

On some nonlinear partial differential equations involving the 1-Laplacian

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Abstract

Let Ω be a smooth bounded domain in \mathbb{R}^N , $N > 1$ and let $n \in \mathbb{N}^*$. We prove here the existence of nonnegative solutions u_n in $BV(\Omega)$, to the problem

$$(P_n) \begin{cases} -\operatorname{div} \sigma + 2n \left(\int_{\Omega} u - 1 \right) \operatorname{sign}^+(u) = 0 & \text{in } \Omega, \\ \sigma \cdot \nabla u = |\nabla u| & \text{in } \Omega, \\ u \text{ is not identically zero, } -\sigma \cdot \vec{n} u = u & \text{on } \partial\Omega, \end{cases}$$

where \vec{n} denotes the unit outer normal to $\partial\Omega$, and $\operatorname{sign}^+(u)$ denotes some $L^\infty(\Omega)$ function defined as:

$$\operatorname{sign}^+(u).u = u^+, \quad 0 \leq \operatorname{sign}^+(u) \leq 1.$$

Moreover, we prove the tight convergence of u_n towards one of the first eigenfunctions for the first 1-Laplacian Operator $-\Delta_1$ on Ω when n goes to $+\infty$.

Key words and phrases: BV functions, 1-Laplacian Operator.

Résumé

Soit Ω un domaine borné et lisse dans \mathbb{R}^N , $N > 1$ et soit $n \in \mathbb{N}^*$. On montre dans ce papier l'existence de solutions positives u_n dans $BV(\Omega)$, au problème

$$(P_n) \begin{cases} -\operatorname{div} \sigma + 2n \left(\int_{\Omega} u - 1 \right) \operatorname{sign}^+(u) = 0 & \text{in } \Omega, \\ \sigma \cdot \nabla u = |\nabla u| & \text{in } \Omega, \\ u \text{ is not identically zero, } -\sigma \cdot \vec{n} u = u & \text{on } \partial\Omega, \end{cases}$$

où \vec{n} est le vecteur normal sortant de $\partial\Omega$, et $\operatorname{sign}^+(u)$ est une fonction dans $L^\infty(\Omega)$ définie par:

$$\operatorname{sign}^+(u).u = u^+, \quad 0 \leq \operatorname{sign}^+(u) \leq 1.$$

De plus, on montre la convergence de u_n vers une des valeurs premières de l'opérateur 1-Laplacien $-\Delta_1$ sur Ω quand n tend vers $+\infty$.

Mots clés: Fonctions BV, 1-Laplacien.

1 Introduction

Recent works about the operator 1-Laplacien revealed the existence of a least eigenvalue for this operator on a bounded smooth set. More precisely this first eigenvalue is well defined as the infimum of

$$\inf_{u \in W_0^{1,1}(\Omega), \int_{\Omega} |u|=1} \int_{\Omega} |\nabla u|.$$

This value denoted as λ_1 is positive by Poincaré's inequality. Unfortunately, since $W^{1,1}(\Omega)$ is not a reflexive space, it is not possible to prove the existence of solutions in $W_0^{1,1}(\Omega)$. The convenient space in which one must look for a minimizer is the space $BV(\Omega)$ which is the weak closure of $W^{1,1}(\Omega)$. Moreover, since the trace map which is well defined on $BV(\Omega)$ is not weakly continuous, one is led to replace the problem by the relaxed following form

$$\inf_{u \in BV(\Omega), \int_{\Omega} |u|=1} \int |\nabla u| + \int_{\partial\Omega} |u|.$$

This problem has an infimum equal to λ_1 . Classical arguments in the theory of BV functions allow then to prove the existence of a minimizer. Moreover using either the fact that there exist non negative solutions and duality in convex analysis, or using an approximation with the more regular problem

$$\inf_{u \in W_0^{1,1+\varepsilon}(\Omega), \int_{\Omega} |u|=1} \int_{\Omega} |\nabla u|^{1+\varepsilon},$$

one can obtain that the solution u satisfies the singular PDE

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda_1,$$

in a sense which must of course be precised, and is detailed in the present paper. Let us note that it is proved in [9] that there are characteristic functions of sets which are solutions, they are consequently called eigensets. Another approach is used in [1] [2], where the authors use the concept of Cheeger sets. In these papers, the authors present a remarkable construction of eigenset in the case $N = 2$ and for convex sets Ω . Among their results there is the uniqueness of eigen sets in this case. Our aim in the present paper is to propose an approach of the first eigenvalue and first eigenfunction, using a penalization method. This method has an obvious numerical advantage : The constraint $\int_{\Omega} |u| = 1$ has a higher cost than the introduction of the penalization term $n(\int_{\Omega} |u| - 1)^2$. In the same time one gets a new proof of the existence of nonnegative function.

2 Preliminaries

Let Ω be a smooth bounded domain in \mathbb{R}^N , $N > 1$, whose boundary is piecewise \mathcal{C}^1 , and let us define for all $n \in \mathbb{N}^*$, the following functional

$$I_{n,0}(u) = \int_{\Omega} |\nabla u| + n \left(\int_{\Omega} |u| - 1 \right)^2 .$$

It is clear, using Poincaré's inequality, that there exists some constant $c > 0$, such that for all $u \in W_0^{1,1}(\Omega)$ and for all $n \in \mathbb{N}^*$,

$$\int_{\Omega} |\nabla u| + n \left(\int_{\Omega} |u| - 1 \right)^2 \geq c \|u\|_{W^{1,1}(\Omega)}. \quad (2.1)$$

We look for $u \in W_0^{1,1}(\Omega)$, nonnegative which satisfies:

$$\begin{cases} -\operatorname{div} \sigma + 2n \left(\int_{\Omega} u - 1 \right) \operatorname{sign}^+(u) = 0 & \text{in } \Omega, \\ \sigma \in L^\infty(\Omega, \mathbb{R}^n), \\ \sigma \cdot \nabla u = |\nabla u| & \text{in } \Omega, \\ u \text{ is not identically zero, } u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

In order to find solutions to (2.2) one can consider the following minimisation problem

$$\inf_{u \in W_0^{1,1}(\Omega)} \left\{ \int_{\Omega} |\nabla u| + n \left(\int_{\Omega} |u| - 1 \right)^2 \right\}, \quad (2.3)$$

We denote by $\lambda_n(\Omega)$ the value of this infimum. In the following, we shall prove that, if $u_n \in W_0^{1,1}(\Omega)$ realizes the minimum defined in (2.3), it is a non trivial solution of (2.2).

