0^\infty \frac{dk}{\sqrt{2\pi k}} e^{-\frac{k}{2}} < \infty
\end{aligned}$$

We next consider the required integral near 0, or over the interval $[0,1]$. We have an expression of the form

$$\begin{aligned}
&\int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(e^{-yC}\sqrt{I(y)} - 1\right)^2 \qquad , C = \frac{B^2-A^2}{2} \\
I(y) &= \left(B\sqrt{y}\right) \int_0^\infty \frac{dh}{\sqrt{2\pi}} e^{-\frac{B^2 y}{2}h} \frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y}{2}}}
\end{aligned}$$

We now isolate the exponential by writing

$$\begin{aligned}
&\int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(e^{-yC}\sqrt{I(y)} - 1\right)^2 = \int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(\left(e^{-yC}-1\right)\sqrt{I(y)} + \sqrt{I(y)} - 1\right)^2 \\
\leq \; & 2\left(\int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(e^{-yC}-1\right)^2 + \int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(\sqrt{I(y)}-1\right)^2\right)
\end{aligned}$$

The exponential term is of order y near zero and hence this first integral is finite. For the second one we write

$$\begin{aligned}
&\int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(\sqrt{I(y)}-1\right)^2 \\
= &\int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left[\frac{\left(\sqrt{I(y)}-1\right)\left(\sqrt{I(y)}+1\right)}{\left(\sqrt{I(y)}+1\right)}\right]^2 \\
\leq &\int_0^1 \frac{dy}{y^{\frac{Y}{2}+1}} \left(I(y)-1\right)^2
\end{aligned}$$

For the finiteness of this integral we analyse the behavior of $(1 - I(y))$ near zero. For this we analyse $I(y) = J(yB^2)$ where

$$\begin{aligned}
J(y) &= \sqrt{y}\int_0^\infty \frac{dh}{\sqrt{2\pi}} e^{-\frac{yh}{2}} \frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y}{2}}} \\
&= \int_0^\infty \frac{dk}{\sqrt{2\pi k}} e^{-\frac{k}{2}} \Phi\left(\frac{k}{y}\right) \\
&= 2\int_0^\infty \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi\left(\frac{x^2}{y}\right) \\
&\equiv \int_{-\infty}^\infty \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi\left(\frac{x^2}{y}\right)
\end{aligned}$$

where

$$\Phi(\xi) = \left(\frac{\xi}{1+\xi}\right)^{\frac{Y}{2}}$$

Lemma 1 *The function $\Phi(\xi)$ is the distribution function of a random variable V that can also be realized as the ratio of two independent gamma variates, specifically*

$$V \overset{(d)}{=} \frac{\gamma_{\frac{Y}{2}}}{\gamma_{\frac{1}{2}}}$$

where γ_a is the gamma variate of parameter a. In particular V has finite moments of all orders $m < 1$ and

$$E[V^m] = \frac{\Gamma\left(\frac{Y}{2} + m\right)}{\Gamma\left(\frac{Y}{2}\right)} \Gamma(1 - m)$$

Proof. We note that Φ is the distribution function of a random variable V

where for a uniform variate U we have

$$\begin{aligned}
P(V \le \xi) &= P\left(U \le \left(\frac{\xi}{1+\xi}\right)^{\frac{Y}{2}}\right) \\
&= P\left(U^{\frac{2}{Y}} \le \frac{\xi}{1+\xi}\right) \\
&= P\left((1+\xi)U^{\frac{2}{Y}} \le \xi\right) \\
&= P\left(U^{\frac{2}{Y}} \le \xi\left(1 - U^{\frac{2}{Y}}\right)\right) \\
&= P\left(\frac{U^{\frac{2}{Y}}}{1 - U^{\frac{2}{Y}}} \le \xi\right)
\end{aligned}$$

so that V is the random variable

$$V = \frac{U^{\frac{2}{Y}}}{1 - U^{\frac{2}{Y}}}$$

From the Beta-Gamma algebra we deduce that V is

$$V = \frac{\gamma_{\frac{Y}{2}}}{\gamma_1}$$

Consequently V has finite moments for all powers below unity. In particular for $m < 1$

$$E[V^m] = \frac{\Gamma\left(\frac{Y}{2} + m\right)}{\Gamma\left(\frac{Y}{2}\right)} \Gamma\left(1 - m\right)$$

