EMERGING PRACTICES FOR RECORDING WRITTEN SOLUTIONS IN A CAS CLASSROOM

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Written solutions for mathematical problems have the potential to be very different in a CAS classroom compared to a non-CAS classroom. New opportunities in a classroom where CAS is the normal technology may require a rethinking of what constitutes a good written record of a solution for a problem. This paper reports on the approaches for modelling good written records in class examples that were used by three teachers teaching year 12 mathematics with CAS for the first time. It was found that the teachers tended to use standard mathematical notation in demonstrating solutions. Student practice appeared to be developing independently of the teachers with written solutions incorporating notation that suggested CAS use.

CAS IMPACTS ON WRITTEN RECORDS AND THE CLASSROOM

When teachers are working in classrooms where CAS is the normal technology, how do they model what a good written record may be for a solution to a problem? Having CAS available provides new opportunities for solving problems, often involving less steps than would be required in a by-hand solution and teachers will need to provide guidance to their students about how to record their working. This paper will consider how three teachers modelled written records of solutions in CAS classrooms.

When teachers are teaching with CAS there are a range of issues that they expect to have to deal with such as teaching students how to use syntax correctly, dealing with an explosion of methods (Artigue, 2001), teaching students to interpret CAS outputs, helping students become effective users of CAS (Pierce & Stacey, in press), instrumentation issues (Lagrange, 1999; Guin & Trouche, 1999; Artigue, 2001), developing students' algebraic insight (Pierce & Stacey, 2002) and assessment issues (Flynn & Asp, 2002). As teachers become familiar with the resources and constraints offered by CAS their focus may move to consideration of the potential of the technology and consideration of pedagogical changes to address some of the issues stated previously.

One issue which has become increasingly apparent during the CAS-CAT study (see CAS-CAT Research Project, n.d. for more details), particularly once teachers move beyond issues associated with teaching how to use CAS, is the issue of what students need to document in a written record to adequately communicate solutions for problems (see for example, Ball & Stacey, 2003; Ball, 2003). The students in this study were able to use CAS calculators for all work and they were allowed to use CAS calculators in their end of year external examinations. As a consequence, in addition to teaching students how to use CAS and how to approach solving problems with CAS, teachers also needed to provide models for how students could record their work to adequately communicate their mathematical thinking.

This paper is not about whether or not teachers are showing students how to use CAS to solve problems, but how they are choosing to demonstrate written solutions, given that they know that students have CAS and that the students are allowed to use CAS in their examinations and hence need to learn to make decisions about what to record in this case. The study teachers found their own ways to deal with the issue of recording. This paper will consider how three teachers from two different schools developed practices for recording of written solutions in CAS classrooms and will then discuss whether students' practice on selected questions in an end of year examination reflected the practice that the teachers had demonstrated in class.

THE THREE TEACHERS

The three teachers discussed in this paper, Ken, Lucy and Meg[1], who are all experienced at teaching with graphics calculators taught with CAS in year 12 mathematics for the first time during 2002, having taught with CAS in year 11 the previous year. The three teachers largely worked in non-CAS classrooms in the past.

Ken and Lucy used one brand of CAS calculator with their classes, while Meg used another different brand. The fact that students in this study were allowed to use CAS in high stakes assessment meant that teachers really had to consider how to help their students use CAS to best advantage and also how to clearly communicate mathematical thinking. This was particularly important as the end of year examinations were marked by external assessors who had knowledge of CAS, but not necessarily the specific syntax of the CAS calculator being used by any particular student.

Due to the students being able to use a CAS calculator in the examinations, teachers needed to model good written records for solutions when one line of CAS entry could replace many of the by-hand procedures that would normally be recorded. Additionally, when CAS is available there is the option to work entirely within the CAS when solving a problem and then go back through the screen history to decide what to record as a written solution. This situation requires students to be able to make sensible decisions about what to record. Students will be looking to their teachers for guidance about what they should be writing down as solutions to problems. Teachers need to model what they believe to be good records of solutions when they work through examples in class. Decisions need to be made about the use of CAS syntax or mathematical notation to communicate mathematical thinking and about the extent and structure of written records.