Since classical methods in the calculus of variations cannot be applied to solve (2.3), we approximate it by the following formulation: for $\varepsilon > 0$, we define

$$\lambda_{n,\varepsilon}(\Omega) = \inf_{u \in W_0^{1,1+\varepsilon}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^{1+\varepsilon} + n \left(\int_{\Omega} |u|^{1+\varepsilon} - 1 \right)^2 \right\}.$$

Note that a non trivial, nonnegative minimizer $u_{n,\varepsilon}$ for this problem solves the following partial differential equation:

$$\begin{cases} -\operatorname{div}(|\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}) + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) u_{n,\varepsilon}^\varepsilon = 0 & \text{in } \Omega, \\ u_{n,\varepsilon} \in W_0^{1,1+\varepsilon}(\Omega). \end{cases}$$

A solution of (2.2) will be obtained, letting ε tend to 0. Let us observe that, in particular, regularity and other properties of u_n can be derived from a priori estimates on $u_{n,\varepsilon}$. Of course, passing to the limit when $\varepsilon \rightarrow 0$ will lead us to consider $BV(\Omega)$ in place of $W^{1,1}(\Omega)$, and to give sense to some expressions as $\sigma_n \cdot \nabla u_n$ when ∇u_n is only a measure, and $\sigma_n \in L^\infty(\Omega)$, $\operatorname{div} \sigma_n \in L^N(\Omega)$.

As the “limit” will be obtained by weak convergence in $BV(\Omega)$, we shall be led to overcome the lack of continuity of the trace map of this space for weak topology. This can be done by introducing the concept of “relaxed problem”: these problems are used in the theory of minimal surfaces and plasticity, and with a slightly different meaning, in the theory of weakly harmonic functions. Here the relaxed problem is defined as:

$$\inf_{u \in BV(\Omega)} \left\{ \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| + n \left(\int_{\Omega} |u| - 1 \right)^2 \right\}. \quad (2.4)$$

We shall prove in the sequel that (2.4) has the same infimum as (2.3) and that it possesses a solution u in $BV(\Omega)$ which satisfies an equation as (2.2), extended to BV -functions, as it is done in [12]. Of course, to prove this, and as we pointed out before, one must give sense to the product “ $\sigma \cdot \nabla u$ ” when ∇u is only a measure.

Proposition 1. *Let σ be in $L^\infty(\Omega)$ and $\operatorname{div} \sigma \in L^N(\Omega)$, and define the distribution $\sigma \cdot \nabla u$ by the formula : For $\varphi \in \mathcal{D}(\Omega)$, $u \in BV(\Omega)$,*

$$\langle \sigma \cdot \nabla u, \varphi \rangle = - \int_{\Omega} \operatorname{div} \sigma u \varphi - \int_{\Omega} \sigma \cdot \nabla \varphi u. \quad (2.5)$$

Then

$$|\langle \sigma \cdot \nabla u, \varphi \rangle| \leq |\sigma|_\infty \langle |\nabla u|, |\varphi| \rangle.$$

In particular, $\sigma \cdot \nabla u$ is a bounded measure on Ω which is absolutely continuous with respect to $|\nabla u|$. In addition, if $\varphi \in \mathcal{C}(\overline{\Omega}) \cap C^1(\Omega)$, the following Green’s Formula holds

$$\langle \sigma \cdot \nabla u, \varphi \rangle = - \int_{\Omega} \operatorname{div} \sigma u \varphi - \int_{\Omega} \sigma \cdot \nabla \varphi u + \int_{\partial\Omega} \sigma \cdot \vec{n} u \varphi. \quad (2.6)$$

Suppose that $U \in BV(\mathbb{R}^N - \overline{\Omega})$, and define for $u \in BV(\Omega)$ the functional \tilde{u} as:

$$\tilde{u} = \begin{cases} u & \text{in } \Omega, \\ U & \text{in } \mathbb{R}^N - \overline{\Omega}, \end{cases}$$

then $\tilde{u} \in BV(\mathbb{R}^N)$ and

$$\nabla \tilde{u} = \nabla u \chi_{\Omega} + \nabla U \chi_{(\mathbb{R}^N - \overline{\Omega})} + (U - u)_{|\Omega} \delta_{\partial\Omega},$$

where $U_{|\Omega}$ and $u_{|\Omega}$ denote the trace of U and u on $\partial\Omega$, $\delta_{\partial\Omega}$ denotes the uniform Dirac measure on $\partial\Omega$ and \vec{n} is the unit outer normal to $\partial\Omega$. Finally, we introduce the measure $\sigma \cdot \nabla \tilde{u}$ on $\overline{\Omega}$ by the formula

$$(\sigma \cdot \nabla \tilde{u}) = (\sigma \cdot \nabla u) \chi_{\Omega} + \sigma \cdot \vec{n} (U - u) \delta_{\partial\Omega}.$$

Then $\sigma \cdot \nabla \tilde{u}$ is absolutely continuous with respect to $|\nabla \tilde{u}|$, with the inequality

$$|\sigma \cdot \nabla \tilde{u}| \leq |\sigma|_\infty |\nabla \tilde{u}|.$$

For a proof the reader can consult ([8],[17],[19]).

