Notes on Contraction Theory

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Abstract

These notes derive a number of technical results on nonlinear contraction theory, a comparatively recent tool for system stability analysis. In particular, they provide new results on the preservation of contraction through system combinations, a property of interest in modelling biological systems.

Key words: contraction theory, centralized feedback, hierarchies, attractors

1 Introduction

Nonlinear contraction theory [Lohmiller and Slotine, 1998] is a comparatively recent tool for system stability analysis. These notes derive a number of technical results motivated by the theory. In particular, they provide new results on the preservation of contraction through system combinations, a property of interest in modelling biological systems.

Section 2 analyzes the preservation of contraction through generalized negative feedback between contracting systems. Section 3 describes a new system combination, centralized contraction, which also preserves contraction by aggregation. Section 4 uses standard results from computer science to simplify the general structure of arbitrary system combinations, and in particular to exploit intrinsic hierarchical properties. Section 5 discusses some applications to nonlinear attractors, while section 6 describes the estimation of the successive derivatives of a vector using composite variables.

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2 Negative feedback

This section analyzes feedback connections which automatically give rise to contraction with regards to a predefined metric.

Consider two contracting systems, of possibly different dimensions and metrics, and connect them in feedback, in such a way that the overall virtual dynamics is of the form

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 & -\mathbf{G}(\mathbf{z},t)\mathbf{B} \\ \mathbf{G}(\mathbf{z},t)^T\ \mathbf{A}^T & \mathbf{F}_2 \end{pmatrix} \begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix}$$

with $\mathbf{A}, \mathbf{B}$ two square matrices. The overall system is contracting if

(1) $\mathbf{A}$ and $\mathbf{B}$ are symmetric positive definite, and
(2) there exists $\beta > 0$ such that
$\dot{\mathbf{A}} + \mathbf{A}.\mathbf{F_1} + \mathbf{F_1}^{\mathbf{T}}\mathbf{A} \leq -\beta\ \mathbf{A}$
$\dot{\mathbf{B}} + \mathbf{B}.\mathbf{F_2} + \mathbf{F_2}^{\mathbf{T}}\mathbf{B} \leq -\beta\ \mathbf{B}$

Indeed, we can define the metric

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$$

We have

$$\frac{d}{dt}(\delta\mathbf{z}^T\mathbf{M}\delta\mathbf{z}) = \mathbf{MF} + \mathbf{F}^T\mathbf{M} + \dot{\mathbf{M}}$$

where the matrix $\mathbf{MF}$ is of the form,

$$\mathbf{MF} = \begin{pmatrix} \mathbf{AF}_1 & -\mathbf{AG}(\mathbf{z},t)\mathbf{B} \\ \mathbf{BG}(\mathbf{z},t)^T\mathbf{A} & \mathbf{B}.\mathbf{F_2} \end{pmatrix}$$

Thus

$$\frac{d}{dt}(\delta\mathbf{z}^T\mathbf{M}\delta\mathbf{z}) = \delta\mathbf{z}^T \begin{pmatrix} \dot{\mathbf{A}} + \mathbf{AF_1} + \mathbf{F_1}^{\mathbf{T}}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{B}} + \mathbf{BF_2} + \mathbf{F_2}^{\mathbf{T}}\mathbf{B} \end{pmatrix} \delta\mathbf{z} \leq -\beta\delta\mathbf{z}^T\mathbf{M}\delta\mathbf{z}$$

by hypothesis, which implies that $\delta\mathbf{z}^T\mathbf{M}\delta\mathbf{z}$ tends exponentially to zero. Since $\mathbf{M}$ is positive definite, this in turn implies that $\delta\mathbf{z}^T\delta\mathbf{z}$ tends exponentially to zero.

3 Centralized contraction

We extend here the class of combinations of contracting systems described in [Lohmiller and Slotine, 1998]. From a practical point of view, the condition given for feedback combinations may be hard to deal with, whereas hierarchical combinations are much simpler but not general enough. It is therefore of interest to find a combination having a strong expressiveness together with an automatic guarantee of contraction.

An almost hierarchical feedback combination. The basic idea lies in the following remark. When looking at the combination depicted in figure 3, the loop between F_1 and F_2 seems to be illusory, as the domain and co-domain in F_2 are disjoint. Let's try to formalize this idea.

We consider two contracting system connecting in feedback in such a way that the second system can be split into $\mathbf{z_2^1}$ and $\mathbf{z_2^2}$ so to write

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z_1} \\ \delta\mathbf{z_2^1} \\ \delta\mathbf{z_2^2} \end{pmatrix} = \begin{pmatrix} \mathbf{F_1} & \mathbf{G_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F_2^{11}} & \mathbf{F_2^{12}} \\ \mathbf{G_2} & \mathbf{F_2^{21}} & \mathbf{F_2^{22}} \end{pmatrix} \begin{pmatrix} \delta\mathbf{z_1} \\ \delta\mathbf{z_2^1} \\ \delta\mathbf{z_2^2} \end{pmatrix}$$

Then, a following our idea that this almost represents a hierarchical combination, we apply the metric

$$\mathbf{\Theta} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \epsilon^{-1}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \epsilon\mathbf{I} \end{pmatrix}$$