■

As a consequence for $m = \frac{1}{2}$ we have that

$$E[\sqrt{V}] = \frac{\Gamma\left(\frac{Y+1}{2}\right)}{\Gamma\left(\frac{Y}{2}\right)} \sqrt{\pi}.$$

Furthermore we have that as

$$\begin{aligned} 1 - J(y) &= P\left(|G| \le \sqrt{Vy}\right) \\ &\sim \sqrt{\frac{2}{\pi}} \sqrt{y} E\left[\sqrt{V}\right] \end{aligned}$$

So the order of convergence of $1 - I(y) = 1 - J(yB^2)$ is always $\alpha = \frac{1}{2}$ and so

$$\frac{Y}{2} < 2\alpha \equiv 1$$

for all $Y < 2$. The desired absolute continuity result is established.

We also observe that

$$I(y) = J(yB^2) = P\left(|G| \ge B\sqrt{Vy}\right) = P\left(\frac{G^2}{B^2 V} \ge y\right) \le 1$$

3.2.3 A Further analysis of $I(y)$

We now write the Lévy measure of the $CGMY$ subordinator in the form

$$\frac{K}{y^{1+\frac{Y}{2}}} E[e^{-yZ}]$$

for some random variable Z.

For a fixed constant B the Lévy measure of our subordinator in the symmetric case is

$$\begin{aligned}
R &= \frac{KBe^{-B^2\frac{y}{2}}}{y^{\frac{Y+1}{2}}}\int_0^\infty \frac{dh}{\sqrt{2\pi}}e^{-\frac{yB^2h}{2}}\frac{h^{\frac{Y-1}{2}}}{(1+h)^{\frac{Y}{2}}} \\
&= \frac{KBe^{-B^2\frac{y}{2}}}{y^{\frac{Y+1}{2}}}\int_0^\infty \frac{dh}{\sqrt{2\pi h}}e^{-\frac{yB^2h}{2}}P\left(V\le h\right) \\
&= \frac{KBe^{-B^2\frac{y}{2}}}{y^{\frac{Y+1}{2}}}\int_0^\infty \frac{dk}{\sqrt{2\pi}}e^{-\frac{yB^2k^2}{2}}P\left(V\le k^2\right) \\
&= \frac{Ke^{-B^2\frac{y}{2}}}{y^{\frac{Y}{2}+1}}2\int_0^\infty \frac{dz}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}P\left(V\le \frac{z^2}{B^2y}\right) \\
&= \frac{Ke^{-B^2\frac{y}{2}}}{y^{\frac{Y}{2}+1}}P\left(\frac{G^2}{VB^2}\ge y\right)
\end{aligned}$$

We also know that

$$V \overset{(d)}{=} \frac{\gamma_{\frac{Y}{2}}}{\gamma_1}$$

with two independent gamma variables. Thus we may write

$$\begin{aligned}
R &= \frac{Ke^{-B^2\frac{y}{2}}}{y^{\frac{Y}{2}+1}}P\left(\gamma_1\ge \frac{yB^2\gamma_{\frac{Y}{2}}}{G^2}\right) \\
&= \frac{Ke^{-B^2\frac{y}{2}}}{y^{\frac{Y}{2}+1}}E\left[\exp\left(-\frac{yB^2\gamma_{\frac{Y}{2}}}{G^2}\right)\right] \\
&= \frac{K}{y^{\frac{Y}{2}+1}}E\left[\exp\left(-yB^2\frac{\gamma_{\frac{Y}{2}}+\frac{y}{2}G^2}{G^2}\right)\right]
\end{aligned}$$