PROJECT MEETINGS AND INTERVIEWS

During 2002 one focus of discussions among the study teachers and research team was how students could communicate solutions when CAS is used for solving. The teachers in 2002 had been participants in the CAS-CAT study since 2000 and during that time there had been a number of meetings where issues associated with CAS were discussed. The research team worked very closely with the teachers to assist in the planning of some teaching material and the provision of classroom support material (CAS-CAT research project, n.d.) however each teacher was responsible for deciding on their own teaching approaches and sequence in the course. In 2002 there were a number of meetings to discuss issues related to implementing the CAS course. Meetings and interviews were audio taped. In addition to meetings with the entire project team, the researchers also had regular contact with teachers via email, phone conversations and classroom visits. There was frequent communication between the author and the teachers in the project.

Discussions at meetings, interviews and informal communication showed that Ken, Lucy and Meg had given a lot of consideration to the issues associated with implementation of CAS in their classrooms. This is also true of the fourth teacher in the project, Neil[2], who is not discussed in this paper as the author did not do classroom observations in Neil's classes.

At a two day meeting at the start of 2002 the research team and teachers discussed recording of written solutions, then examples were developed to illustrate appropriate recording when CAS was available followed by discussion. The main issues discussed related to recording of solutions are shown in Figure 1. Clearly the teachers we were working with were thinking about issues associated with recording at this early stage of the year. The discussions considered

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[1] Ken, Lucy and Meg are not their real names. The teachers' anonymity has been protected by changing their names and possibly gender.

[2] Neil is not his real name. The teacher's anonymity has been protected by changing his name and possibly gender.

communication with assessors, the elements of a good written record, what might be important in a mathematical solution and the use of calculator syntax.

- Students need to consider what is important in a written mathematical solution.
- Students need to think about who they are communicating to when they write solutions and keep in mind that the assessor may not be familiar with the student's CAS.
- Assessors need to be able to understand written solutions regardless of the CAS used.
- RIPA: Inputs essential, give reasons and the plan (Ball & Stacey, 2003). Every intermediate step of working does not need to be shown and students do not need to copy down every output from the CAS for their written record.
- Process for solving being written in words.
- Use of calculator specific syntax and whether this is appropriate.
- What key answers need to be recorded?
- CAS shifts the emphasis - mathematical thinking still required in solving problems, however the extent of by-hand work may be less.
- CAS students need the overview of solving in a written record, but might not have specific calculations.
- Teachers wanted to see models of good written records.

Figure 1: Main points of discussion about recording written solutions at Feb 2002 meeting

STYLES OF WRITTEN SOLUTIONS BEING MODELLED BY THE TEACHERS

The author observed three teachers, Ken, Lucy and Meg over a ten month period during 2002. During 2002, Ken was observed for five lessons, Lucy for eight and Meg for nine lessons. The lesson observations occurred from March (towards the start of the school year) to October (a few weeks before the final examination) at times convenient to the teachers. Lesson observations occurred when the teachers were "teaching" students rather than lessons when students were doing tests or assessment tasks. During lesson observations field notes were taken which included a copy of everything written on the board by the teacher as well as comments on CAS use. The lessons were audio taped.

This paper will analyse the written records that the teachers wrote for worked examples in demonstrating how to solve problems on the board. Each written record was classified according to two features, the first being whether the record contained standard mathematical notation (M) or whether there was evidence of non-standard notation (M′). The written records were then classified as either containing words (W) or not containing any words (W′) in the mathematical working. Words that were given in the statement of the answer were not categorised as 'words' in the mathematical working. These two features together determine four possible styles of written records (see Table 1).

Table 1 shows the percentage of worked examples of each style for each teacher. Overall the teachers are predominantly using standard mathematical notation in demonstrating written records in class, assuming that the classroom observations were typical lessons. Based on conversations with the teachers, it is reasonable to view these lessons as typical. There is some use of words in written solutions by teachers, however this is limited. Most non-standard notation was used in class to demonstrate a CAS procedure. Clearly it is not institutionalised that CAS syntax or non-standard notation be recorded when communicating working.

In class Ken spoke about using the calculator and he would ask students what the output was when entering something into their calculator. In terms of recording of written work, Ken mainly used standard mathematical notation in the classes that were observed. The only exception was when Ken was showing how to use the calculator to anti-differentiate and in this case he wrote the calculator syntax on the board (style M′+W′). For other examples, even though students may have generated answers using the calculator, Ken, in recording student responses from the CAS calculator, wrote these on the board using standard mathematical notation.