This gives rise to the generalized Jacobian

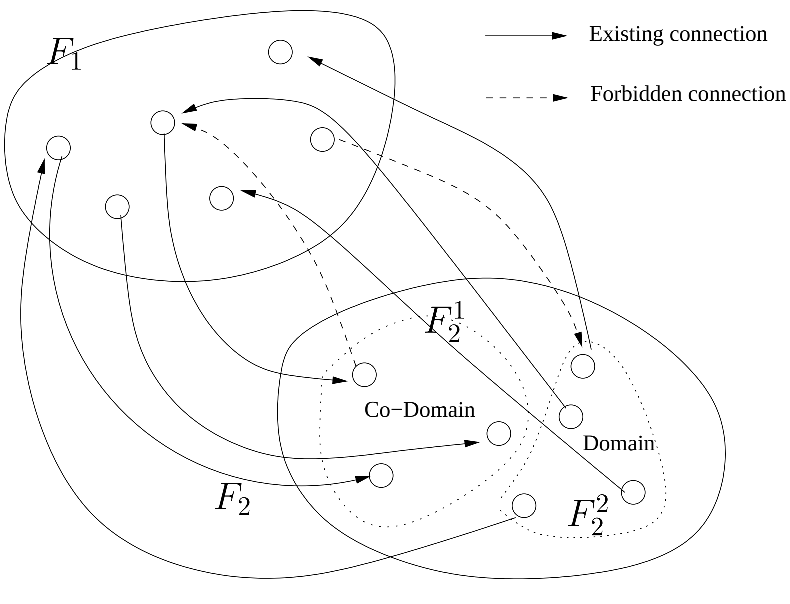

Fig. 1. A combination that seems to give rise to automatic contraction

$$\frac{d}{dt}\begin{pmatrix}\delta\mathbf{z_1}\\ \delta\mathbf{z_2^1}\\ \delta\mathbf{z_2^2}\end{pmatrix} = \begin{pmatrix}\mathbf{F_1} & \epsilon\,\mathbf{G_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{F_2^{11}} & \epsilon^{-2}\,\mathbf{F_2^{12}}\\ \epsilon\,\mathbf{G_2} & \epsilon^2\,\mathbf{F_2^{21}} & \mathbf{F_2^{22}}\end{pmatrix}\begin{pmatrix}\delta\mathbf{z_1}\\ \delta\mathbf{z_2^1}\\ \delta\mathbf{z_2^2}\end{pmatrix}$$

This says that, as long as $\mathbf{G_1}$ and $\mathbf{G_2}$ are bounded, they are negligible. However, we have no more guarantee on the contraction of $\mathbf{F_2}$ as the matrix of feedback $\mathbf{F_2^{12}}$ and $\mathbf{F_2^{21}}$ have been perturbed by ϵ.

This tells us that we have to restrict this intuition to a particular kind of feedback within $\mathbf{F_2}$.

3.1 Orientable systems

The first step is to master the metric used in each local feedback. Indeed, as in the case of feedback combination, to apply the combination recursively, we need some guarantees of non interference between the metric. Then idea is to require that the metric use for each combination only acts on the peripheral system and not on the centralizer. This leads to the notion of *orientable system*.

Definition 1 *A combination between two systems is said to be* orientable *if a metric that makes the generalized Jacobian negative definite can be written*

$$\mathbf{M} = \begin{pmatrix}\mathbf{M}' & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}$$

Small gain. Consider two contracting systems, of possibly different dimensions and metrics, and connect them in feedback, in such a way that the overall virtual dynamics is of the form

$$\frac{d}{dt}\begin{pmatrix}\delta\mathbf{z}_1\\ \delta\mathbf{z}_2\end{pmatrix} = \begin{pmatrix}\mathbf{F}_1 & \mathbf{B}\mathbf{G}(\mathbf{z},t)\\ \mathbf{G}(\mathbf{z},t)^T\,\mathbf{A}^T & \mathbf{F}_2\end{pmatrix}\begin{pmatrix}\delta\mathbf{z}_1\\ \delta\mathbf{z}_2\end{pmatrix}$$

with $\mathbf{A}$, $\mathbf{B}$ two square matrices. Note that in this form, $\mathbf{A}$ and $\mathbf{B}$ must have the same dimension.

Assume now that $\mathbf{A}$ and $\mathbf{B}$ satisfy

- $\mathbf{B}$ is invertible
- $\mathbf{A}\mathbf{B}^{-1}$ is constant and symmetric positive definite

We can then define the metric

$$\mathbf{M} = \begin{pmatrix} \mathbf{AB}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

which can be rewritten $\mathbf{M} = \mathbf{\Theta}^T\mathbf{\Theta}$ with

$$\mathbf{\Theta} = \begin{pmatrix} \sqrt{\mathbf{AB}^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

Using the fact that

$$\sqrt{\mathbf{AB}^{-1}}\mathbf{B} = (\sqrt{\mathbf{AB}^{-1}})^{-1}\mathbf{A}$$

we have

$$\begin{pmatrix} \sqrt{\mathbf{AB}^{-1}}\mathbf{F_1}(\sqrt{\mathbf{AB^{-1}}})^{\mathbf{-1}} & \sqrt{\mathbf{AB}^{-1}}\mathbf{BG}(\mathbf{z},t) \\ \mathbf{G}(\mathbf{z},t)^T(\sqrt{\mathbf{AB}^{-1}}\mathbf{B})^T & \mathbf{F_2} \end{pmatrix}$$

Applying a standard result for small gain feedback (see [Slotine, 2003]), we can conclude on the contraction of the system if $\sqrt{\mathbf{AB}^{-1}}\mathbf{F_1}(\sqrt{\mathbf{AB^{-1}}})^{\mathbf{-1}}$ is negative definite and the following inequality holds

$$\sigma^2(\sqrt{\mathbf{AB}^{-1}}\mathbf{BG}(\mathbf{z},t)) < \lambda((\sqrt{\mathbf{AB}^{-1}}\mathbf{F_1}(\sqrt{\mathbf{AB^{-1}}})^{\mathbf{-1}})_{\mathbf{s}})\lambda((\mathbf{F_2})_{\mathbf{s}}) \qquad (1)$$

We can conclude that this system is an orientable scaling-robust system.

Note that the above assumptions on $\mathbf{A}$ and $\mathbf{B}$ are verified in the common case that $\mathbf{A}$ is constant and symmetric positive definite and $\mathbf{B} = \lambda\mathbf{I}$ with constant $\lambda > 0$.