3 Existence's Theorem

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^N, N > 1$, which is a piece-wise \mathcal{C}^1 set. Let λ_n be defined as in (2.3). There exists a nonnegative solution $u_n \in BV(\Omega)$ to the problem (2.2) which realizes the following partial differential equation:*

$$\begin{cases} -\operatorname{div}\sigma_n + 2n \left(\int_{\Omega} u_n - 1 \right) \operatorname{sign}^+(u_n) = 0 & \text{in } \Omega, \\ \sigma_n \in L^\infty(\Omega, \mathbb{R}^N), |\sigma_n|_\infty \leq 1, \\ \sigma_n \cdot \nabla u_n = |\nabla u_n| & \text{in } \Omega, \\ u \text{ is not identically zero, } -\sigma_n \cdot \vec{n}(u_n) = u_n & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

\vec{n} denotes the unit outer normal to $\partial\Omega$, and $\sigma_n \cdot \nabla u_n$ is the measure defined in proposition 1.

Remark 1. From proposition 1 (with $U = 0$) the conditions:

$$\sigma_n \cdot \nabla u_n = |\nabla u_n| \quad \text{in } \Omega, \quad -\sigma_n \cdot \vec{n}(u_n) = u_n \quad \text{on } \partial\Omega$$

are equivalent to

$$\sigma \cdot \nabla \tilde{u} = |\nabla \tilde{u}| \quad \text{on } \bar{\Omega}$$

Remark 2. Eq (3.1) can be written:

$$\operatorname{div}\sigma_n = -2n \left(\int_{\Omega} u_n - 1 \right) \operatorname{sign}^+(u_n)$$

and since the right-hand side is an element of $L^\infty(\Omega)$, $\sigma_n \cdot \nabla u_n$ is well-defined.

Proof of Theorem 1. We split the proof of Theorem 1 into several steps:

Step 1:

We begin by approximating (2.4) with the following minimization problem

$$\lambda_{n,\varepsilon}(\Omega) = \inf_{u \in W_0^{1,1+\varepsilon}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^{1+\varepsilon} + n \left(\int_{\Omega} |u|^{1+\varepsilon} - 1 \right)^2 \right\}, \quad (3.2)$$

where ε is some positive parameter. This problem can be solved by classical methods in the calculus of variations, since the compact embedding of $W_0^{1,1+\varepsilon}(\Omega)$ into $L^q(\Omega)$ holds for all $q < (1 + \varepsilon)^*$. Furthermore, there exists a nonnegative solution to the problem (3.2), since if $u_{n,\varepsilon} \in W_0^{1,1+\varepsilon}(\Omega)$, so is $|u_{n,\varepsilon}|$ and

$$|\nabla |u_{n,\varepsilon}|| = |\nabla u_{n,\varepsilon}|.$$

By regularity results, as developed by Guedda-Veron [16], (see also Tolksdorf [20]), one gets that $u_{n,\varepsilon} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}), \alpha \in (0, 1)$ and by Vazquez' Strict Maximum Principle [21], one gets that $u_{n,\varepsilon} > 0$ in Ω . Let $u_{n,\varepsilon}$ be a solution of (3.2) which is

positive, then $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$ satisfies the following partial differential equations:

$$\begin{cases} -\operatorname{div} \sigma_{n,\varepsilon} + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) u_{n,\varepsilon}^{\varepsilon} = 0 & \text{in } \Omega, \\ \sigma_{n,\varepsilon} \cdot \nabla u_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{1+\varepsilon} & \text{in } \Omega, \\ u_{n,\varepsilon} > 0, \quad u_{n,\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.3)$$

Remark 3. The solution $u_{n,\varepsilon}$ is unique.

Indeed, let u and v two positive solutions of (3.3), then we have:

$$-\operatorname{div} \sigma_{\varepsilon}(u) + 2n \left(\int_{\Omega} u^{1+\varepsilon} - 1 \right) u^{\varepsilon} = 0. \quad (3.4)$$

$$-\operatorname{div} \sigma_{\varepsilon}(v) + 2n \left(\int_{\Omega} v^{1+\varepsilon} - 1 \right) v^{\varepsilon} = 0. \quad (3.5)$$

Let us note:

$$\alpha(u) = \int_{\Omega} u^{1+\varepsilon} - 1.$$

Subtracting (3.5) from (3.4), one gets:

$$-\operatorname{div} (\sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(v)) + 2n (\alpha(u) - \alpha(v)) u^{\varepsilon} + 2n \alpha(v) (u^{\varepsilon} - v^{\varepsilon}) = 0. \quad (3.6)$$

Case 1: $\|u\|_{1+\varepsilon}^{1+\varepsilon} = \|v\|_{1+\varepsilon}^{1+\varepsilon}$.

Let us multiply (3.6) by $(u - v)$ then integrate, we get that:

$$\int_{\Omega} (\sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(v)) \cdot \nabla (u - v) + 2n \alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v) = 0.$$

We know that

$$\int_{\Omega} (\sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(v)) \cdot \nabla (u - v) \geq 0. \quad (3.7)$$

On the other hand it is clear that

$$\int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v) \geq 0. \quad (3.8)$$

So, we can conclude that $\int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v) = 0$, and this implies that $u = v$ a.e.

Case 2: $\|u\|_{1+\varepsilon}^{1+\varepsilon} \geq \|v\|_{1+\varepsilon}^{1+\varepsilon}$.