But

$$\frac{1}{2}G^2 \overset{(d)}{=} \gamma_{\frac{1}{2}}$$

so that we get

$$R = \frac{Ke^{-B^2\frac{y}{2}}}{y^{\frac{Y}{2}+1}}E\left[\exp\left(-y\frac{B^2}{2}\frac{\gamma_{\frac{Y}{2}}}{\gamma_{\frac{1}{2}}}\right)\right] \qquad (12)$$

We now have identified the two Lévy measures as

$$\nu_0(dy) = \frac{Kdy}{y^{\frac{Y}{2}+1}}$$

and

$$\begin{aligned}
\nu_1(dy) &= \nu_0(dy)e^{-\frac{B^2y}{2}}E\left[\exp\left(-yZ\right)\right] \\
Z &= \frac{B^2}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}.
\end{aligned}$$

3.2.4 Evaluating explicitly the LT of Z

There is an additional randomness in the simulation if the expectation

$$E[e^{-y\frac{B^2}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}}]$$

is evaluated by simulation. It is helpful to explicitly evaluate this function. We begin with

$$\phi_{a,b}(\lambda) = E\left[\exp\left(-\lambda\frac{\gamma_a}{\gamma_b}\right)\right]$$

Now we have that

$$\begin{aligned}
e^{-\lambda}\phi_{a,b}(\lambda) &= E\left[\exp\left(-\frac{\lambda}{\beta(b,a)}\right)\right] \\
&= \frac{1}{B(b,a)}\int_0^1 (1-x)^{a-1}x^{b-1}e^{-\frac{\lambda}{x}}dx \\
&= \frac{1}{B(b,a)}\int_1^\infty \frac{dy}{y^2}\left(\frac{1}{y}\right)^{b-1}\left(1-\frac{1}{y}\right)^{a-1}e^{-\lambda y} \\
&= \frac{1}{B(b,a)}\int_1^\infty \frac{dy}{y^{a+b}}(y-1)^{a-1}e^{-\lambda y}
\end{aligned}$$

Hence we have that

$$\phi_{a,b}(\lambda) = \frac{1}{B(a,b)}\int_0^\infty \frac{z^{a-1}}{(1+z)^{a+b}}e^{-\lambda z}dz$$

We are interested in the case $a = \frac{Y}{2}$, $b = \frac{1}{2}$ and so we write

$$\phi_{\frac{Y}{2},\frac{1}{2}}(\lambda) = \frac{1}{B\left(\frac{Y}{2},\frac{1}{2}\right)}\int_0^\infty dx\ x^{\frac{Y}{2}-1}(1+x)^{-\frac{Y}{2}-\frac{1}{2}}e^{-\lambda x}$$

From Gradshetyn and Ryzhik (1995) (3.38) (7) Page 319 we have

$$\begin{aligned}
\int_0^\infty dx\ x^{\frac{Y}{2}-1}(1+x)^{-\frac{Y}{2}-\frac{1}{2}}e^{-\lambda x} &= 2^{\frac{Y}{2}}\Gamma\left(\frac{Y}{2}\right)e^{\frac{\lambda}{2}}D_{-Y}\left(\sqrt{2\lambda}\right) \\
&= 2^{\frac{Y}{2}}\Gamma\left(\frac{Y}{2}\right)h_{-Y}\left(\sqrt{2\lambda}\right)
\end{aligned}$$

where $h_\nu(x)$ is the Hermite function of index ν.

We have related the Hermite functions to the functions

$$I(\nu,a,\lambda) = \int_0^\infty x^{\nu-1}e^{-ax-\lambda x^2}dx = (2\lambda)^{-\nu/2}\Gamma(\nu)h_{-\nu}\left(\frac{a}{\sqrt{2\lambda}}\right)$$

in Carr, Geman, Madan and Yor (2005).