Negative feedback. In the same way, for a system of the form

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 & -\mathbf{BG}(\mathbf{z},t) \\ \mathbf{G}(\mathbf{z},t)^T\ \mathbf{A}^T & \mathbf{F}_2 \end{pmatrix}\begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix}$$

we can construct a orientable metric such that the system is contracting if $\sqrt{\mathbf{AB}^{-1}}\mathbf{F_1}(\sqrt{\mathbf{AB^{-1}}})^{\mathbf{-1}}$ is negative definite.

3.2 Orientable scaling-robust systems

Definition 2 *A combination between two systems is said to be* orientable scaling-robust *if transforming the system by the metric* $D_\epsilon = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \epsilon\,\mathbf{I} \end{pmatrix}$ *(*$\epsilon > 0$*) leads to an* *orientable combination with metric* $\begin{pmatrix} M_\epsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$. *We further require that* $M_\epsilon \xrightarrow{\epsilon \to 0} 0$.

Remark 3 *The definition above says that the dynamics*

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{x^1} \\ \delta\mathbf{x^2} \end{pmatrix} = \begin{pmatrix} \mathbf{F^{11}} & \epsilon^{-1}\,\mathbf{F^{12}} \\ \epsilon\,\mathbf{F^{21}} & \mathbf{F^{22}} \end{pmatrix}\begin{pmatrix} \delta\mathbf{x^1} \\ \delta\mathbf{x^2} \end{pmatrix}$$

are orientable for all ϵ.

Small gain and negative feedback. Let us come back to the two previous examples. It is clear that applying the metric $\mathbf{D}_\epsilon = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \epsilon\,\mathbf{I} \end{pmatrix}$ for $\epsilon > 0$ leads to another contracting system with metric

$$\mathbf{M}_\epsilon = \begin{pmatrix} \epsilon\,\mathbf{A}\mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

So an orientable small gain (resp. negative feedback) is automatically scaling-robust.

3.3 Centralized contraction

Assume that we have, as in figure 3.3, n systems connecting to a particular system called the *center* in such a way that every connection to the center is *orientable* and *scaling-robust*. Assume also that the connection between the different peripheral systems is hierarchical.

That kind of system can be rewritten

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z_6} \\ \delta\mathbf{z_5} \\ \delta\mathbf{z_4} \\ \delta\mathbf{z_3} \\ \delta\mathbf{z_2} \\ \delta\mathbf{z_1} \\ \delta\mathbf{z_C} \end{pmatrix} = \begin{pmatrix} \mathbf{F_6} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{F_6^2} \\ \mathbf{0} & \mathbf{F_5} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{F_5^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{F_4} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{F_4^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F_3} & \mathbf{X} & \mathbf{X} & \mathbf{F_3^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F_2} & \mathbf{X} & \mathbf{F_2^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F_1} & \mathbf{F_1^2} \\ \mathbf{F_6^1} & \mathbf{F_5^1} & \mathbf{F_4^1} & \mathbf{F_3^1} & \mathbf{F_2^1} & \mathbf{F_1^1} & \mathbf{C} \end{pmatrix}\begin{pmatrix} \delta\mathbf{z_6} \\ \delta\mathbf{z_5} \\ \delta\mathbf{z_4} \\ \delta\mathbf{z_3} \\ \delta\mathbf{z_2} \\ \delta\mathbf{z_1} \\ \delta\mathbf{z_C} \end{pmatrix}$$

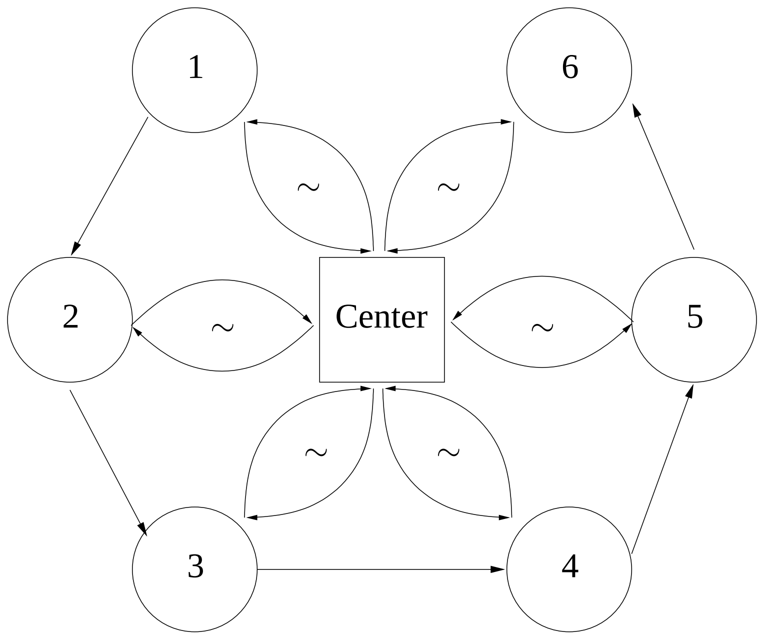

Fig. 2. A centralized combination of contracting systems

For that particular virtual system, let us use the metric

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \epsilon^{-1}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \epsilon^{-2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \epsilon^{-3}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \epsilon^{-4}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \epsilon^{-5}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \epsilon\mathbf{I} \end{pmatrix}$$

