

ALMOST COMPLEX STRUCTURES ON THE COTANGENT BUNDLE

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ABSTRACT. We construct some lift of an almost complex structure to the cotangent bundle, using a connection on the base manifold. This unifies the complete lift defined by I.Satô and the horizontal lift introduced by S.Ishihara and K.Yano. We study some geometric properties of this lift and its compatibility with symplectic forms on the cotangent bundle.

INTRODUCTION

Analysis on almost complex manifolds recently became an indispensable tool in symplectic geometry with the celebrated work of M.Gromov in [3]. The local existence of pseudoholomorphic discs proved by A.Nijenhuis-W.Woolf in their famous paper [6], allows to lead some local analysis on such manifolds. There is a natural and deep connection between local analysis on complex and almost complex manifolds and canonical bundles. For instance, the cotangent bundle is tightly related to extension of biholomorphisms and to the study of stationary discs. Moreover, it is well known that the cotangent bundle plays a very important role in symplectic geometry and its applications, since this carries a canonical symplectic structure induced by the Liouville form.

Several lifts of an almost complex structure on a base manifold are constructed on the cotangent bundle. These are essentially due to I.Satô in [7] and S.Ishihara-K.Yano in [10]. I.Satô defined a lift of the ambient structure as a correction of the *complete lift*; S.Ishihara-K.Yano introduced the *horizontal lift* obtained via a symmetric connection. The aim of the present paper is to unify and to generalize these lifts by introducing a more natural almost complex lift called the *generalized horizontal lift*.

It turns out that our construction depends on the introduction of some connection : we study the dependence of the lift on it. Our main result states that the structure defined by I.Satô and the *horizontal lift* are special cases of our general construction, obtained by particular choices of connections (Theorem 2.1). We establish some geometric properties of this general lift (Theorems 3.1 and 3.2). Then we characterize generically the structure constructed by I.Satô by the holomorphicity of the lift of a given diffeomorphism on the bases and by the holomorphicity of the complex fiberwise multiplication (Corollary 3.1 and Corollary 3.2).

Finally, we study the compatibility between lifted almost complex structures and symplectic forms on the cotangent bundle. The conormal bundle of a strictly pseudoconvex hypersurface is a totally real maximal submanifold in the cotangent bundle endowed with the structure defined by I.Satô. This was proved by S.Webster ([9]) for the standard complex structure, and by A.Spiro ([8]), and independently by H.Gaussier-A.Sukhov ([2]), for the almost complex case. One can search for a symplectic proof of this, since every Lagrangian submanifold in a symplectic manifold is totally real for almost complex structures compatible with the symplectic form. We prove that for every almost complex manifold and every symplectic form on T^*M compatible with the *generalized horizontal lift*, the conormal bundle of a strictly pseudoconvex hypersurface is not Lagrangian (Proposition 4.1).

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1. PRELIMINARIES

Let M be a real smooth manifold of even dimension n . We denote by TM and T^*M the tangent and cotangent bundles over M , by $\Gamma(TM)$ and $\Gamma(T^*M)$ the sets of sections of these bundles and by $\pi : T^*M \rightarrow M$ the fiberwise projection. We consider local coordinates systems (x_1, \dots, x_n) in M and $(x_1, \dots, x_n, p_1, \dots, p_n)$ in T^*M . We do not write any sum symbol; we use Einstein summation convention.

1.1. Almost complex structures.

Definition 1.1. *An almost complex structure on M is a tensor field J of type $(1,1)$ which satisfies $J^2 = -Id$. The pair (M, J) is called an almost complex manifold.*

In local coordinates, J is given by $J_l^k dx^l \otimes \partial x_k$.

We say that a map $f : (M, J) \rightarrow (M', J')$ between two almost complex manifolds is (J, J') -holomorphic if :

$$J'(f(x)) \circ d_x f = d_x f \circ J(x), \text{ for every } x \in M.$$

If $f : (M, J) \rightarrow (M', J')$ is a diffeomorphism, we define the direct image of J by f by :

$$f_* J(y) := d_{f^{-1}(y)} f \circ J(f^{-1}(y)) \circ d_y f^{-1}, \text{ for every } y \in M'.$$

The tensor field $f_* J$ is an almost complex structure on M' for which f is $(J, f_* J)$ -holomorphic.

We recall that the Nijenhuis tensor of the almost complex structure J is defined by :

$$N_J(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \text{ for } X, Y \in \Gamma(TM).$$

1.2. Tensors and contractions. Let θ be the Liouville form on T^*M . This one-form is locally given by $\theta = p_i dx^i$. The two-form $\omega_{st} := d\theta$ is the canonical symplectic form on the cotangent bundle, with local expression $\omega_{st} = -dx^k \wedge dp^k$. We stress out that these forms do not depend on the choice of coordinates on T^*M .

We denote by $T_q^r M$ the space of q covariant and r contravariant tensors on M . For positive q , we consider the contraction map $\gamma : T_q^1 M \rightarrow T_{q-1}^1(T^*M)$ defined by :

$$\gamma(R) := p_k R_{i_1, \dots, i_q}^k dx^{i_1} \otimes \dots \otimes dx^{i_{q-1}} \otimes \partial p_{i_q}$$

for $R = R_{i_1, \dots, i_q}^k dx^{i_1} \otimes \dots \otimes dx^{i_q} \otimes \partial x_k$.

We also define a q -form on T^*M by $\theta(R) := p_k R_{i_1, \dots, i_q}^k dx^{i_1} \otimes \dots \otimes dx^{i_q}$ for a tensor $R \in T_q^1 M$ on M . We notice that $\theta(R)(X_1, \dots, X_q) = \theta(R(d\pi(X_1), \dots, d\pi(X_q)))$ for $X_1, \dots, X_q \in \Gamma(T^*M)$.