Lucy's recording style in class seems to be more varied than the other two teachers and she is inclined to include more words in her solutions than the other two teachers, although there is little use of non-standard notation. Lucy's practice was very reflective of her beliefs stated in interviews where by-hand techniques were 'privileged' (Kendal & Stacey, 2002) for introducing new content as she believed this was necessary to develop student understanding. On the other hand, Lucy does have a focus on development of judicious use of CAS (Ball & Stacey, in press) among her students and both used and had her students use CAS in class frequently, discussing merits of different methods. Given the high level of use of CAS in Lucy's class the fact that Lucy predominantly used mathematical notation could suggest a belief that students should be recording working using mathematical notation. Lucy incorporated words into her solutions. These were usually used in the same way as in a by-hand solution unless the word was being used to describe non-standard notation, such as when she used the word 'solve' in writing a CAS procedure, for example, Solve(p(1)=10000 and p(5)=8000, {a, b}).

Table 1

Percentage of worked examples of each style by teacher

Notation used in recording worked examples during classroom observations	**Ken (N=14)**	**Lucy (N=36)**	**Meg (N=39)**
M+W′	93	69	90
M+W	0	19	3
M′+W	0	8	3
M′+W′	7	3	5

N=number of worked examples analysed

The classroom observations suggest that the teachers mainly used mathematical notation to model written records for solutions to problems. Where words were used they were generally used with standard mathematical notation (style M+W) using terms such as 'substitute' or 'derivative' (for example, in the context of setting the derivative equal to zero). Generally, if a CAS was used then it was either demonstrated without recording what was done and the written record contained standard mathematical notation, either with or without words (M+W or M+W′). Where non-standard notation was present in a worked example, it was generally in the context of showing students how to use their CAS to perform a procedure, for example, in showing the syntax for finding an anti-derivative of a function, Ken wrote (∫f′(x),x). This non-standard notation most likely appeared as a consequence of having CAS as there would be no purpose for using this if CAS was not being used.

The small number of instances of new non-standard notation, or even use of a combination of mathematical notation and words (refer to Table 1) observed during classroom observations (N=22) suggested that these three teachers expected that the students in their classes would use mathematical notation to communicate solutions. This could be related to beliefs about how students learn mathematics. For a teacher who believed that students learn mathematics through performance of by-hand routines, it may be that they demonstrated all procedures by-hand, using mathematical notation. Lucy was an example of this type of teacher as her comments in interviews suggested that she believed students needed to be able to perform simple routines by-hand for understanding. Lucy generally introduced new topics using by-hand procedures. However, Lucy was also very willing to use CAS in her class and to allow her students to use CAS. Lucy both encouraged and discouraged CAS use to promote thinking before reaching for technology. She would often ask students what the CAS answer was and then show by-hand algebraic manipulation to link this to the answer that was obtained using by-hand techniques. There are a number of reasons why teachers might demonstrate by-hand techniques, for the development of understanding of a particular concept by students or because the teachers are very competent at these techniques and find them a quick and efficient way to solve problems. In some ways, the teachers could be choosing the most efficient method available to them, which they also equate with learning mathematics.

In terms of some CAS symbolism becoming normal practice in written examples, from the classroom observations it seemed that this only occurred in Meg's class where EXP(X) was used to represent e^x. This shows the introduction of new ways of writing maths based on a calculator convention. The calculator used by Meg and her students required this entry when working with exponentials.

One other difference in the classroom practice of Meg was that she stated in a meeting that she had students focus on writing down procedures rather than showing step by step working:

> … write down the procedure…write it down, solve for x, solve for k, solve for whatever…

This would seem to be reinforced by an example in the classroom observations where Meg demonstrated how to solve simultaneous equations (Figure 2) which involved finding values for *a*, *b* and *c* given initial conditions for a quadratic function. Meg included a verbal description of the entry into her calculator, describing the process for solving simultaneous equations. She described the plan used for solving for *a*, *b* and *c*, and then wrote down the answer. This written record would be classified as M+W.

	$y=ax^2+bx+c$
(3,1)	11=9a+3b+c (1)
(-1, 7)	7=a-b+c (2)
(9, 107)	107=81a+9b+c (3)

(2) – (1) – eliminate c → (4)

(3) – (1) – eliminate c → (5

(4), (5) → find (a) (b)

$y=1.5x^2-2x+3.5$

Figure 2: Meg's written record for solving simultaneous equations

In discussions in November, both Ken and Meg believed that their students were having difficulty reconciling the practice in previous years, where teachers had expected working out to include all intermediate steps, to the practice being suggested by them where not all steps needed to be recorded. They both felt that students were feeling unsure about not having to write down every step in their solutions when teachers in previous years had required this for written records.