Let us multiply (3.6) by $(u - v)^+$ then integrate. It is clear that

$$2n (\alpha(u) - \alpha(v)) \int_{\Omega} u^{\varepsilon} (u - v)^+ \geq 0.$$

So we get that:

$$\int_{\Omega} (\sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(v)) \cdot \nabla(u - v)^+ + 2n \alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v)^+ \leq 0. \quad (3.9)$$

By (3.7) and (3.8), we have that:

$$\int_{\Omega} (\sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(v)) \cdot \nabla(u - v)^+ + 2n \alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v)^+ \geq 0. \quad (3.10)$$

So from (3.9) and (3.10), we obtain that

$$\int_{\Omega} (\sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(v)) \cdot \nabla(u - v)^+ + 2n \alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v)^+ = 0.$$

Then we can conclude that $\int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v)^+ = 0$, and this implies that $(u - v)^+ = 0$, which gives us that $u \leq v$.

Recall that $\|u\|_{1+\varepsilon}^{1+\varepsilon} \geq \|v\|_{1+\varepsilon}^{1+\varepsilon}$, hence $u = v$ a.e.

Case 3: $\|u\|_{1+\varepsilon}^{1+\varepsilon} \leq \|v\|_{1+\varepsilon}^{1+\varepsilon}$.

We use the same arguments then the Case 2, replacing $(u - v)^+$ by $(v - u)^+$.

Step 2:

Proposition 2.

$$\overline{\lim}_{\varepsilon \rightarrow 0} \lambda_{n,\varepsilon}(\Omega) \leq \lambda_n(\Omega).$$

Proof. Let $I_{n,\varepsilon}(\varphi) = \int_{\Omega} |\nabla\varphi|^{1+\varepsilon} + n \left(\int_{\Omega} |\varphi|^{1+\varepsilon} - 1 \right)^2$, and $\delta > 0$ be given and $\varphi \in \mathcal{D}(\Omega)$ such that

$$\int_{\Omega} |\nabla\varphi| + n \left(\int_{\Omega} |\varphi| - 1 \right)^2 \leq \lambda_n + \delta,$$

For ε close to 0, $|I_{n,\varepsilon}(\varphi) - I_{n,0}(\varphi)| < \delta$, hence

$$\overline{\lim}_{\varepsilon \rightarrow 0} \lambda_{n,\varepsilon} \leq \lambda_n + \delta,$$

δ being arbitrary, we get $\overline{\lim}_{\varepsilon \rightarrow 0} \lambda_{n,\varepsilon} \leq \lambda_n$. □

Let now $u_{n,\varepsilon}$ be a positive solution of (3.3). Then, it is bounded in $W_0^{1,1+\varepsilon}(\Omega)$.

Using Hölder's inequality, we get

$$\int_{\Omega} u_{n,\varepsilon} dx \leq \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}}, \quad (3.11)$$

and thus $(u_{n,\varepsilon})_{\varepsilon>0}$ is bounded in $L^1(\Omega)$. By the same arguments we prove that $(|\nabla u_{n,\varepsilon}|)_{\varepsilon>0}$ is also bounded in $L^1(\Omega)$. Hence, $(u_{n,\varepsilon})_{\varepsilon>0}$ is bounded in $BV(\Omega)$. Therefore, we may extract from it a subsequence, still denoted $(u_{n,\varepsilon})$, such that

$$u_{n,\varepsilon} \rightarrow u_n \quad \text{in } L^k(\Omega), \forall k < 1^*,$$

$$\nabla u_{n,\varepsilon} \rightharpoonup \nabla u_n \quad \text{in } M^1(\Omega) \quad \text{weakly,}$$

We need now to recall a result of concentration compactness, which is a consequence of the concentration compactness theory of P.L.Lions [18].

Lemma 1. *Suppose that Ω is an open bounded set in \mathbb{R}^N , $N > 1$, and that $u_{n,\varepsilon}$ is bounded in $W_0^{1,1+\varepsilon}(\Omega)$, then if $u_{n,\varepsilon} \rightharpoonup u_n \in BV(\Omega)$ weakly, there exists some nonnegative bounded measure μ on Ω , a numerable set $\{x_i\}_{i \in \mathbb{N}} \in \overline{\Omega}$, and some numbers $\mu_i \geq 0$ such that*

$$|\nabla u_{n,\varepsilon}| \rightharpoonup \mu \geq |\nabla u_n| + \sum_i \mu_i \delta_{x_i} \quad \text{in } M^1(\Omega) \quad \text{weakly,} \quad (3.12)$$

where δ_{x_i} denotes the Dirac mass on x_i .

Step 3:

we obtain $\sigma_n = \frac{\nabla u_n}{|\nabla u_n|}$ as the weak limit of $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$.

Let $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$. Then $\sigma_{n,\varepsilon}$ belongs to $L^{\frac{1+\varepsilon}{\varepsilon}}(\Omega)$. By passing to the limit when ε goes to 0, one obtains that $\sigma_{n,\varepsilon}$ tends to σ_n weakly in $L^q(\Omega)$, for all $q < \infty$. We need to prove that $|\sigma_n|_\infty \leq 1$.