Since the canonical symplectic form ω_{st} establishes a correspondence between q -forms and $T_{q-1}^1 M$, one may define the contraction map γ using the Liouville form θ and ω_{st} by setting, for $X_1, \dots, X_q \in \Gamma(T^*M)$:

$${}^t(\theta(R))(X_1, \dots, X_q) = -\omega_{st}(X_1, \gamma(R)(X_2, \dots, X_q)),$$

where ${}^t(\theta(R))(X_1, \dots, X_q) = \theta(R)(X_2, \dots, X_q, X_1)$.

For a tensor $R \in T_2^1 M$, we have a matricial interpretation of the contraction γ ; if $R_{i,j}^k$ are the coordinates of R then $\gamma(R)$ is given by :

$$\gamma(R) = \begin{pmatrix} 0 & 0 \\ a_j^i & 0 \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}), \text{ with } a_j^i = p_k R_{j,i}^k.$$

1.3. Connections. Let ∇ be a connection on an almost complex manifold (M, J) . We denote by $\Gamma_{i,j}^k$ its Christoffel symbols defined by $\nabla_{\partial x_i} \partial x_j = \Gamma_{i,j}^k \partial x_k$. Let also $\Gamma_{i,j}$ defined in local coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$ on T^*M by the equality $p_k \Gamma_{i,j}^k = \Gamma_{i,j}$.

The torsion T of ∇ is defined by :

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], \text{ for every } X, Y \in \Gamma(TM).$$

There are ‘‘natural’’ families of connections on an almost complex manifold.

Definition 1.2. A connection ∇ on M is called :

- (1) almost complex when $\nabla_X(JY) = J\nabla_X Y$ for every $X, Y \in \Gamma(TM)$,
- (2) minimal when its torsion T is equal to $\frac{1}{4}N_J$,
- (3) symmetric when its torsion T is identically zero.

A.Lichnerowicz proved, in [5], that for any almost complex manifold, the set of almost complex and minimal connections is nonempty. This fact is crucial in the following.

We introduce a tensor $\nabla J \in T_2^1 M$ which measures the lack of complexity of the connection ∇ :

$$(1.1) \quad (\nabla J)(X, Y) := \nabla_X JY - J\nabla_X Y \text{ for every } X, Y \in \Gamma(TM).$$

Locally we have $(\nabla J)_{i,j}^k = \partial x_i J_j^k - J_l^k \Gamma_{i,j}^l + J_j^l \Gamma_{i,l}^k$.

To the connection ∇ we associate three other connections :

- $\bar{\nabla} := \nabla - T$. The Christoffel symbols $\bar{\Gamma}_{i,j}^k$ of $\bar{\nabla}$ are given by $\bar{\Gamma}_{i,j}^k = \Gamma_{j,i}^k$.
- $\tilde{\nabla} := \nabla - \frac{1}{2}T$. The connection $\tilde{\nabla}$ is a symmetric connection and its Christoffel symbols $\tilde{\Gamma}_{i,j}^k$ are given by : $\tilde{\Gamma}_{i,j}^k = \frac{1}{2}(\Gamma_{i,j}^k + \Gamma_{j,i}^k)$.
- a connection on (M, T^*M) , still denoted by ∇ , and defined by :

$$(\nabla_X s)(Y) := X.s(Y) - s\nabla_X Y \text{ for every } X, Y \in \Gamma(TM) \text{ and } s \in \Gamma(T^*M).$$

Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. The horizontal distribution H^∇ of ∇ is defined by :

$$H_\xi^\nabla := \{d_x s(X), X \in T_x M, s \in \Gamma(T^*M), s(x) = \xi, \nabla_X s = 0\} \subseteq T_\xi T^*M.$$

We recall that $d_\xi \pi$ induces an isomorphism between H_ξ^∇ and $T_x M$. Moreover we have the following decomposition : $T_\xi T^*M = H_\xi^\nabla \oplus T_x^*M$. So an element $Y \in T_\xi T^*M$ decomposes as $Y = (X, v^\nabla(Y))$, where $v^\nabla : T_\xi T^*M \rightarrow T_x^*M$ is the projection on the vertical space T_x^*M parallel to H_ξ^∇ .

2. GENERALIZED HORIZONTAL LIFT ON THE COTANGENT BUNDLE

Let (M, J) be an almost complex manifold. We first recall the definitions of the structures constructed by I.Satô and S.Ishihara-K.Yano. Then we introduce a new almost complex lift of J to the cotangent bundle T^*M over M and we prove that this unifies the complete lift and the horizontal lift.

2.1. Complete and horizontal lifts. We consider the complete lift denoted by J^c and defined by I.Satô in [7] as follows : let $\theta(J)$ be the one-form on T^*M with local expression $\theta(J) = p_k J_l^k dx^l$. We define J^c by the identity $d(\theta(J)) = \omega_{st}(J^c, \cdot)$. Then J^c is locally given by :

$$J^c = \begin{pmatrix} J_j^i & 0 \\ p_k(\partial x_j J_i^k - \partial x_i J_j^k) & J_i^j \end{pmatrix}.$$

The complete lift J^c is an almost complex structure on T^*M if and only if J is an integrable structure on M , that is if and only if M is a complex manifold. Introducing a correction term which involves the non integrability of J , I.Satô obtained an almost complex structure on the cotangent bundle ([7]); this is given by :

$$\tilde{J} := J^c - \frac{1}{2}\gamma(JN_J).$$

For convenience we will also call \tilde{J} the *complete lift* of J . The coordinates of JN_J are given by :

$$JN_J(\partial x_i, \partial x_j) = [-\partial x_j J_i^k + \partial x_i J_j^k + J_s^k J_i^q \partial x_q J_j^s - J_s^k J_j^q \partial x_q J_i^s] dx^k.$$