We may therefore write

$$h_{-Y}(\sqrt{2\lambda}) = (2\lambda)^{\frac{Y}{2}}\frac{1}{\Gamma(Y)}I(Y,2\lambda,\lambda)$$

It follows that

$$\int_0^\infty dx\ x^{\frac{Y}{2}-1}(1+x)^{-\frac{Y}{2}-\frac{1}{2}}e^{-\lambda x} = 2^{\frac{Y}{2}}\Gamma\left(\frac{Y}{2}\right)\frac{(2\lambda)^{\frac{Y}{2}}}{\Gamma(Y)}I(Y,2\lambda,\lambda)$$
$$= 2^Y\lambda^{\frac{Y}{2}}\frac{\Gamma\left(\frac{Y}{2}\right)}{\Gamma(Y)}I(Y,2\lambda,\lambda)$$

It follows that

$$\phi_{\frac{Y}{2},\frac{1}{2}}(\lambda) = 2^Y\lambda^{\frac{Y}{2}}\frac{\Gamma\left(\frac{Y}{2}+\frac{1}{2}\right)}{\Gamma(Y)\Gamma(\frac{1}{2})}I\left(Y,2\lambda,\lambda\right)$$

We therefore evaluate

$$E\left[e^{-y\frac{B^2}{2}\frac{\gamma_{\frac{Y}{2}}}{\gamma_{\frac{1}{2}}}}\right] = \frac{\Gamma\left(\frac{Y}{2}+\frac{1}{2}\right)}{\Gamma(Y)\Gamma(\frac{1}{2})}2^Y\left(\frac{B^2y}{2}\right)^{\frac{Y}{2}}I\left(Y,B^2y,\frac{B^2y}{2}\right) \tag{13}$$

Putting together the result of equation (13) and equation (12) we get the results for the $CGMY$ subordinator (9).

4 Simulating CGMY using Rosinski Rejection

We suppose that we have two Lévy measures $Q(dx), Q_0(dx)$ with the property that

$$\frac{dQ}{dQ_0} \leq 1;$$

and this is our case, then it is shown in Rosinski that we may simulate the paths of Q from those of Q_0 by only accepting all jumps x in the paths of Q_0 for which

$$\frac{dQ}{dQ_0}(x) > w$$

where w is an independent draw from a uniform distribution.

For our case we have that

$$\frac{d\nu_1}{d\nu_0} = E\left[e^{-yZ}\right] < 1$$

and so accept all jumps in the paths of ν_0 for which

$$E\left[e^{-yZ}\right] > w$$

The detailed algorithm is for parameters C, G, M, Y to first define the time step to be C,

$$t = C.$$

Then we let

$$\begin{aligned} A &= \frac{G-M}{2} \\ B &= \frac{G+M}{2} \end{aligned}$$

We next simulate at time t from the one-sided stable subordinator with Lévy measure

$$\frac{1}{y^{\frac{Y}{2}+1}}dy$$

For this we let $\varepsilon = .0001$ and truncate jumps below ε replacing them by their expected value at a rate of

$$\begin{aligned} d &= \int_0^{\varepsilon} y \frac{1}{y^{\frac{Y}{2}+1}} dy \\ &= \frac{\varepsilon^{1-\frac{Y}{2}}}{1-\frac{Y}{2}} \end{aligned}$$

For the arrival rate of jumps we have an arrival rate λ of

$$\begin{aligned} \lambda &= \int_{\varepsilon}^{\infty} \frac{1}{y^{\frac{Y}{2}+1}} dy \\ &= \frac{2}{Y}\frac{1}{\varepsilon^{\frac{Y}{2}}} \end{aligned}$$

The interval jump times are exponential and are simulated by

$$t_i = -\frac{1}{\lambda}\log\left(1-u_{2i}\right)$$

for an independent uniform sequence u_{2i}. The actual jump times are

$$\Gamma_j = \sum_{i=1}^{j} t_i$$

For the jump magnitude we simulate from the normalized Lévy measure the jump size y_j given by

$$y_j = \frac{\varepsilon}{(1-u_{1j})^{\frac{2}{Y}}}$$

for an independent uniform sequence u_{1j}.