For that aim, let η be in $\mathcal{D}(\Omega, \mathbb{R}^N)$. Then

$$\begin{aligned} \left| \int_\Omega \sigma_n \cdot \eta \right| &\leq \liminf_{\varepsilon \rightarrow 0} \left| \int_\Omega \sigma_{n,\varepsilon} \cdot \eta \right| \leq \liminf_{\varepsilon \rightarrow 0} \int_\Omega |\nabla u_{n,\varepsilon}|^\varepsilon |\eta| \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left(\int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_\Omega |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \liminf_{\varepsilon \rightarrow 0} (C)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_\Omega |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \int_\Omega |\eta|. \end{aligned}$$

On the other hand, we prove that u_n^ε converges weakly to some α in $L^\infty(\Omega)$, $\alpha \in [0, 1]$. Indeed, for $\eta \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \left| \int_\Omega u_n^\varepsilon \cdot \eta \right| &\leq \left(\int_\Omega |u_n^\varepsilon|^{\frac{1+\varepsilon}{\varepsilon}} \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_\Omega |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \liminf_{\varepsilon \rightarrow 0} (C)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_\Omega |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \int_\Omega |\eta|. \end{aligned}$$

This implies that $|\sigma_n|_\infty \leq 1$. On the other hand by passing to the limit in (3.3), one gets:

$$-\operatorname{div}\sigma_n + 2n \left(\int_{\Omega} u_n - 1 \right) \alpha = 0. \quad (3.13)$$

Step 4:

Extension of $u_{n,\varepsilon}$ outside Ω and convergence towards a solution of (3.3).

We shall need in this part the Proposition 1 and a classical result in the theory of BV -functions:

Lemma 2. *Assume that Ω is an open bounded set in \mathbb{R}^N , $N > 1$, and that $u \in BV(\Omega)$. Then if $x_0 \in \Omega$, $|\sigma \cdot \nabla u|(\{x_0\}) = 0$.*

The proof of this result can be found in [14].

Let $\tilde{u}_{n,\varepsilon}$ be the extension of $u_{n,\varepsilon}$ by 0 in $\mathbb{R}^N - \overline{\Omega}$. Then $\tilde{u}_{n,\varepsilon} \in W^{1,1+\varepsilon}(\mathbb{R}^N)$, since $u_{n,\varepsilon} = 0$ on $\partial\Omega$, and $(\tilde{u}_{n,\varepsilon})$ is bounded in $BV(\mathbb{R}^N)$. Then one may extract from it a subsequence, still denoted $(\tilde{u}_{n,\varepsilon})$ such that

$$\tilde{u}_{n,\varepsilon} \rightarrow v_n \quad \text{in } L^k(\mathbb{R}^N), \forall k < \frac{N}{N-1},$$

with $v_n = 0$ outside of $\overline{\Omega}$. We denote by u_n the restriction of v_n to Ω . In addition:

$$\nabla \tilde{u}_{n,\varepsilon} \rightharpoonup \nabla v_n \quad \text{in } M^1(\mathbb{R}^N) \quad \text{weakly,}$$

$$\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon} \rightharpoonup \sigma_n \quad \text{in } L^q(\Omega), \forall q < \infty.$$

Using Concentration Compactness Lemma, there exists a non negative measure μ , with support in $\overline{\Omega}$, a numerable set $\{x_i\}_i$ in $\overline{\Omega}$ and some non negative reals μ_i , such that

$$|\nabla \tilde{u}_{n,\varepsilon}|^{1+\varepsilon} \rightharpoonup \mu \geq |\nabla v_n| + \sum_i \mu_i \delta_{x_i}.$$

Multiplying (3.3) by $\tilde{u}_{n,\varepsilon} \varphi$ where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, and integrating by parts, one obtains:

$$\int_{\overline{\Omega}} \sigma_{n,\varepsilon} \cdot \nabla(\tilde{u}_{n,\varepsilon} \varphi) + 2n \left(\int_{\Omega} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1 \right) \int_{\Omega} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} \varphi = 0,$$

or equivalently

$$\int_{\mathbb{R}^N} |\nabla(\tilde{u}_{n,\varepsilon})|^{1+\varepsilon} \varphi + \int_{\mathbb{R}^N} \sigma_{n,\varepsilon} \tilde{u}_{n,\varepsilon} \cdot \nabla \varphi + 2n \left(\int_{\mathbb{R}^N} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1 \right) \int_{\mathbb{R}^N} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} \varphi = 0.$$

Since $\sigma_{n,\varepsilon} \rightharpoonup \sigma_n$ in $L^q(\Omega)$ for all q , in particular for some $\alpha > 0$, $\sigma_{n,\varepsilon}$ tends weakly towards σ_n in $L^{N+\alpha}(\Omega)$, and then, since $\tilde{u}_{n,\varepsilon}$ tends strongly towards v_n in L^k , $k < \frac{N}{N-1}$, one obtains that:

$$\int_{\mathbb{R}^N} \sigma_{n,\varepsilon} \tilde{u}_{n,\varepsilon} \cdot \nabla \varphi \longrightarrow \int_{\mathbb{R}^N} \sigma_n v_n \cdot \nabla \varphi, \quad \text{when } \varepsilon \rightarrow 0.$$

By passing to the limit in the last equation above, one obtains:

$$\langle \mu, \varphi \rangle + \int_{\Omega} \sigma_n u_n \cdot \nabla \varphi + 2n \left(\int_{\Omega} u_n - 1 \right) \int_{\Omega} u_n \varphi = 0. \quad (3.14)$$

Using generalised Green's Formula in Proposition 1 and (3.13), we have

$$\begin{aligned} \int_{\Omega} \sigma_n u_n \cdot \nabla \varphi &= - \int_{\Omega} \operatorname{div} \sigma_n u_n \varphi - \int_{\Omega} \sigma_n \cdot \nabla u_n \varphi + \int_{\partial\Omega} \sigma_n \cdot \vec{n} u_n \varphi \\ &= -2n \left(\int_{\Omega} u_n - 1 \right) \int_{\Omega} u_n \varphi - \int_{\overline{\Omega}} \sigma_n \cdot \nabla v_n \varphi. \end{aligned} \quad (3.15)$$