Thus we have the following local expression of \tilde{J} :

$$\tilde{J} = \begin{pmatrix} J_j^i & 0 \\ B_j^i & J_i^j \end{pmatrix}, \text{ with } B_j^i = \frac{pk}{2} [\partial x_j J_i^k - \partial x_i J_j^k + J_s^k J_i^q \partial x_q J_j^s - J_s^k J_j^q \partial x_q J_i^s].$$

We now recall the definition of the horizontal lift of an almost complex structure. Let ∇ be a connection on M and $\tilde{\nabla} := \nabla - \frac{1}{2}T$. The horizontal lift of J is defined in [10] by :

$$J^{H,\nabla} := J^c + \gamma([\tilde{\nabla}J]),$$

where the tensor $[\tilde{\nabla}J] \in T_2^1 M$ is given by :

$$[\tilde{\nabla}J](X, Y) := -(\tilde{\nabla}J)(X, Y) + (\tilde{\nabla}J)(Y, X), \text{ for every } X, Y \in \Gamma(TM) \text{ (}\tilde{\nabla}J \text{ is defined in (1.1)).}$$

S.Ishihara and K.Yano proved that $J^{H,\nabla}$ is an almost complex structure on T^*M . It is important to notice that without symmetrizing ∇ , the horizontal lift of J is not an almost complex structure. The structure $J^{G,\nabla}$ is locally given by :

$$J^{H,\nabla} = \begin{pmatrix} J_j^i & 0 \\ \tilde{\Gamma}_{i,l} J_j^l - \tilde{\Gamma}_{j,l} J_i^l & J_i^j \end{pmatrix}.$$

The complete and the horizontal lifts are both a correction of J^c . Our aim is to unify and to characterize these two almost complex structures.

2.2. Construction of the generalized horizontal lift. Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. Assume that H is a distribution satisfying the local decomposition $T_\xi T^*M = H_\xi \oplus T_x^*M$. From an algebraic point of view it is natural to lift the almost complex structure J as a product structure, that is $J \oplus {}^t J$ with respect to $H_\xi \oplus T_x^*M$. Since any such distribution determines and is determined by a unique connection one may define a lifted almost complex structure using a connection (this point of view is inspired by P.Gauduchon in [1]).

Let ∇ be a connection on M . We consider the connection induced by ∇ on (M, T^*M) , defined in subsection 2.3. For a vector $Y = (X, v^\nabla(Y)) \in T_\xi T^*M = H_\xi^\nabla \oplus T_x^*M$, we define :

$$J^{G,\nabla}(Y) = (JX, {}^t J(v^\nabla(Y))),$$

where $JX = (d_\xi \pi|_{H_\xi^\nabla})^{-1}(J(x)d_\xi \pi(X))$.

Definition 2.1. *The almost complex structure $J^{G,\nabla}$ is called the generalized horizontal lift of J associated to the connection ∇ .*

We first study the dependence of $J^{G,\nabla}$ on the connection ∇ .

Proposition 2.1. *Assume that ∇ and ∇' are two connections on (M, J) . Then $J^{G,\nabla} = J^{G,\nabla'}$ if and only if the tensor $L := \nabla' - \nabla$ satisfies $L(J, \cdot) = L(\cdot, J)$.*

Proof. . Let ∇ and ∇' be two connections on (M, J) and let $L \in T_2^1(M)$ be the tensor defined by $L := \nabla' - \nabla$. We notice that, considering the induced connections on (M, T^*M) , we have :

$$\nabla'_X s = \nabla_X s - s(L(X, \cdot)).$$

Moreover :

$$v^{\nabla'}(Y) = v^\nabla(Y) - \xi(L(d_\xi \pi(X), \cdot)),$$

where $Y = (X, v^\nabla(Y)) \in T_\xi T^*M$.

A vector $Y \in T_\xi T^*M$ can be written $Y = (X, v^\nabla(Y))$ in the decomposition $H_\xi^\nabla \oplus T_x^*M$ of $T_\xi T^*M$ and $Y = (X', v^{\nabla'}(Y))$ in $H_\xi^{\nabla'} \oplus T_x^*M$, with $d_\xi \pi(X) = d_\xi \pi(X')$. By construction we have

$d_\xi\pi(JX) = d_\xi\pi(JX')$. Thus $J^{G,\nabla'} = J^{G,\nabla}$ if and only if $v^\nabla(J^{G,\nabla'}Y) = v^\nabla(J^{G,\nabla}Y)$ for every $\xi \in T^*M$ and $Y \in T_\xi T^*M$. Let us compute $v^\nabla(J^{G,\nabla'}Y)$:

$$\begin{aligned} v^\nabla(J^{G,\nabla'}Y) &= v^{\nabla'}(J^{G,\nabla'}Y) + \xi(L(Jd_\xi\pi(X), \cdot)) \\ &= {}^tJ(v^{\nabla'}(Y)) + \xi(L(Jd_\xi\pi(X), \cdot)) \\ &= {}^tJ(v^\nabla(Y)) - {}^tJ\xi(L(d_\xi\pi(X), \cdot)) + \xi(L(Jd_\xi\pi(X), \cdot)) \\ &= v^\nabla(J^{G,\nabla}Y) - \xi(L(d_\xi\pi(X), J)) + \xi(L(Jd_\xi\pi(X), \cdot)). \end{aligned}$$

So $J^{G,\nabla'} = J^{G,\nabla}$ if and only if $L(d_\xi\pi(X), J) = L(Jd_\xi\pi(X), \cdot)$. Since $d_\xi\pi|_{H_\xi^\nabla}$ is a bijection between H_ξ^∇ and T_xM , we obtain the result. \square

A consequence of Proposition 2.1 is the following Corollary :

Corollary 2.1. *Let ∇ and ∇' be two minimal almost complex connections. One has $J^{G,\nabla'} = J^{G,\nabla}$.*

Proof. Since ∇ and ∇' have the same torsion, the tensor $L := \nabla - \nabla'$ is symmetric. Moreover, since ∇ and ∇' are almost complex, we have $L(\cdot, J) = JL(\cdot, \cdot)$. Thus $L(J, \cdot) = JL(\cdot, \cdot) = L(\cdot, J)$. \square

We see from Corollary 2.1 that minimal almost complex connections are ‘‘natural’’ connections in almost complex manifolds, to construct generalized horizontal lifts.