which leads to the system

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z_6} \\ \delta\mathbf{z_5} \\ \delta\mathbf{z_4} \\ \delta\mathbf{z_3} \\ \delta\mathbf{z_2} \\ \delta\mathbf{z_1} \\ \delta\mathbf{z_C} \end{pmatrix} = \begin{pmatrix} \mathbf{F_6} & \epsilon\mathbf{X} & \epsilon^{2}\mathbf{X} & \epsilon^{3}\mathbf{X} & \epsilon^{4}\mathbf{X} & \epsilon^{5}\mathbf{X} & \epsilon^{-1}\mathbf{F_6^2} \\ \mathbf{0} & \mathbf{F_5} & \epsilon\mathbf{X} & \epsilon^{2}\mathbf{X} & \epsilon^{3}\mathbf{X} & \epsilon^{4}\mathbf{X} & \epsilon^{-2}\mathbf{F_5^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{F_4} & \epsilon\mathbf{X} & \epsilon^{2}\mathbf{X} & \epsilon^{3}\mathbf{X} & \epsilon^{-3}\mathbf{F_4^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F_3} & \epsilon\mathbf{X} & \epsilon^{2}\mathbf{X} & \epsilon^{-4}\mathbf{F_3^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F_2} & \epsilon\mathbf{X} & \epsilon^{-5}\mathbf{F_2^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F_1} & \epsilon^{-6}\mathbf{F_1^2} \\ \epsilon\mathbf{F_6^1} & \epsilon^{2}\mathbf{F_5^1} & \epsilon^{3}\mathbf{F_4^1} & \epsilon^{4}\mathbf{F_3^1} & \epsilon^{5}\mathbf{F_2^1} & \epsilon^{6}\mathbf{F_1^1} & \mathbf{C} \end{pmatrix}\begin{pmatrix} \delta\mathbf{z_6} \\ \delta\mathbf{z_5} \\ \delta\mathbf{z_4} \\ \delta\mathbf{z_3} \\ \delta\mathbf{z_2} \\ \delta\mathbf{z_1} \\ \delta\mathbf{z_C} \end{pmatrix}$$

Since all the couple (F_i^1, F_i^2) are assumed to be orientable scaling-robust, for any small ϵ, there is a constant metric $\mathbf{M}$ such that the symmetric part of the generalized Jacobian can be written, when ϵ tends to zero, as

$$\begin{pmatrix} \mathbf{H_6} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K_6} \\ \mathbf{0} & \mathbf{H_5} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K_5} \\ \mathbf{0} & \mathbf{0} & \mathbf{H_4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K_4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H_3} & \mathbf{0} & \mathbf{0} & \mathbf{K_3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H_2} & \mathbf{0} & \mathbf{K_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H_1} & \mathbf{K_1} \\ \mathbf{K_6^T} & \mathbf{K_5^T} & \mathbf{K_4^T} & \mathbf{K_3^T} & \mathbf{K_2^T} & \mathbf{K_1^T} & \mathbf{C} \end{pmatrix}$$

where the matrices

$$\begin{pmatrix} \mathbf{H_i} & \mathbf{K_i} \\ \mathbf{K_i^T} & \mathbf{C} \end{pmatrix}$$

are all negative definite.

Applying a basic result of matrix analysis thus yields the condition

$$\mathbf{C} < \sum_i \mathbf{K_i^T H_i^{-1} K_i}$$

which is equivalent to

$$\sum_i \lambda(\mathbf{K_i^T H_i^{-1} K_i C^{-1}}) < 1$$

A sufficient condition is thus

$$\sum_i \sigma(\mathbf{K_i})^2 \lambda(\mathbf{H_i})^{-1} < \lambda(\mathbf{C})$$

or even the less general but easier to verify inequality

$$\sum_i \sigma(\mathbf{K_i})^2 < \lambda(\mathbf{C}) \min_i \lambda(\mathbf{H_i})$$

We call this case *centralized contraction*.

3.4 Going further

The structure of centralized contraction is a general scheme that can be extended to more complex structures. We present here two basic extensions.

Multiple layers We can apply the result of centralized contraction even if the peripheral system is composed of different layers. For example, the system described in figure 3 is automatically contracting providing that the red connections are orientable scaling robust.

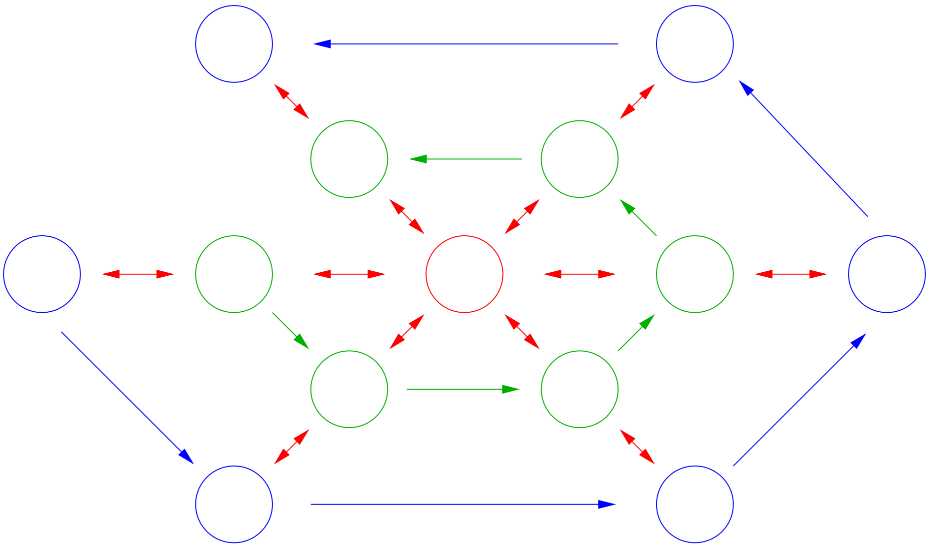

Fig. 3. Centralized contraction for multiple layers

Multiple centers In biological systems, it is often of interest to consider a centralizer which is composed of multiple systems. In that case, the contraction can be guaranteed if all connections to the center considered as a whole are orientable scaling robust.