Subtracting (3.15) from (3.14), one gets for $\varphi \in \mathcal{D}(\overline{\Omega})$

$$\langle \mu, \varphi \rangle - \int_{\overline{\Omega}} \sigma_n \cdot \nabla v_n \varphi = 0. \quad (3.16)$$

Let now h be a $|\nabla v_n|$ -measurable function and μ^\perp be a measure orthogonal to $|\nabla v_n|$, such that, according to the Radon-Nikodym decomposition, one has

$$\mu = h|\nabla v_n| + \mu^\perp. \quad (3.17)$$

By Lemma 2 and the analogous of (3.12) of Lemma 1, one has

$$h|\nabla v_n| \geq |\nabla v_n|, \quad (3.18)$$

and

$$\mu^\perp \geq \sum_i \mu_i \delta_{x_i}. \quad (3.19)$$

Using (3.18) and (3.19) in equation (3.16) one gets that

$$h|\nabla v_n| = \sigma_n \cdot \nabla v_n \quad \text{on } \overline{\Omega}, \quad (3.20)$$

and

$$\mu^\perp = \sum_i \mu_i \delta_{x_i} \leq \sigma_n \cdot \nabla v_n. \quad (3.21)$$

Using Lemma 2 one gets that $\mu_i = 0 \forall i$. And from (3.20), $|\sigma_n| \leq 1$ and Proposition 1, one obtains that in the sense of measures:

$$|\sigma_n \cdot \nabla v_n| \leq |\nabla v_n| \quad \text{on } \overline{\Omega},$$

$$\mu = \sigma_n \cdot \nabla v_n,$$

and then

$$\sigma_n \cdot \nabla v_n = |\nabla v_n| \quad \text{on } \overline{\Omega},$$

and $h = 1$, $|\nabla v_n|$ -almost everywhere. Using this in equation (3.14) with $\varphi = 1$, we have:

$$\int_{\Omega} |\nabla v_n| + \int_{\partial\Omega} v_n + 2n \left(\int_{\Omega} v_n - 1 \right) \int_{\Omega} v_n = 0.$$

Recalling that we have from Proposition 1 that:

$$\nabla v_n = \nabla v_n \chi_{\Omega} - v_n \delta_{\partial\Omega} \vec{n},$$

$$\sigma_n \cdot \nabla v_n = \sigma_n \cdot \nabla v_n \chi_{\Omega} - \sigma_n \cdot \vec{n} v_n \delta_{\partial\Omega}.$$

This implies that $\sigma_n \cdot \nabla v_n = |\nabla v_n|$, on $\Omega \cup \partial\Omega$. This condition can be splitted in the two equations

$$\begin{cases} \sigma_n \cdot \nabla u_n = |\nabla u_n| & \text{in } \Omega, \\ \sigma_n \cdot \vec{n} u_n = -u_n & \text{on } \partial\Omega. \end{cases}$$

Then u_n is a nonnegative solution of (2.2). Moreover, the convergence of $|\nabla \tilde{u}_{n,\varepsilon}|$ is tight on $\overline{\Omega}$ which means that

$$\int_{\Omega} |\nabla u_{n,\varepsilon}| \longrightarrow \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n, \quad \text{when } \varepsilon \rightarrow 0.$$

Indeed, one has $\int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \rightarrow \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n$ and using the lower semi-continuity for the extension $v_{n,\varepsilon}$, we get first

$$\int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n \leq \underline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{n,\varepsilon}|,$$

and secondly, using Hölder's inequality $\int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \geq (\int_{\Omega} |\nabla u_{n,\varepsilon}|)^{\frac{1}{1+\varepsilon}} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}}$ which gives by passing to the limit the reverse inequality.

Step 5: u_n is a solution of (2.4)

Let us recall the relaxed form of (2.3)

$$\inf_{u \in BV(\Omega)} \left\{ \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| + n \left(\int_{\Omega} |u| - 1 \right)^2 \right\}. \quad (1.4)$$

We prove now that the solution u_n obtained in the two previous steps is a nonnegative solution of (2.4). For that aim, let us recall that by using the lower semi-continuity for the weak topology of $BV(R^N)$, we have:

$$\begin{aligned}
\lambda_n(\Omega) &= \int_{\mathbb{R}^n} |\nabla v_n| + n \left(\int_{\Omega} v_n - 1 \right)^2 \\
&\leq \underline{\lim}_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} |\nabla \tilde{u}_{n,\varepsilon}|^{1+\varepsilon} + n \left(\int_{\Omega} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1 \right)^2 \\
&\leq \overline{\lim}_{\varepsilon \rightarrow 0} \lambda_{n,\varepsilon}(\Omega) \leq \lambda_n(\Omega).
\end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}^n} |\nabla v_n| = \int_{\Omega \cup \partial\Omega} |\nabla v_n| = \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n.$$

One obtains that u_n is a nonnegative solution of the relaxed problem (2.4) and in the same time we get

$$\lim_{\varepsilon \rightarrow 0} \lambda_{n,\varepsilon} = \lambda_n.$$

Moreover, one has:

$$\int_{\Omega} |u_{n,\varepsilon}| \longrightarrow \int_{\Omega} u_n, \quad \text{when } \varepsilon \rightarrow 0.$$

Then, we conclude that:

$$\int_{\Omega} |\nabla u_{n,\varepsilon}| \longrightarrow \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n, \quad \text{when } \varepsilon \rightarrow 0.$$

Hence we get the tight convergence of $u_{n,\varepsilon}$ towards u_n in $BV(\overline{\Omega})$. □

4 Convergence result

We begin to recall some properties of the first eigenvalue for the 1-Laplacian operator, see e.g. [9]