The links between the generalized horizontal lift $J^{G,\nabla}$, the complete lift \tilde{J} , and the horizontal lift $J^{H,\nabla}$ are given by the following Theorem :

Theorem 2.1. *We have :*

- (1) $J^{G,\nabla} = \tilde{J}$ if and only if $S = -\frac{1}{2}JN_J$, where $S(X, Y) = -(\nabla J)(X, Y) + (\nabla J)(Y, X) + T(JX, Y) - JT(X, Y)$,
- (2) $J^{G,\nabla} = J^{H,\nabla}$ if and only if $T(J, \cdot) = T(\cdot, J)$ and,
- (3) For every almost complex and minimal connection, we have $J^{G,\nabla} = \tilde{J} = J^{H,\nabla}$.

2.3. Proof of Theorem 2.1. The main idea of the proof is to find a tensorial expression of the generalized horizontal structure $J^{G,\nabla}$, involving J^c . In that way, we first describe locally the horizontal distribution H^∇ :

Lemma 2.1. *We have $H_\xi^\nabla = \left\{ \left(\begin{array}{c} X \\ \Gamma_{j,k} X^j \end{array} \right), X \in T_xM \right\}$ for $\xi \in T^*M$ such that $\pi(\xi) = x$.*

Proof. Let us prove that $H_\xi^\nabla \subseteq \left\{ \left(\begin{array}{c} X \\ \Gamma_{j,k} X^j \end{array} \right), X \in T_xM \right\}$. Let $Y \in H_\xi^\nabla$; Y is equal to $d_x s(X)$ where $X \in T_xM$ and s is a section of the cotangent bundle such that $\nabla_X s = 0$. Locally we have $s = s_i dx^i$, $X = X^i \partial x_i$ and so :

$$Y = \left(\begin{array}{c} X \\ X^j \partial x_j s_i \end{array} \right).$$

Since $\nabla_X s = 0$ we obtain :

$$0 = X^j \nabla_{\partial x_j} (s_i dx^i) = X^j s_i \nabla_{\partial x_j} dx^i + X^j \partial x_j s_i dx^i = -X^j s_i \Gamma_{j,k}^i dx^k + X^j \partial x_j s_k dx^k.$$

Therefore $X^j \partial x_j s_k = X^j s_i \Gamma_{j,k}^i = X^j \Gamma_{j,k}^i$. This proves the inclusion.

Moreover the following decomposition insures the equality :

$$T_\xi T^*M = \left\{ \left(\begin{array}{c} X \\ \Gamma_{j,k} X^j \end{array} \right), X \in T_xM \right\} \oplus T_x^*M.$$

\square

The following Proposition gives the local expression of the generalized horizontal lift which is necessary to obtain the desired tensorial expression stated in part (2).

Proposition 2.2.

(1) With respect to the local coordinates system $(x_1, \dots, x_n, p_1, \dots, p_n)$, $J^{G, \nabla}$ is given by :

$$J^{G, \nabla} = \begin{pmatrix} J_j^i & 0 \\ \Gamma_{l,i} J_j^l - \Gamma_{j,l} J_i^l & J_i^j \end{pmatrix}.$$

(2) We have $J^{G, \nabla} = J^c + \gamma(S)$ with $S(X, Y) = -(\nabla J)(X, Y) + (\nabla J)(Y, X) + T(JX, Y) - JT(X, Y)$.

Proof. We first prove part (1). We denote by δ_j^i the Kronecker symbol. With respect to the local coordinates system $(x_1, \dots, x_n, p_1, \dots, p_n)$, the structure $J^{G, \nabla}$ is locally given by :

$$J^{G, \nabla} = \begin{pmatrix} J_j^i & 0 \\ a_j^i & J_i^j \end{pmatrix}.$$

Since $\begin{pmatrix} \delta_j^i \\ \Gamma_{i,j} \end{pmatrix} \in H_\xi^\nabla$, it follows from Lemma 2.1, that for every $i \in \{1, \dots, n\}$:

$$J^{G, \nabla} \begin{pmatrix} \delta_j^i \\ \Gamma_{i,j} \end{pmatrix} = \begin{pmatrix} J_i^j \\ \Gamma_{k,j} J_i^k \end{pmatrix}.$$

Hence we have : $a_j^i = \Gamma_{l,i} J_j^l - \Gamma_{j,l} J_i^l$. This concludes the proof of part (1).