In comparing another brand of CAS to the brand his class used, Ken believed that his students weren't using syntax as such because, Ken believed:

> See the [CAS calculator brand] doesn't have that calculator language.

Ken did not think that the calculator he was using had any specific calculator syntax. He had however noted a change in his practice and this was in terms of how he defined functions:

> … I used to say, maybe, let y equal f(x), now you say DEFINE or something.

It is interesting that Ken had noted that he had started to use the word DEFINE as a normal part of his mathematical language given that this word was calculator syntax for the CAS being used, particularly given that he did not feel that his CAS calculator had a specific 'calculator language'. In some ways Ken had institutionalised the use of the term DEFINE and did not view this as calculator syntax. Would his students think of DEFINE as standard mathematical notation and would this be reflected in how they recorded their solutions?

Lucy had a different view to Ken and believed that her students were influenced by the particular syntax of the calculator and that they had started using syntax in their language. For example, Lucy believed:

> It was more natural, I think, for the kids to say things like 'SOLVE f(x)=3, x....

Lucy felt that her students were getting better at having a more macro view of solving with an increased focus on choosing procedures rather than carrying them out. She believed that students were able to consider a problem and think about the overall steps required to solve based on their knowledge of solving similar types of problems. She characterized a student solving a problem as saying:

> ...Oh, I can see that what I need to do here is [as follows]. I have two equations and two unknowns… I know that I've got to solve. I've got to define this function that way and I'm going to solve it for this…

Overall the practice of the teachers in terms of what they were recording as written solutions for problems had not appeared to change significantly due to having CAS available.

COMPARISON OF STUDENTS' STYLES OF RECORDING IN EXAMINATIONS WITH TEACHERS' STYLES IN WORKED EXAMPLES

The following section will consider the practice of the students from these two schools in the end of year examinations and discuss whether or not students' practice reflected what the teachers had demonstrated in class. If the expectation was that students would follow the models presented by their teachers for written records then we would expect most students to use standard mathematical notation in recording their solutions, with some words.

For this section the responses of students to four question parts (1b, 1ci, 3ai and 3bi) on examination two were considered. Figure 3 shows the questions that students' responses were analysed for. These questions were selected as they were questions where CAS could be used for algebraic work. Table 2 gives a summary of students' styles used in written responses for these questions, as well as a summary of teachers' styles used in written worked examples in class for each CAS. It is worth noting that it was evident from students' solutions that a large number of students used CAS to solve each question.

In the examination, Ken's and Lucy's students used the same type of CAS calculator while Meg's students used a different brand of calculator. For this paper Ken's and Lucy's students are considered as one cohort as it was not possible to separate the examination scripts into two classes based on the information supplied to the research team. Even though there is a difference in the extent of words and new non-standard notation used by Ken and Lucy, the amount of mathematical notation is relatively high for both of them individually and also when they are considered together.

… According to Fitts' Law, for a fixed distance traveled by the mouse, the time taken, in seconds, is given by $a - b\log_e(x)$, $0 < x \leq 5$, where x cm is the button width and a and b are positive constants for a particular user… 1. b. Mickey decides to find the values of a and b for his use. He finds that when x is 1, his time is 0.5 seconds and when x is 1.5, his time is 0.3 seconds. Find the exact values of a and b for Mickey.	3.a.i. The polynomial $2x^4 - x^3 - 5x^2 + 3x$ can be factorised as $x(2x-3)(ax^2+bx+c)$. Find the values of a, b and c.
1. c. i. Solve the equation $k\left(1.1 - 0.5\log_e(x)\right) = T$ for x, where k and T are positive real numbers	Students were shown the graph of $y = \frac{1}{2}\left(2x^4 - x^3 - 5x^2 + 3x\right)$ and a normal to the curve at $x=1$. 3.b.i. Show that the equation of this normal is $y = x - 1.5$

Figure 3: From VCE Mathematical Methods (CAS) Pilot Study Examination 2, Questions 1 & 3 (abbreviated)