The process $S(t)$ for the stable subordinator is given by

$$S(t) = dt + \sum_{j=1}^{\infty} y_j \mathbf{1}_{\Gamma_j < t}$$

We now get the $CGMY$ subordinator $H(t)$ by

$$\begin{aligned} H(t) &= dt + \sum_{j=1}^{\infty} y_j \mathbf{1}_{\Gamma_j < t} \mathbf{1}_{h(y) > u_{3j}} \\ h(y) &= e^{-\frac{B^2 y}{2}} \frac{\Gamma\left(\frac{Y}{2}+\frac{1}{2}\right)}{\Gamma(Y)\Gamma(\frac{1}{2})} 2^Y \left(\frac{B^2 y}{2}\right)^{\frac{Y}{2}} I\left(Y, B^2 y, \frac{B^2 y}{2}\right) \end{aligned}$$

for an independent uniform sequence u_{3j}

Finally we simulate the $CGMY$ random variable by

$$X = AH(t) + \sqrt{H(t)}z$$

for a draw z of a standard normal random variable.

5 The Meixner Process as a Time Changed Brownian Motion

We consider the Meixner Process (Schoutens and Teugels (1998), Pitman and Yor (2003)) as a time changed Brownian motion. The Lévy measure of the Meixner process is

$$k(x) = \delta \frac{\exp\left(\frac{b}{a}x\right)}{x \sinh\left(\frac{\pi x}{a}\right)}$$

The characteristic function is given by

$$\begin{aligned} \phi_{Meixner}(u) &= E[e^{iuX_1}] \\ &= \left(\frac{\cos(b/2)}{\cosh(au - ib)/2)}\right)^{2\delta} \end{aligned}$$

To see this process as a time changed Brownian motion we wish to identify $l(u)$ the Lévy measure of a subordinator such that

$$\begin{aligned} k(x) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{(x-Ay)^2}{2y}\right) l(y)dy \\ &= e^{Ax} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y} - \frac{A^2 y}{2}\right) l(y)dy \end{aligned}$$

Hence we set

$$A = \frac{b}{a}$$

and seek to write

$$\delta \frac{1}{x \sinh\left(\frac{\pi x}{a}\right)} = \int_0^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y} - \frac{A^2 y}{2}\right) l(y)dy \qquad (14)$$

We transform the left hand side of (14) as follows.

We recall that
$$\frac{Cx}{\sinh(Cx)} = E\left[\exp\left(-\frac{x^2}{2}T_C^{(3)}\right)\right]$$
where $T_C^{(3)} = \inf\left\{t|R_t^{(3)} = C\right\}$ for $R_t^{(3)}$ the $BES(3)$ process.

Then we write

$$\begin{aligned}
\delta\frac{1}{x\sinh\left(\frac{\pi x}{a}\right)} &= \frac{\delta\left(\frac{\pi x}{a}\right)}{\left(\frac{\pi x^2}{a}\right)\sinh\left(\frac{\pi x}{a}\right)} \\
&= \frac{\delta a}{\pi}\frac{1}{x^2}E\left[\exp\left(-\frac{x^2}{2}T_C^{(3)}\right)\right] \\
&= \frac{\delta a}{\pi}\frac{1}{x^2}E\left[\exp\left(-\frac{x^2C^2}{2}T_1^{(3)}\right)\right]
\end{aligned}$$

with $C = \frac{\pi}{a}$. Denote by $\theta(h)dh$ the law of $T_1^{(3)}$. We may then write