Then we prove part (2). Using the local expression of J^c , we have :

$$J^{G, \nabla} = J^c + \begin{pmatrix} 0 & 0 \\ -p_k \partial x_j J_i^k + p_k \partial x_i J_j^k + \Gamma_{l,i} J_j^l - \Gamma_{j,l} J_i^l & 0 \end{pmatrix}.$$

Since $\nabla_{\partial x_i}(J \partial x_j) = \partial x_i J_j^k \partial x_k + \Gamma_{i,l}^k J_j^l \partial x_k$, it follows that :

$$-p_k \partial x_j J_i^k + p_k \partial x_i J_j^k + \Gamma_{l,i} J_j^l - \Gamma_{j,l} J_i^l = p_k dx^k [-\nabla_{\partial x_j}(J \partial x_i) + \overline{\nabla}_{\partial x_i}(J \partial x_j)].$$

We define $S'(X, Y) := -\nabla_X(JY) + \overline{\nabla}_Y(JX) = -\nabla_X(JY) + \nabla_Y JX + T(JX, Y)$ and we notice that $S'(\partial x_i, \partial x_j) = -\nabla_{\partial x_i}(J \partial x_j) + \overline{\nabla}_{\partial x_j}(J \partial x_i)$. We point out that S' is not a tensor. However with a correction term, we obtain the tensor S :

$$\begin{aligned} S(X, Y) &= S'(X, Y) + J[X, Y] \\ &= -\nabla_X(JY) + \nabla_Y(JX) + T(JX, Y) + J\nabla_X Y - J\nabla_Y X - JT(X, Y) \\ &= -(\nabla J)(X, Y) + (\nabla J)(Y, X) + T(JX, Y) - JT(X, Y). \end{aligned}$$

The components of S are given by $S(\partial x_i, \partial x_j) = S'(\partial x_i, \partial x_j)$ and so $J^{G, \nabla} = J^c + \gamma(S)$. \square

Hence we may compare the three lifted structures via their intrinsic expressions given by :

- $J^{G, \nabla} = J^c + \gamma(S)$ (Proposition 2.2),
- $\tilde{J} = J^c - \frac{1}{2}\gamma(JN_J)$ (see subsection 2.2) and,
- $J^{H, \nabla} = J^c + \gamma([\tilde{\nabla} J])$ with $[\tilde{\nabla} J](X, Y) = -(\tilde{\nabla} J)(X, Y) + (\tilde{\nabla} J)(Y, X)$ (see subsection 2.2).

The lecture of the two first expressions gives part (1) of Theorem 2.1.

To prove (2), we notice that :

$$\begin{aligned} [\tilde{\nabla} J](X, Y) &= -(\tilde{\nabla} J)(X, Y) + (\tilde{\nabla} J)(Y, X) \\ &= -(\nabla J)(X, Y) + (\nabla J)(Y, X) + \frac{1}{2}T(X, JY) + \frac{1}{2}T(JX, Y) - JT(X, Y). \end{aligned}$$

Let us prove part (3) of Theorem 2.1. The equality $J^{G, \nabla} = \tilde{J}$ follows from the fact that $\nabla J = 0$ because the connection ∇ is almost complex and from the equality $-T(J, \cdot) + JT(\cdot, \cdot) = \frac{1}{4}JN_J + \frac{1}{4}JN_J = \frac{1}{2}JN_J$. Since $T = \frac{1}{4}N_J$ and $N_J(J, \cdot) = N_J(\cdot, J)$ we have $J^{G, \nabla} = J^{H, \nabla}$.

The proof of Theorem 2.1 is now achieved. \square

We end this section with :

Corollary 2.2. *We have $J^{H,\nabla} = J^{G,\tilde{\nabla}}$.*

Proof. This is a direct consequence of Theorem 2.1 since $J^{H,\nabla} = J^{H,\tilde{\nabla}}$ and $J^{G,\tilde{\nabla}} = J^{H,\tilde{\nabla}}$ by part (2). \square

We point out that Corollary 2.2 may also be proved using Lemma 2.1 and the distribution D of horizontal lifted vectors defined by S.Ishihara and K.Yano as follows : let $x \in M$ and $\xi \in T^*M$ such that $\pi(\xi) = x$. Assume $X^{H,\nabla}$ is the horizontal lift of $X \in T_x M$ on the cotangent bundle given in [10] by :

$$X^{H,\nabla} = \begin{pmatrix} X \\ \tilde{\Gamma}_{j,k} X^j \end{pmatrix} \in T_\xi T^*M.$$

Then the distribution D of horizontal lifted vectors is defined by $D_\xi = \{X^{H,\nabla}, X \in T_x M\}$. S.Ishihara and K.Yano proved that $J^{H,\nabla} = J \oplus {}^t J$ in the decomposition $T_\xi T^*M = D_\xi \oplus T_x^*M$. From Lemma 2.1 we have $D = H^{\tilde{\nabla}}$ and finally $J^{H,\nabla} = J \oplus {}^t J = J^{G,\tilde{\nabla}}$ with respect to the decomposition $T_\xi T^*M = D_\xi \oplus T_x^*M = H_\xi^{\tilde{\nabla}} \oplus T_x^*M$.

3. GEOMETRIC PROPERTIES OF THE GENERALIZED HORIZONTAL LIFT

3.1. Lift Properties. In Theorem 3.1 we state the lift properties of the generalized horizontal lift of an almost complex structure.

Theorem 3.1.

- (1) *The projection $\pi : T^*M \longrightarrow M$ is $(J^{G,\nabla}, J)$ -holomorphic.*
- (2) *The zero section $s : M \longrightarrow T^*M$ is $(J, J^{G,\nabla})$ -holomorphic.*
- (3) *The lift of a diffeomorphism $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ to the cotangent bundle is $(J_1^{G,\nabla_1}, J_2^{G,\nabla_2})$ -holomorphic if and only if f is a (J_1, J_2) -holomorphic map satisfying $f_*S_1 = S_2$.*

We recall that the lift \tilde{f} of a diffeomorphism $f : M_1 \longrightarrow M_2$ to the cotangent bundle is defined by $\tilde{f} = (f, {}^t(df)^{-1})$ and that the differential $d\tilde{f}$ is locally given by :

$$d\tilde{f} = \begin{pmatrix} df & 0 \\ (*) & {}^t(df)^{-1} \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}),$$

where $(*)$ denotes a $(n \times n)$ block of derivatives of f with respect to (x_1, \dots, x_n) .