Proposition 3. *Suppose that $\lambda > 0$ is such that there exists σ , $|\sigma|_{\infty} \leq 1$, and $u \geq 0$ in $BV(\Omega)$ with*

$$\begin{cases} -\operatorname{div} \sigma = \lambda, & u \geq 0, \quad u \not\equiv 0, \quad u \in BV(\Omega), \\ \sigma \cdot \nabla u = |\nabla u| & \text{in } \Omega, \quad |\sigma|_{L^{\infty}(\Omega)} \leq 1, \\ \sigma \cdot \vec{n}(-u) = u & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

Then $\lambda = \lambda_1$ where

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ \|u\|_1 = 1}} \int_{\Omega} |\nabla u|,$$

λ_1 is called the first eigenvalue for $-\Delta_1$ on Ω . Moreover

$$\lambda_1 = \inf_{\substack{u \in BV(\Omega) \\ \|u\|_1=1}} \left\{ \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| \right\}, \quad (4.2)$$

and this last infimum is achieved on some u which satisfies (4.1). Among the “eigenfunctions” there exist characteristic functions of Cacciopoli sets.

Theorem 2. *Let u_n be a function for which λ_n is achieved, then, up to a subsequence, (u_n) converges to $u \in BV(\Omega)$, $u \geq 0$, $u \not\equiv 0$, which realizes the minimum defined in (4.2). Moreover*

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda_1.$$

Proof of the Theorem 2. For λ_n and λ_1 defined as above, it is clear that we have:

$$\overline{\lim}_{n \rightarrow \infty} \lambda_n \leq \lambda_1. \quad (4.3)$$

Let $(u_n)_n$ be a sequence of solutions of the relaxed problem (2.4). We begin to prove that $(u_n)_n$ is bounded in $BV(\Omega)$. For that aim let us note that by (4.3), one gets that $n \left(\int_{\Omega} u_n - 1 \right)^2$ is bounded by λ_1 , which implies that $\lim_{n \rightarrow \infty} \left(\int_{\Omega} u_n - 1 \right)^2 = 0$. Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n = 1,$$

Hence, $(u_n)_n$ is bounded in $L^1(\Omega)$.

Using once more (4.3) for $|\nabla(u_n)|_n$ one can conclude that $(u_n)_n$ is bounded in $BV(\Omega)$. Then, the extension of u_n by zero outside of $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. One can then extract from it a subsequence, still denoted u_n , such that

$$u_n \rightharpoonup u \quad \text{in } BV(\mathbb{R}^N) \quad \text{weakly,}$$

Obviously $u = 0$ outside of $\overline{\Omega}$. By compactness of the Sobolev embedding from $BV(\Omega)$ into $L^1(\Omega)$, one has $|u|_{L^1(\Omega)} = 1$. Using lower semi continuity, one has

$$\begin{aligned} \lambda_1 &\leq \int_{\mathbb{R}^N} |\nabla u| \leq \int_{\mathbb{R}^N} |\nabla u| + n \left(\int_{\mathbb{R}^N} u - 1 \right)^2 \\ &\leq \underline{\lim}_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n| + n \left(\int_{\mathbb{R}^N} u_n - 1 \right)^2 \right) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \lambda_n \leq \lambda_1. \end{aligned}$$

Then one gets that

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda_1.$$

Since $u = 0$ outside of $\overline{\Omega}$, one has $\nabla u = \nabla u \chi_\Omega - u \cdot \vec{n} \delta_{\partial\Omega}$ on $\partial\Omega$, and then

$$\int_{\mathbb{R}^N} |\nabla u| = \int_{\Omega} |\nabla u| + \int_{\partial\Omega} u.$$

Moreover, one obtains that:

$$\lim_{n \rightarrow \infty} n \left(\int_{\Omega} u_n - 1 \right)^2 = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n| = \int_{\Omega} |\nabla u|.$$

Then, we get the tight convergence of u_n to u in $BV(\overline{\Omega})$.

Hence by passing to the limit in (3.13) when $n \rightarrow \infty$ one gets that:

$$-2n \left(\int_{\Omega} u_n - 1 \right) \longrightarrow \lambda_1, \quad \text{when } n \rightarrow \infty.$$

□

5 Minima as Cacciopoli sets

Let us introduce $\lambda_{0,n}$ as the value of the infimum

$$\lambda_{0,n} = \inf_{\substack{E, E \text{ is Cacciopoli set} \\ E \subset \subset \Omega}} \left\{ \int_{\Omega} |\nabla \chi_E| + n \left(\int_{\Omega} \chi_E - 1 \right)^2 \right\}, \quad (5.1)$$

(let us recall that a Cacciopoli set in Ω is merely a set whose characteristic function belongs to $BV(\Omega)$). We have the following.

Theorem 3. *One has*

$$\begin{aligned} \lambda_{0,n} &= \inf_{\substack{E, E \text{ is Cacciopoli set} \\ E \subset \subset \Omega}} \left\{ \int_{\Omega} |\nabla \chi_E| + n \left(\int_{\Omega} \chi_E - 1 \right)^2 \right\} \\ &= \inf_{\substack{E, E \text{ is Cacciopoli set} \\ E \text{ set in } \mathbb{R}^n}} \left\{ \int_{\Omega} |\nabla \chi_E| + \int_{\partial\Omega} \chi_E + n \left(\int_{\Omega} \chi_E - 1 \right)^2 \right\} \\ &= \inf_{\substack{E, E \text{ is Cacciopoli set} \\ E \text{ set in } \mathbb{R}^n}} P(E, \Omega) + |E \cap \partial\Omega| + n (|E \cap \Omega| - 1)^2 \end{aligned}$$

and

$$\lambda_{0,n} \geq \lambda_{1,n}.$$

Remark 4. $P(E, \Omega)$ is the perimeter of E in Ω , see ([6], [15]).