$$\begin{aligned}
\delta\frac{1}{x\sinh\left(\frac{\pi x}{a}\right)} &= \frac{\delta a}{\pi}\int_0^\infty \frac{du}{2}\exp\left(-\frac{x^2u}{2}\right)E\left[\exp\left(-\frac{x^2C^2}{2}T_1^{(3)}\right)\right] \\
&= \frac{\delta a}{2\pi}\int_0^\infty duE\left[\exp\left(-\frac{x^2}{2}\left(u + C^2T_1^{(3)}\right)\right)\right] \\
&= \frac{\delta a}{2\pi}\int_0^\infty du\int_0^\infty \theta(t)dt\exp\left(-\frac{x^2}{2}(u+C^2t)\right) \\
&= \frac{\delta a}{2\pi}\int_0^\infty du\int_u^\infty \frac{dv}{C^2}\exp\left(-\frac{x^2v}{2}\right)\theta\left(\frac{v-u}{C^2}\right) \\
&= \frac{\delta a}{2\pi}\int_0^\infty dv\exp\left(-\frac{x^2v}{2}\right)\int_0^v \frac{du}{C^2}\theta\left(\frac{v-u}{C^2}\right) \\
&= \frac{\delta a}{2\pi}\int_0^\infty dv\exp\left(-\frac{x^2v}{2}\right)\int_0^{\frac{v}{C^2}} dh\theta(h) \\
&= \int_0^\infty dv\exp\left(-\frac{x^2v}{2}\right)\widehat{\theta}(v)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\theta}(v) &= \frac{\delta a}{2\pi}\int_0^{\frac{v}{C^2}}\theta(h)dh \\
&= \frac{\delta a}{2\pi}P\left(T_1^{(3)} \leq \frac{v}{C^2}\right) \\
&= \frac{\delta a}{2\pi}P\left(Max_{t\leq\frac{v}{C^2}}R_t^{(3)} \geq 1\right)
\end{aligned}$$

We recall that
$$T_1^{(3)} \stackrel{(law)}{=} \frac{1}{\left(\max_{t\leq 1}R_t^{(3)}\right)^2}$$

We now transform the right hand side of (14) to write

$$\int_0^\infty \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y} - \frac{A^2 y}{2}\right) l(y)dy = \int_0^\infty \frac{1}{\sqrt{2\pi v^3}} \exp\left(-\frac{x^2 v}{2} - \frac{A^2}{2v}\right) l\left(\frac{1}{v}\right) dv$$

From the uniqueness of Laplace transforms we deduce that

$$\widehat{\theta}(v) = \frac{1}{\sqrt{2\pi v^3}} \exp\left(\frac{A^2}{2v}\right) l\left(\frac{1}{v}\right)$$

or

$$\begin{aligned} l(u) &= \sqrt{\frac{2\pi}{u^3}}\widehat{\theta}\left(\frac{1}{u}\right) \exp\left(-\frac{A^2 u}{2}\right) \\ &= \sqrt{\frac{2\pi}{u^3}}\frac{\delta a}{2\pi} P\left(M_1^{(3)} \geq C\sqrt{u}\right) \exp\left(-\frac{A^2 u}{2}\right) \\ &= \frac{\delta a}{\sqrt{2\pi u^3}} P\left(M_1^{(3)} \geq C\sqrt{u}\right) \exp\left(-\frac{A^2 u}{2}\right) \\ &= \frac{\delta a}{\sqrt{2\pi u^3}} g(u) \end{aligned}$$

where

$$g(u) = P\left(M_1^{(3)} \geq C\sqrt{u}\right) \exp\left(-\frac{A^2 u}{2}\right)$$

For the absolute continuity of our subordinator with respect to the one sided stable $\frac{1}{2}$ subordinator we require that

$$\int \frac{1}{\sqrt{u^3}} \left(\sqrt{g(u)} - 1\right)^2 du < \infty.$$

For this we observe that

$$\begin{aligned} \left(\sqrt{g(u)} - 1\right)^2 &\leq |g(u) - 1| \\ &= 1 - g(u) \\ &= 1 - P\left(M_1^{(3)} \geq C\sqrt{u}\right) \exp\left(-\frac{A^2 u}{2}\right) \\ &= 1 - \exp\left(-\frac{A^2 u}{2}\right) + \exp\left(-\frac{A^2 u}{2}\right) \left(1 - P\left(M_1^{(3)} \geq C\sqrt{u}\right)\right) \end{aligned}$$