Proof of Theorem 3.1. Parts (1) and (2) are consequences of Proposition 2.2 (part (1)).

Let us prove part (3). Assume that $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ is a (J_1, J_2) -holomorphic diffeomorphism satisfying $\tilde{f}_*S_1 = S_2$ and let \tilde{f} be its lift to the cotangent bundle. According to Proposition 2.2, we have $J^{G,\nabla_i} = J^c + \gamma(S_i)$ for $i = 1, 2$. We denote by θ_i and $\omega_{i,st}$ the Liouville form and the canonical symplectic form of T^*M_i . The invariance by lifted diffeomorphisms of these forms insure that $\tilde{f}_*\theta_1 = \theta_2$ and $\tilde{f}_*\omega_{1,st} = \omega_{2,st}$. We also recall that ${}^t(\theta_i(S_i)) = -\omega_{i,st}(\cdot, \gamma(S_i))$.

Let us establish the following equality $\tilde{f}_*(J_1^{G,\nabla_1}) = J_2^{G,\nabla_2}$. The first step consists in proving that the direct image of J_1^c by \tilde{f} is J_2^c . By the nondegeneracy of $\omega_{2,st}$, it is equivalent to obtain the equality $\omega_{2,st}(\tilde{f}_*J_1^c, \cdot) = \omega_{2,st}(J_2^c, \cdot)$:

$$\begin{aligned} \omega_{2,st}(\tilde{f}_*J_1^c, \cdot) &= \omega_{2,st}(d\tilde{f} \circ J_1^c \circ (d\tilde{f})^{-1}, \cdot) \\ &= \omega_{1,st}(J_1^c \circ (d\tilde{f})^{-1}, (d\tilde{f})^{-1}) \\ &= \tilde{f}_*(\omega_{1,st}(J_1^c, \cdot)) \\ &= \tilde{f}_*d(\theta_1(J_1)), \\ \text{and, } \omega_{2,st}(J_2^c, \cdot) &= d(\theta_2(J_2)). \end{aligned}$$

So let us prove that the pull-back of $\theta_2(J_2)$ by \tilde{f} is $\theta_1(J_1)$. According to the local expression of $d\tilde{f}$, we have $\tilde{f}^*(\theta_2(J_2)) = \theta_2(J_2 \circ df)$ and then :

$$\tilde{f}^*(\theta_2(J_2)) = \theta_2(df \circ J_1) = (\tilde{f}^*\theta_2)(J_1) = \theta_1(J_1).$$

Thus we obtain $\tilde{f}_*d(\theta_1(J_1)) = d(\theta_2(J_2))$, that is $\tilde{f}_*J_1^c = J_2^c$.

To show the result, we may prove that the direct image of $\gamma(S_1)$ by \tilde{f} is $\gamma(S_2)$. We prove more generally that $f_*(S_1) = S_2$ if and only if $\tilde{f}_*(\gamma(S_1)) = \gamma(S_2)$ which is equivalent to prove that $f_*(S_1) = S_2$ if and only if $\omega_{2,st}(\cdot, \tilde{f}_*(\gamma(S_1))) = \omega_{2,st}(\cdot, \gamma(S_2))$. We have :

$$\begin{aligned} \omega_{2,st}(\cdot, \tilde{f}_*\gamma(S_1)) &= \omega_{2,st}(\cdot, d\tilde{f} \circ \gamma(S_1) \circ (d\tilde{f})^{-1} \cdot) \\ &= \omega_{1,st}((d\tilde{f})^{-1} \cdot, \gamma(S_1) \circ (d\tilde{f})^{-1} \cdot) \\ &= \tilde{f}_*(\omega_{1,st}(\cdot, \gamma(S_1))) \\ &= -\tilde{f}_*({}^t\theta_1(S_1)). \end{aligned}$$

Let us check that $f_*(S_1) = S_2$ if and only if $\tilde{f}_*{}^t(\theta_1(S_1)) = {}^t(\theta_2(S_2))$. We have :

$$\tilde{f}^*(\theta_2(S_2)) = \theta_2(S_2(df, df)) \text{ and } \theta_1(S_1) = (\tilde{f}^*\theta_2)(S_1) = \theta_2(df \circ S_1).$$

According to this fact and the definition of $\theta(R)$, where $R \in T_2^1M$ is given in the section 1.2, it follows that $f_*S_1 = S_2$ if and only if $\theta_2(S_2(df, df)) = \theta_2(df \circ S_1)$. So $f_*(S_1) = S_2$ if and only if $\tilde{f}_*(\gamma(S_1)) = \gamma(S_2)$. Finally we have proved that if $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ is a (J_1, J_2) -holomorphic diffeomorphism satisfying $f_*S_1 = S_2$ then \tilde{f} is $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphic.

Reciprocally if \tilde{f} is $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphic then f is (J_1, J_2) -holomorphic. Indeed the zero section $s_1 : M_1 \longrightarrow T^*M_1$ is (J_1, J_1^{G, ∇_1}) -holomorphic by part (2) of Theorem 3.1, the projection $\pi_2 : T^*M_2 \longrightarrow M_2$ is (J_2^{G, ∇_2}, J_2) -holomorphic by part (1) of Theorem 3.1 and we have the equality $f = \pi_2 \circ \tilde{f} \circ s_1$. Since f is (J_1, J_2) -holomorphic we have $\tilde{f}_*J_1^c = J_2^c$. Then the $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphicity of \tilde{f} implies the equality $\tilde{f}_*(\gamma(S_1)) = \gamma(S_2)$, that is $f_*S_1 = S_2$. \square

As a Corollary, we obtain the lift properties of the complete and the horizontal lifts by considering special connections. We point out that Theorem 3.1 and Corollary 3.1 characterize the complete lift via the lift of diffeomorphisms.