4 Strongly connected components

In computational neuroscience as in many biological fields, we have to deals with large systems. Here we exploit a standard algorithm from computer science [Knuth, 1997] to decompose a large system into sub-systems, in a such way that the contraction of the overall system can be deduced from the contraction of the smaller sub-systems.

Definition 4 *A* strongly connected component *of a directed graph* $G = (V, E)$ *is a maximal set of vertices* $U \subset V$ *such that for all* $u, v \in U$, u *is reachable from* v *and* v *is reachable from* u.

Proposition 5 *Any directed graph is a union of* strongly connected components *plus edges to join the components together.*

Thus, we are able to distinguish between micro-systems which are connected in feedback combination or not. Indeed, we can state the proposition:

Proposition 6 *Two sub-systems of large systems are in feedback combination iff those two systems belongs to the same strongly connected component.*

Let us now describe the algorithm to compute such a decomposition.

Algorithm. *Strongly_connected_components*(G)

(1) Use the Depth-First-Search (DFS) algorithm to compute $f[U]$ the finishing time of u
(2) Compute $G^T = (V, E)$ where $E^T = \{(u, v)|(v, u) \in E\}$
(3) Execute DFS on G^T by grabbing vertices in the order of decreasing $f[u]$ as computed in step 1.
(4) Output the vertices of each tree in the depth-first forest of step 3. as a separate strongly connected component

Complexity. This algorithm runs twice the time of DFS(G) which is $\Theta(|V|+|E|)$

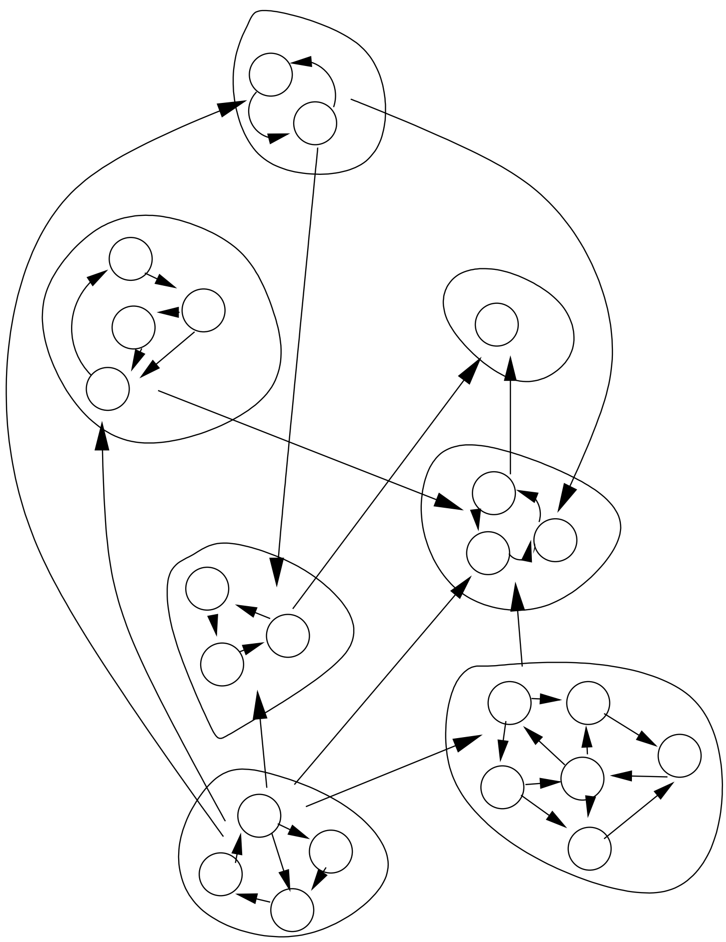

Fig. 4. The strongly connected components of a large system

4.1 *Topological sort of graph*

If the system consists of a directed acyclic graph (DAG), we can compute the topological sort of this graph in order to have its hierarchical combination.

Algorithm. *Topological_sort*(G)

(1) Call DFS(G) to compute $f[u]$, the finishing time of u
(2) As each vertices is finished, put it into the front of a linked list
(3) Return the linked list of vertices

Complexity. Since DFS(G) takes $\Theta(|V| + |E|)$ and insertion into linked list cost $\theta(1)$ for each vertex, topological sort costs only $\Theta(|V| + |E|)$.

4.2 *Filtering large systems*

Once we have computed the *strongly connected components* of the large system G, we can consider the graph $G' = (V', E')$ consisting of the strongly connected components of G as vertices and $E' = \{(C_1, C_2) | \exists u \in C_1, v \in C_2 (u, v) \in E\}$.

Proposition 7 *G' is a directed acyclic graph.*

Thus we can compute the topological sort of G' which gives rise to the hierarchical structure of the large system G.