Table 2 allows for comparison of the practice used by students in the examination for selected questions to the practice being demonstrated by the teachers in class. The 'percentage of worked examples for each style' combined value for Ken and Lucy was found by averaging their individual percentages for each style. The two classes had approximately the same number of students, so for this group of students approximately half had been taught by each teacher and hence had observed their particular practice for recording. Table 2 shows that students are developing new practice for recording irrespective of the fact that their teachers mainly used standard mathematical notation in the examples demonstrated in class. The extent of use of standard notation (without words) by students was much lower than the use by their teachers. The main difference between the student and teacher written records appeared to be in the use of a combination of mathematical notation and words and there was some new notation that was appearing, mainly in the form of calculator syntax for solving. There are a number of factors that could contribute to students' use of words in their written solutions. It could be that they equate words with processes they assign to CAS or it could be that because they are using words for CAS inputs. For both of the CAS calculators used by these students the syntax for the questions considered required words to indicate the input, for example, to solve question 1c (Figure 3) the following syntax could be used: solve(k(1.1− 0.5loge(x))=T,x).

Students may be more prone to thinking about what they are doing eg solve or substitute (as they consider what technology feature may be appropriate to use) rather than carrying out the computation by hand and as a result they may equate the mathematical process with the calculator input. Students are developing a practice where they write solutions using a combination of mathematical notation, words and calculator syntax. Often what appeared to be calculator syntax was a description of what they had input into the calculator, rather than an accurate description of the syntax that was required for the particular CAS being used. For example a student who recorded "solve $f(1)=0.5$ and $f(1.5)=0.3$ for a and b" which described how they could use their CAS to solve this problem did not write the specific calculator input that would be required to find a and b. In some ways students who write this are giving their plan for solving. This practice was different overall to the practice modelled by the teachers in class.

Table 2

Comparison of use of styles by teachers and students.

	Percentage of worked examples of each style		Percentage of examination questions of each style	
	Meg	Ken and Lucy	Meg's students' scripts (N=13)	Ken's and Lucy's students' scripts (N=35)
M+W′	90	81	46	40
M+W	3	10	44	41
M′+W	3	4	4	12
M′+W′	5	5	0	1
No response			6	6

N=number of examination scripts

IMPLICATIONS FOR TEACHERS' PRACTICE

This paper has considered the practice of writing solutions in a CAS classroom. It was found that there was a disparity between the practice of the three teachers observed and their students in terms of ways to communicate mathematical thinking in written solutions. It appears that a number of students are developing their own practice for recording, often independent of the models for written records presented by their teachers. But this does not mean that students have not demonstrated the capacity to clearly communicate their thinking. For the examination questions considered the students seemed able to document their solutions clearly when they used CAS.

There are a number of questions that arise as a result of the findings. For example, what purpose do students see for the worked examples that teachers write on the board? Do students see worked examples as 'teacher work', done in class but not necessarily impacting on what they are expected to do when solving problems? Do they even view worked examples as models for writing solutions, or only as examples of methods for solving or illustrations of mathematical concepts? In considering the marked differences between the written records demonstrated in class and the practice demonstrated by the students it would seem that students might not be viewing teacher worked examples as models of written records.

Another issue to be considered is the type of written record being demonstrated. Teachers on the whole did not record what they did with their CAS when solving problems, whereas students clearly did. This is evident through the amount of mathematical notation and words (M+W) students used to describe the procedures they inputted into their CAS. As teachers favoured the use of standard mathematical notation in recording did students equate this with use of by-hand techniques for solving? Students' interpretation might be that when problems are being solved by-hand they should follow the examples of how to write solutions demonstrated by their teacher, however when using CAS, given that models of written records weren't demonstrated they should find their own way to communicate their thinking.

One response to the obvious differences between teacher and student written records could be for teachers to think that the model for written records that they provide is irrelevant, particularly if students are able to communicate their solutions clearly. An alternative response would be to consider how to include a balance between written records involving by-hand techniques and those acknowledging CAS use. This would bring teacher written records more in line with observed student practice and hence more able to lead it. Teachers need to help students develop good practice in using a combination of words, standard mathematical notation and possibly even new notation in their solutions by discussing when each is more appropriate. Teachers could give models of written records for by-hand and CAS solutions, highlighting differences of each, but emphasising the key elements of a good solution in both cases. This would enable students to reflect on written records where steps carried out using CAS are evident. Students need guidance related to what an acceptable solution may be, particularly in the case where they might be using CAS and some procedures are assigned to the CAS.

The emphasis for written records must remain on clear communication of mathematical thinking regardless of the methods used to solve. If students are going to develop a new practice for recording purely because they have a CAS then teachers should incorporate written records acknowledging CAS use into their classroom worked examples.

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