Corollary 3.1.

- (1) *The lift of a diffeomorphism $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ to the cotangent bundle is $(\tilde{J}_1, \tilde{J}_2)$ -holomorphic if and only if f is (J_1, J_2) -holomorphic.*
- (2) *The lift of a diffeomorphism $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ to the cotangent bundle is $(J_1^{H, \nabla_1}, J_2^{H, \nabla_2})$ -holomorphic if and only if f is a (J_1, J_2) -holomorphic map satisfying $f_*[\tilde{\nabla}_1 J_1] = [\tilde{\nabla}_2 J_2]$.*

Proof. To prove part (1), we consider almost complex and minimal connections ∇_1 and ∇_2 on M_1 and M_2 . Hence $\tilde{J}_1 = J^{G, \nabla_1} = J_1^c + \gamma(S_1)$ and $\tilde{J}_2 = J^{G, \nabla_2} = J_2^c + \gamma(S_2)$. We have $S_1 = -\frac{1}{2}J_1N_{J_1}$ and $S_2 = -\frac{1}{2}J_2N_{J_2}$. We notice that if $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ is a (J_1, J_2) -holomorphic diffeomorphism then $f_*N_{J_1} = N_{J_2}$ and then $f_*J_1N_{J_1} = J_2N_{J_2}$. According to Theorem 3.1 the lift of a diffeomorphism f to the cotangent bundle is $(\tilde{J}_1, \tilde{J}_2)$ -holomorphic if and only if f is (J_1, J_2) -holomorphic.

Finally, part (2) follows from the equality $J^{G, \nabla} = J^{H, \nabla}$ obtained in Corollary 2.2 and from Theorem 3.1. \square

We point out that the projection (resp. the zero section) is (J', J) -holomorphic (resp. (J, J') -holomorphic) for $J' = \tilde{J}, J^{H, \nabla}$ due to local expressions of the complete lift and of the horizontal lift.

3.2. Fiberwise multiplication. We consider the multiplication map $Z : T^*M \longrightarrow T^*M$ by a complex number $a + ib$ with $b \neq 0$ on the cotangent bundle. This is locally defined by $Z(x, p) = (x, (a + b^t J(x))p)$. For $(x, p) \in T^*M$ we have $d_{(x,p)}Z = \begin{pmatrix} Id & 0 \\ C & aId + b^t J \end{pmatrix}$, where $C_j^i = bp_k \partial x_j J_i^k$.

Theorem 3.2. *The multiplication map Z is $J^{G,\nabla}$ -holomorphic if and only if $(\nabla J)(J, \cdot) = (\nabla J)(\cdot, J)$.*

Proof. Let us evaluate $d_{(x,p)}Z \circ J^{G,\nabla}(x, p) - J^{G,\nabla}(x, ap + b^t Jp) \circ d_{(x,p)}Z$. This is equal to :

$$\begin{pmatrix} 0 & 0 \\ CJ + (aId + b^t J)B(x, p) - B(x, ap + b^t Jp) - {}^t JC & 0 \end{pmatrix},$$

where $B_j^i(x, p) = p_k(\Gamma_{l,i}^k J_j^l - \Gamma_{j,l}^k J_i^l)$.

We first notice that $aB_j^i(x, p) - B_j^i(x, ap + b^t Jp) = -bp_k J_s^k(\Gamma_{l,i}^s J_j^l - \Gamma_{j,l}^s J_i^l)$. Let us compute $D = CJ + (aId + b^t J)B(x, p) - B(x, ap + b^t Jp) - {}^t JC$:

$$D_j^i = bp_k \left[\underbrace{J_j^l \partial x_l J_i^k}_{(1)} + \underbrace{J_i^l \Gamma_{s,l}^k J_j^s}_{(2)} - \underbrace{J_i^l \Gamma_{j,s}^k J_l^s}_{(2')} - \underbrace{J_s^k \Gamma_{l,i}^s J_j^l}_{(3)} + \underbrace{J_s^k \Gamma_{j,l}^s J_i^l}_{(3')} - \underbrace{J_i^l \partial x_j J_l^k}_{(1')} \right].$$

We obtain $(1) + (2) + (3) = J_j^l(\partial x_l J_i^k + J_i^s \Gamma_{l,s}^k - J_s^k \Gamma_{l,i}^s)$ and $(1)' + (2)' + (3)' = J_i^l(\partial x_j J_l^k + J_l^s \Gamma_{j,s}^k - J_s^k \Gamma_{j,l}^s)$. We recognize the coordinates of the tensor ∇J (section 1.3) :

$$\partial x_l J_i^k - J_s^k \Gamma_{l,i}^s + J_i^s \Gamma_{l,s}^k = (\nabla J)_{l,i}^k \quad \text{and} \quad \partial x_j J_l^k - J_s^k \Gamma_{j,l}^s + J_l^s \Gamma_{j,s}^k = (\nabla J)_{j,l}^k.$$

Finally $D_j^i = bp_k [J_j^l (\nabla J)_{l,i}^k - J_i^l (\nabla J)_{j,l}^k]$. Then Z is $J^{H,\nabla}$ -holomorphic if and only if $J_j^l (\nabla J)_{l,i}^k = (\nabla J)_{j,l}^k J_i^l$. Since $(\nabla J)_{j,l}^k J_i^l \partial x_k = (\nabla J)(\partial x_j, J \partial x_i)$ and $J_j^l (\nabla J)_{l,i}^k \partial x_k = (\nabla J)(J \partial x_j, \partial x_i)$, this concludes the proof of Theorem 3.2. \square

In particular, the almost complex lift \tilde{J} may be characterized generically by the holomorphicity of Z ; more precisely we have :

Corollary 3.2.