The first part is clearly integrable with respect to $\left(\frac{du}{u^{3/2}}\right)$and for the second we observe that as

$$\lambda^k P\left(T \geq \lambda\right) \leq E\left[T^k\right]$$

that

$$P\left(\frac{1}{(M_1^{(3)})^2} \geq \frac{1}{C^2 u}\right) = P\left(T_1^{(3)} \geq \frac{1}{C^2 u}\right) \leq K u^k, \text{ for all } k$$

For the simulation of Meixner as a time changed Brownian motion we would wish to evaluate

$$
\begin{aligned}
P\left(M_1^{(3)} \geq C\sqrt{u}\right) &= P\left(\frac{1}{(M_1^{(3)})^2} \leq \frac{1}{C^2 u}\right) \\
&= P\left(T_1^{(3)} \leq \frac{1}{C^2 u}\right) \\
&= P(\pi^2 T_1^{(3)} \leq \frac{\pi^2}{C^2 u}) \\
&= P(T_\pi^{(3)} \leq \frac{\pi^2}{C^2 u}) \\
&= \sum_{-\infty}^{\infty} (-1)^n e^{-n^2\pi^2/(2C^2 u)}
\end{aligned}
$$

For the last equality we refer to Pitman and Yor (2003).

6 Simulation of the Meixner Process

The simulation strategy is similar to that employed in section 3 for $CGMY$, except that here we simulate first the jumps of the one sided stable $\frac{1}{2}$ with Lévy density

$$k(x) = \frac{\delta a}{\sqrt{2\pi x^3}}, \ x > 0.$$

We approximate the small jumps of the subordinator using the drift

$$\zeta = \delta a \sqrt{\frac{2\varepsilon}{\pi}}$$

The arrival rate for the jumps above ε is

$$\lambda = \delta a \sqrt{\frac{2}{\pi\varepsilon}}$$

and the jump sizes for the one sided stable$\left(\frac{1}{2}\right)$ are

$$y_j = \frac{\varepsilon}{u_j^2}$$

for an independent uniform sequence u_j.

We then evaluate the function $g(y)$ at the point y_j and define the time change variable

$$\tau = \varsigma + \sum_j y_j \mathbf{1}_{g(y_j) > w_j}$$

for yet another independent uniform sequence w_j. We note that the function $g(y)$ only use the parameters a, b and is independent of the parameter d.

The value of the Meixner random variable or equivalently the unit time level of the process is then generated as

$$X = \frac{b}{a}\tau + \sqrt{\tau}z$$

where z is an independent standard normal variate.

7 Results of Simulations

For both the $CGMY$ and $Meixner$ processes we present in this section the results of simulating the processes at typical parameter values obtained on calibrating option prices on the S&P 500 index. The parameter values for the $CGMY$ are $C = 1$, $G = 5$, $M = 10$, and $Y = .5$. The parameters for the $Meixner$ were $a = .25$, $b = -1.5$ and $\delta = 1$.

We present graphs (1,2) for a weekly time step $h = .02$ of the simulated and actual densities as well as chi square tests of the hypothesis that the sample was drawn from the respective densities. The solid lines are the theoretical density while the data points are indicated by dots. The sample sizes in both cases were 5000. The range for both the $CGMY$ and $Meixner$ returns was 25%. In both cases we used 100 cells and employed those with more than five observations for the test. The $CGMY$ had a chisquare statistic of 42.0122 with 56 degrees of freedom and a $p-value$ of .9172. For the $Meixner$ the test statistic was 78.70 with 84 degrees of freedom and a $p-value$ of .6427.

Figure 1: CGMY simulation as time changed Brownian Motion using shaved one sided stable $Y/2$.

Figure 2: Meixner simulation as time changed Brownian motion using shaved one sided stable 1/2.

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