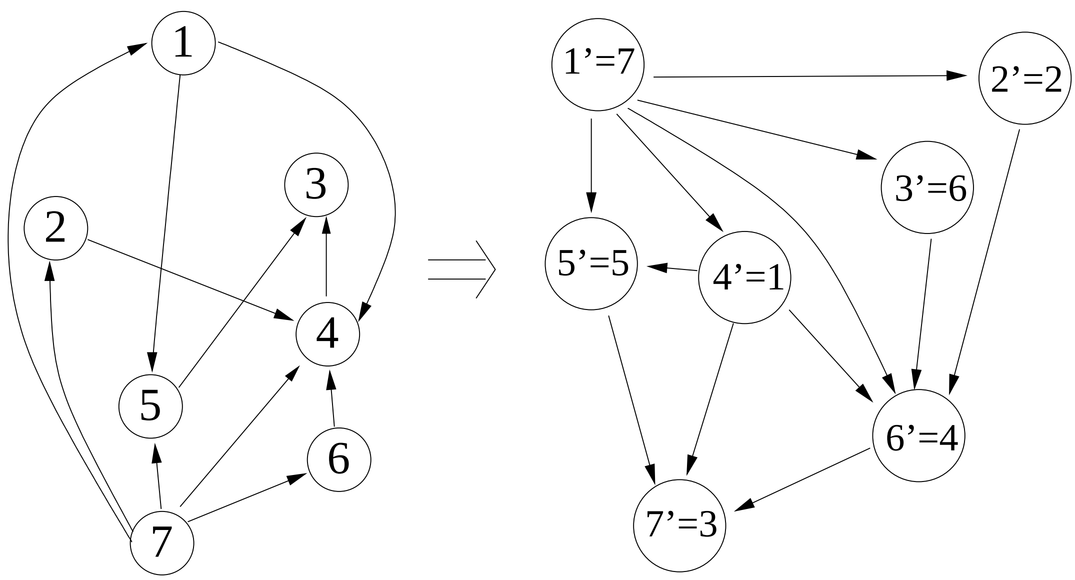

Fig. 5. The topological sort of the strongly connected components generated in figure 1.

Using the basic result on contraction of hierarchies [Lohmiller and Slotine, 1998], this implies that in order to show that a large system is contracting, we only have to show that each *strongly connected component* of the system is contracting, and that the couplings are bounded.

5 Study of time varying hyper-curved attractors

5.1 Line attractor

Consider a system $\dot{\mathbf{x}} = f(\mathbf{x})$ contracting in a constant metric $\mathbf{M}$. Then, the system

$$\begin{cases} \dot{s} &= 0 \\ \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) - \mathbf{g}(s) \end{cases}$$

will be called a line attractor as $\mathbf{x}$ tends exponentially towards $\mathbf{x_0}$ satisfying $\mathbf{f}(\mathbf{x_0}) = \mathbf{g}(s)$.

5.2 Time varying hyper-curved attractor

Consider a system $\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}, t)$ and suppose that there exists an explicit metric in which the system can be rewritten :

$$\begin{cases} \dot{\mathbf{z}}_\mathbf{1} &= \mathbf{s}(\mathbf{z_1}, t) \\ \dot{\mathbf{z}}_\mathbf{2} &= \mathbf{f}(\mathbf{z_2}, t) + \mathbf{g}(\mathbf{z_1}, t) \end{cases}$$

with $\frac{\partial \mathbf{f}}{\partial \mathbf{z_2}}(\mathbf{z_2}, t)$ uniformly negative definite.

Then the system is said to be a time varying hyper-curved attractor as it tends to

$$\mathbf{z}^\infty(\mathbf{z_1}, t) = \begin{pmatrix} \alpha(\mathbf{z_1}, t) \\ \beta(\mathbf{z_1}, t) \end{pmatrix}$$

Define the virtual system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) + \mathbf{g}(\mathbf{z_1}, t)$$

This system is contracting as $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}, t) = \frac{\partial \mathbf{f}}{\partial \mathbf{z_2}}(\mathbf{z_2}, t)$ is uniformly negative definite. So $\mathbf{y}$ tends exponentially to some $\beta(\mathbf{z_1}, t)$. As $\mathbf{z_2}(t)$ is another particular solution, we know from partial contraction that $\mathbf{z_2}(t)$ tends to the same $\beta(\mathbf{z_1}, t)$. □

Remark To know if hypothesis above are true given a system (with $\mathbf{h_1}$ and $\mathbf{h_2}$ assumed to be $\mathcal{C}^2$)

$$\frac{d}{dt}\begin{pmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{pmatrix} = \begin{pmatrix} \mathbf{h_1}(\mathbf{z_1}, \mathbf{z_2}) \\ \mathbf{h_2}(\mathbf{z_1}, \mathbf{z_2}) \end{pmatrix}$$

we only have to check that

$$\frac{\partial\, \mathbf{h_1}}{\partial \mathbf{z_2}}(\mathbf{z_1}, \mathbf{z_2}) = 0 \qquad \frac{\partial^2\, \mathbf{h_2}}{\partial \mathbf{z_1}\mathbf{z_2}}(\mathbf{z_1}, \mathbf{z_2}) = 0$$

Then, from Schwarz theorem, we know that two other equations are true

$$\frac{\partial^2\, \mathbf{h_1}}{\partial \mathbf{z_2}\mathbf{z_1}}(\mathbf{z_1}, \mathbf{z_2}) = 0 \qquad \frac{\partial^2\, \mathbf{h_2}}{\partial \mathbf{z_2}\mathbf{z_1}}(\mathbf{z_1}, \mathbf{z_2}) = 0$$

and so the system can be rewritten in the required form.

Example Consider the system

$$\begin{cases} \dot{s} = \prod_{i=1}^{n}(s_i - s) \\ \dot{\mathbf{x}} = -\mathbf{x} + \mathbf{f}(s) \end{cases}$$

where the s_i's are scalars. This system is an attractor with stable points $(s_i, \mathbf{f}(s_i))$.

6 Composite variables

6.1 Estimation of the successive derivatives of a vector

We show how to compute the n successive derivatives of a given vector only by assuming that the $(n+1)^{th}$ derivative of the vector $\mathbf{x}$ is zero.