- (1) *The multiplication map Z is \tilde{J} -holomorphic and,*
- (2) *Z is $J^{H,\nabla}$ -holomorphic if and only if $(\tilde{\nabla} J)(J, \cdot) = (\tilde{\nabla} J)(\cdot, J)$.*

Proof. Let us prove part (1). Assume ∇ is an almost complex minimal connection on M . We have $\tilde{J} = J^{G,\nabla}$ and by almost complexity of ∇ , ∇J is identically equal to zero. Theorem 3.2 implies the \tilde{J} -holomorphicity of Z .

Part (2) follows from Theorem 3.2 and from the equality $J^{H,\nabla} = J^{G,\tilde{\nabla}}$ stated in Corollary 2.2. \square

Remark 3.1. *In the case of the tangent bundle TM , the fiberwise multiplication is holomorphic for the complete lift of J if and only if J is integrable. More precisely, "the lack of holomorphicity" of this map is measured by the Nijenhuis tensor (see [4]).*

4. COMPATIBLE LIFTED STRUCTURES AND SYMPLECTIC FORMS

Assume (M, J) is an almost complex manifold. Let $\Gamma = \{\rho = 0\}$ be a real smooth hypersurface of M , where $\rho : M \rightarrow \mathbb{R}$ is a defining function of Γ .

Definition 4.1.

- (1) *Let $x \in \Gamma$. The Levi form of Γ at x is defined by $\mathcal{L}_x^J(\Gamma)(X) = -d(J^* d\rho)(X, JX)$ for any $X \in T_x \Gamma$.*
- (2) *The hypersurface $\Gamma = \{\rho = 0\}$ is strictly J -pseudoconvex if its Levi form is positive definite at any point $x \in \Gamma$.*

Let $x \in \Gamma$, we define $N_x^*(\Gamma) := \{p_x \in T_x^*M, (p_x)|_{T_x\Gamma} = 0\}$. The *conormal bundle* over Γ , defined by the disjoint union $N^*(\Gamma) := \bigcup_{x \in \Gamma} N_x^*(\Gamma)$, is a totally real submanifold of T^*M endowed with the complete lift (see [2] and [8]), that is $TN^*(\Gamma) \cap \tilde{J}(TN^*(\Gamma)) = \{0\}$. To look for a symplectic proof of that fact, we search for a symplectic form, ω' , compatible with the complete lift for which $N^*(\Gamma)$ is Lagrangian, that is $\omega'(X, Y) = 0$ for every sections X, Y of $TN^*(\Gamma)$. More generally we are interested in the compatibility with the generalized horizontal lift. Proposition 4.1 states that one cannot find such a form.

Proposition 4.1. *Assume (M, J, ∇) is an almost complex manifold equipped with a connection. Let ω be a symplectic form on T^*M compatible with the generalized horizontal lift $J^{G, \nabla}$. There is no strictly pseudoconvex hypersurface in M whose conormal bundle is Lagrangian with respect to ω .*

Proof. Let Γ be a strictly pseudoconvex hypersurface in M and let $x \in \Gamma$. Since the problem is purely local we can suppose that $M = \mathbb{R}^{2m}$, $J = J_{st} + O(|x_1, \dots, x_{2m}|)$ and $x = 0$. Since Γ is strictly pseudoconvex we can also suppose that $T_0\Gamma = \{X \in \mathbb{R}^{2m}, X_1 = 0\}$. The two-form ω is given by $\omega = \alpha_{i,j} dx^i \wedge dx^j + \beta_{i,j} dp^i \wedge dp^j + \gamma_{i,j} dx^i \wedge dp^j$.

Assume that $\omega(X, Y) = 0$ for every $X, Y \in TN^*(\Gamma)$. We have $N_0^*(\Gamma) = \{p_0 \in T_0^*\mathbb{R}^{2m}, (p_0)|_{T_0\Gamma} = 0\} = \{(P_1, 0, \dots, 0), P_1 \in \mathbb{R}\}$. Then a vector $Y \in T_0N^*(\Gamma)$ can be written $Y = X_2 \partial x_2 + \dots + X_{2m} \partial x_{2m} + P_1 \partial p_1$. So we have for $2 \leq i < j \leq 2m$:

$$\omega_{(0)}(\partial x_i, \partial x_j) = \alpha_{i,j} = 0.$$

Then $w'_{(0)}$ is given by $\omega_{(0)} = \alpha_{1,j} dx^1 \wedge dx^j + \beta_{i,j} dp^i \wedge dp^j + \gamma_{i,j} dx^i \wedge dp^j$.

Since $J_{(0)}^{G, \nabla} = \begin{pmatrix} J_{st} & 0 \\ 0 & J_{st} \end{pmatrix}$ we have $J_{(0)}^{G, \nabla} Y' = \partial x_{2m}$ for $Y' = \partial x_{2m-1} \neq 0 \in T_0(T^*\Gamma)$. Thus $\omega_{(0)}(Y', J_{(0)}^{G, \nabla} Y') = 0$ and so ω is not compatible with $J^{G, \nabla}$. □

Proposition 4.1 is also established for complete and horizontal lifts because $J_{(0)}^{G, \nabla} = \tilde{J}_{(0)} = J_{(0)}^{H, \nabla}$.

Remark 4.1. *Since the conormal bundle of a (strictly pseudoconvex) hypersurface is Lagrangian for the symplectic form ω_{st} on T^*M , Proposition 4.1 shows that ω_{st} and $J^{G, \nabla}$ are not compatible.*

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