Proposition 4. *Let $\lambda_{0,n}$ be defined as in Theorem 3, then $\lambda_{0,n}$ is achieved.*

Proof. Let (E_i) be a subsequence of Cacciopoli sets, $E_i \subset\subset \Omega$ such that

$$\int_{\Omega} |\nabla \chi_{E_i}| + n \left(\int_{\Omega} \chi_{E_i} - 1 \right)^2 \rightarrow \lambda_{0,n}.$$

Then

$$\overline{\lim}_{i \rightarrow \infty} \left\{ \int_{\Omega} |\nabla \chi_{E_i}| + n \left(\int_{\Omega} \chi_{E_i} - 1 \right)^2 \right\} \leq \lambda_{0,n}.$$

It is clear that χ_{E_i} is bounded in $BV(\Omega)$ (same arguments as in Theorem 3). More precisely χ_{E_i} is bounded in $BV(\mathbb{R}^n)$.

Extracting from it a subsequence still denoted χ_{E_i} , one get that

$$\chi_{E_i} \rightharpoonup u \text{ in } BV(\mathbb{R}^n).$$

By construction $u = 0$ outside of $\overline{\Omega}$. Moreover one can assume that χ_{E_i} tends to u a.e, and then u can only takes the values 0 and 1. As a consequence u is the characteristic function of some set E .

By lower semicontinuity, one has that

$$\int_{\mathbb{R}^n} |\nabla \chi_E| + n \left(\int_{\mathbb{R}^n} \chi_E - 1 \right)^2 \leq \underline{\lim}_{i \rightarrow \infty} \left\{ \int_{\Omega} |\nabla \chi_{E_i}| + n \left(\int_{\Omega} \chi_{E_i} - 1 \right)^2 \right\},$$

Then, one obtains that E is a solution for the relaxed problem (5.1). \square

References

- [1] **Alter, F. ; Cazelles, V. ; Chambolle, A.** *A characterization of convex calibrable sets in \mathbb{R}^N* , prepublication.
- [2] **Alter, F. ; Cazelles, V. ; Chambolle, A.** *Evolution of convex sets in the plane by the minimizing totalvariaion flow*, Prépublication.
- [3] **Andreu, F. ; Caselles, V. ; Mazón, J. M.** *A strongly degenerate quasilinear elliptic equation*, Nonlinear Anal. 61 (2005), no. 4, 637–669
- [4] **Bellettini, G. ; Caselles, V. ; Novaga, M.** *Explicit solutions of the eigenvalue problem $-\operatorname{div}\left(\frac{Du}{|Du|}\right) = u$* .
- [5] **Cheeger, J.** *A lower bound for the smallesteigenvalue of the Laplacian in Problems in Analysis, Symposium in honorof Salomon Bochner, Ed : RC Ganning, Princeton Univ. Press (1970), p.195-199.*
- [6] **De Giorgi; Carriero, M.; Leaci, A.;** *Existence Theorem for a minimum problem with a Free dicontinuity set*,A.R.M.A, 108, 195-218, (1989).

- [7] **Demengel, F.** *On Some Nonlinear Partial Differential Equations Involving The 1-Laplacian and Critical Sobolev exponent*, ESAIM: Control, Optimisation and Calculus of Variations, 4 (1999), 667-686
- [8] **Demengel, F.** *Some compactness result for some spaces of functions with bounded derivatives*, A.R.M.A. 105(2) (1989), 123-161.
- [9] **Demengel, F.** *Theoremes d'existence pour des equations avec l'operateur 1-Laplacien, premiere valeur propre pour $-\Delta_1$* , C.R Acad. Sci. Paris, Ser. I334 (2002), 1071-1076.
- [10] **Demengel, F.** *Some existence's results for noncoercive 1-Laplacian operator*, Asymptot. Anal. 43 (2005), no. 4, 287-322.
- [11] **Demengel, F.** *Functions locally almost 1-harmonic*, Applicable Analysis, Vol.83, N°9, September 2004, 865-896.
- [12] **Demengel, F.** *On some nonlinear partial differential equations involving the "1-Laplacian " and critical Sobolev exponent*, ESAIM Control Optim. Calc. Var. 4 (1999), 667-686.
- [13] **Ekeland, I. and Temam, R.** *Convex Analysis and variational problems*, North-Holland, 1976.
- [14] **Giusti, E.** *Minimal surfaces and functions of bounded variation*, Notes de cours rediges pr G.H. Williams. Departement of Mathematics Australian National University, Canberra (1977), et Birkhauser (1984).
- [15] **Giaquinta, M. ; Modica, G. ; and Soucek, J.** *Cartesian Currents in the Calculus of Variations I*, LNM, Vol 37, Springer, 1997.
- [16] **Guedda, M. and Veron, L.** *Quasilinear elliptic equations involving critical sobolev exponents*, Nonlinear Analysis, Theory, Methods and Applications, 13 (1989), 879-902.
- [17] **Kohn, R.V. and Temam, R.** *Dual spaces of stress and strains with applications to Hencky plasticity*, Appl.Math.Optim (10) (1983), 1-35.
- [18] **Lions, P.L.** *The concentration-compactness principle in the calculus of variations. The limit case*, I et II. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145-201.
- [19] **Strang, G. and Temam, R.** *Functions with bounded derivatives*, A.R.M.A. (1980), 493-527.
- [20] **Tolksdorf, P.** *Regularity for a more general class of quasilinear elliptic equations*, Journal of Differential Equations, 51 (1984), 126-150.
- [21] **Vazquez, J.L.** *A Strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (1984), 191-202.