Let $\widehat{\mathbf{x}}_i$ be the estimation of the i^{th} derivative of $\mathbf{x}$. We define each $\widehat{\mathbf{x}}_i$ associated composite variable.

$$\widehat{\mathbf{x}}_i = \overline{\mathbf{x}}_i + \alpha_i \mathbf{x}$$

and define the system :

$$\dot{\widehat{\mathbf{X}}} = \mathbf{J}\,\widehat{\mathbf{X}}$$

With $\mathbf{X} = (x_1, \ldots, x_n)^T$ and

$$\mathbf{J} = \begin{pmatrix} -\alpha_1 & 1 & . & & & . \\ . & 0 & 1 & & & \\ . & & 0 & 1 & & \\ . & & & 0 & 1 & . \\ . & & & . & 0 & 1 \\ -\alpha_n & 0 & & & & 0 \end{pmatrix}$$

Let us rewrite the system in $\widehat{\mathbf{X}}$:

$$\dot{\widehat{\mathbf{X}}} = \mathbf{J}\,(\widehat{\mathbf{X}} - \mathbf{Y})$$

with $\mathbf{Y} = (\dot{\mathbf{x}}, 0, \ldots, 0)^T$. It is clear that the system is contracting iff the companion matrix $\mathbf{J}$ satisfies $\mathbf{J}^T\,M + M\,\mathbf{J} \;<\; 0$ for some metric M.

Now, assuming that the system is contracting, it is easy to see that $\widehat{\mathbf{X}}_i = \mathbf{x}_i,\ \forall i$ is the unique solution of the system. Indeed,

$$\dot{\widehat{\mathbf{X}}} = (\mathbf{x}_2, \ldots, \mathbf{x}_n, 0)^T = \mathbf{J}\,(0, \mathbf{x}_2, \ldots, \mathbf{x}_n)^T = \mathbf{J}\,(\widehat{\mathbf{X}} - \mathbf{Y})$$

So, if the system is contracting, we are sure to converge to the right successive derivatives exponentially.

Note that we can check the contraction of the system above using the Routh-Hurwitz criterion.

6.2 Extension

We can now analyze the case where the $(n+1)^{th}$ derivative of the vector $\mathbf{x}$ is a nonlinear function of $\mathbf{x}, \dot{\mathbf{x}}, \ldots$, that is $\mathbf{x}_{n+1} = f(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$. We just have to replace

$$\dot{\widehat{\mathbf{X}}} = \mathbf{J}\,\widehat{\mathbf{X}} \quad \text{by} \quad \dot{\widehat{\mathbf{X}}} = \mathbf{J}\,\widehat{\mathbf{X}} + (0, \ldots, 0, f(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n))^T$$

The Jacobian of the system is

$$\mathbf{J}_{ext} = \begin{pmatrix} -\alpha_1 & 1 & . & & . \\ . & 0 & 1 & & \\ . & & 0\ 1 & & \\ . & & & 0\ 1 & . \\ . & & & .\ 0 & 1 \\ -\alpha_n + \frac{\partial f}{\partial \widehat{\mathbf{x}}_1} & \frac{\partial f}{\partial \widehat{\mathbf{x}}_2} & . & . & . \frac{\partial f}{\partial \widehat{\mathbf{x}}_n} \end{pmatrix}$$

Note that the result that α_n must be greater than $\frac{\partial f}{\partial \widehat{\mathbf{x}}_n}$ was obtained in [Slotine, 2003] in the particular case where f corresponds to a Van der Pol oscillator. In that case $\frac{\partial f}{\partial \widehat{\mathbf{x}}_i}$ were all null except for $\frac{\partial f}{\partial \widehat{\mathbf{x}}_n} = 1$, and thus the condition was only $\alpha_n > 1$

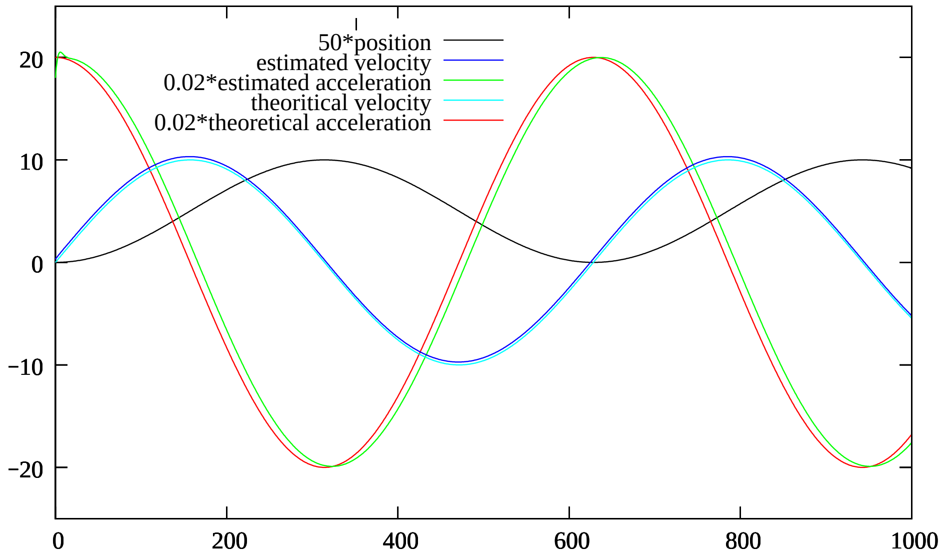

Fig. 6. An estimation of a sinusoidal displacement

Example We compute the velocity and acceleration of the displacement $\frac{1-cos(100x)}{10}$ using the differential equation

$$\frac{d\ddot{x}}{dt} + 5\ddot{x} + 10000\,\dot{x} = 0$$

The result can be seen in figure 6 (we have shifted the estimated curves to facilitate the analysis), where we have used the values $\alpha_1 = 5000$ and $\alpha_2 = 2000$.

6.3 *Estimation of velocity and acceleration using neural net*

Composite variables can be used estimate the velocity and the acceleration of a target given its position using a “neural network”, with potential application in

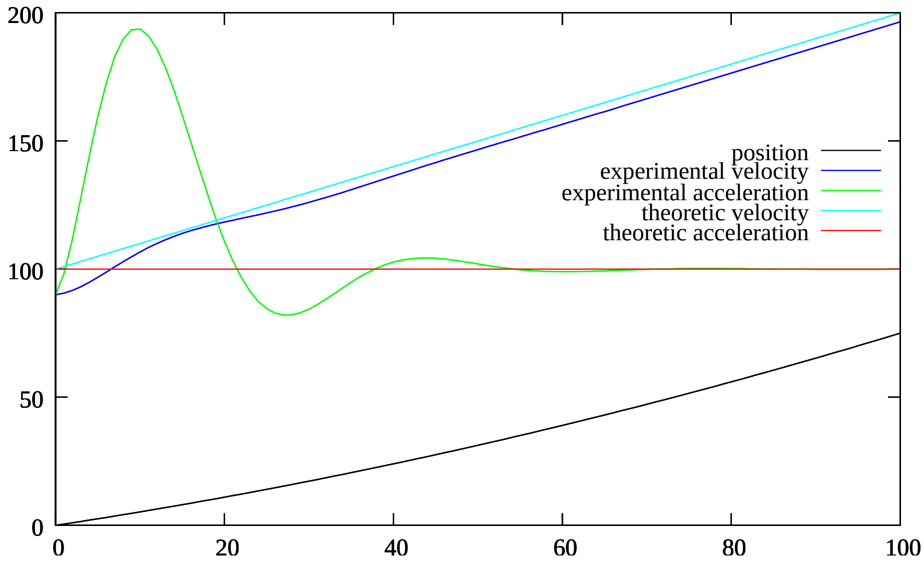

Fig. 7. An estimation of velocity and acceleration of a parabolic trajectory

modelling prediction.

As seen in (6.1), assuming that the acceleration of the target is constant (ie. $\dot{\mathbf{A}} = 0$), we can compute the estimation of velocity and acceleration (resp. $\overline{\mathbf{V}}$ and $\overline{\mathbf{A}}$) using only the position of the target X. For that, we introduce two composite variables $\widehat{\mathbf{V}} = \overline{\mathbf{V}} + \alpha\mathbf{X}$ and $\widehat{\mathbf{A}} = \overline{\mathbf{A}} + \beta\mathbf{X}$ computed by the system :

$$\begin{cases} \dot{\overline{\mathbf{V}}} = -\alpha\,\widehat{\mathbf{V}} + \widehat{\mathbf{A}} = -\alpha\,\overline{\mathbf{V}} + (\beta - \alpha^2)\mathbf{X} + \overline{\mathbf{A}} \\ \dot{\overline{\mathbf{A}}} = -\beta\,\widehat{\mathbf{V}} = -\beta\,\overline{\mathbf{V}} - \beta\,\alpha\mathbf{X} \end{cases}$$

We thus obtain a classical neural network :

$$\tau\,\frac{d}{dt}\begin{pmatrix} \overline{\mathbf{V}} \\ \overline{\mathbf{A}} \\ \widehat{\mathbf{V}} \\ \widehat{\mathbf{A}} \end{pmatrix} = \begin{pmatrix} -\tau\,\alpha & \tau & 0 & 0 \\ -\tau\,\beta & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} . \begin{pmatrix} \overline{\mathbf{V}} \\ \overline{\mathbf{A}} \\ \widehat{\mathbf{V}} \\ \widehat{\mathbf{A}} \end{pmatrix} + \begin{pmatrix} -\tau\,(-\beta + \alpha^2)\mathbf{X} \\ -\tau\,\beta\,\alpha\mathbf{X} \\ \alpha\mathbf{X} \\ \beta\mathbf{X} \end{pmatrix}$$

This network is contracting from section (6.1) and hierarchical analysis.

Example A simulation of this system is presented in figure 7. The small discrepancy between estimated and theoretical values is due to the use of the "neural network".

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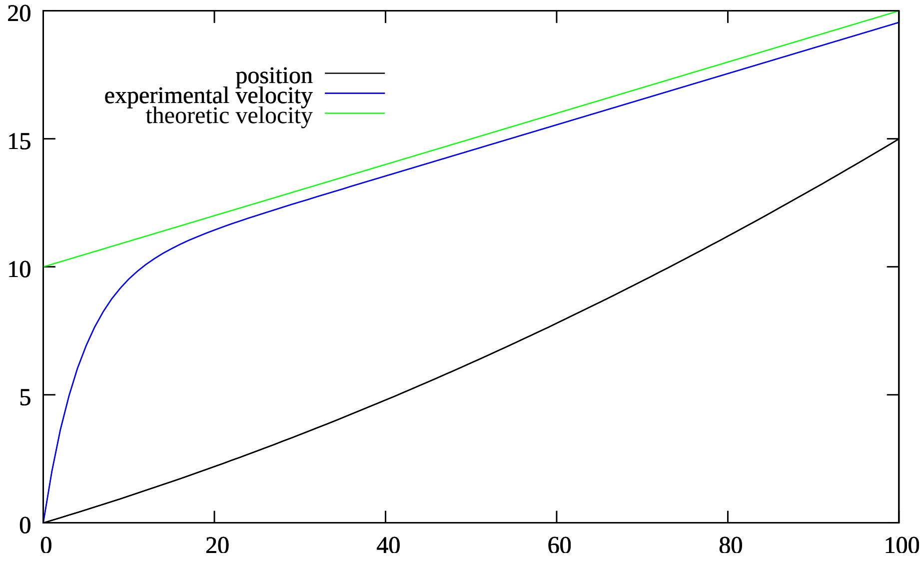

Fig. 8. Converging beyond the scope of the system

References

[Knuth, 1997] Knuth, D. (1997). *The Art of Computer Programming, 3rd Ed.* Addison-Wesley.

[Lohmiller and Slotine, 1998] Lohmiller, W. and Slotine, J. (1998). Contraction analysis for nonlinear systems. *Automatica*, 34(6):683–696.

[Slotine, 2003] Slotine, J. (2003). Modular stability tools for distibuted computation and control. *Journal of Adaptive Control and Signal Processing*, 17(6):397–416.