ANALYSIS VERSUS GEOMETRY: WILLIAM ROWAN HAMILTON, JAMES MACCULLAGH AND THE ELUCIDATION OF THE FRESNEL WAVE SURFACE IN THE THEORY OF DOUBLE REFRACTION

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The following paper is presented in honor of Imre Toth on the occasion of his seventy-fifth birthday. The work represents a hitherto unpublished chapter of my doctoral dissertation 'Humphrey Lloyd (1800-1881) and the Dublin Mathematical School of the Nineteenth Century' (University of Manchester, 1979). When I first met Imre in Manchester during the academic year 1975/76, this was my main research interest. My next meeting with Imre was during a symposium on mathematical schools in the nineteenth century held at the Department of History of Science of the University of Regensburg during the week ending July 4, 1976. I met Imre once again in 1978 when he visited the Technical University of Delft, the Netherlands, where I was a research scholar for the academic year 1977/78. On completion of my Ph.D. dissertation, I was able to spend the summer semester as a scholar of the German Academic Exchange Service (DAAD) under Imre's supervision at the University of Regensburg. I am grateful for the interest and encouragement shown by Imre in my research at that time and during my subsequent career which brought me to the Leibniz-Archiv at Hanover as an editor of the manuscript papers of Leibniz in 1987.
My congratulations and very best wishes for the future to you, dear Imre.

Jim O'Hara

The immediate consequence of Hamilton' s investigation of the Fresnel wave surface in biaxial crystals was the discovery of conical refraction (1832 - 1833). More significant in the long term was the elucidation of the nature and properties of the Fresnel wave surface and the extension and generalization of Huygens' construction. Hamilton's discovery was based on the conception of a theorem of receprocal surfaces which he arrived at by analytical methods. In 1830 and 1833 his colleague James MacCullagh applied inversive geometry to the study of Fresnel's wave surface with considerable success and also conceived the theorem of reciprocal surfaces. This paper considers the contributions of both men in investigating the wave surface and reflects on their respective approaches through analysis and geometry.

Das unmittelbare Ergebnis der Untersuchungen von Hamilton über Fresnels Wellenfläche in zweiachsigen Kristallen war die Entdeckung der konischen Refraktion (1832-1833). Größere Bedeutung kam dabei auf längere Sicht der Erläuterung des Wesens und der Eigenschaften der Wellenfläche von Fresnel und der Erweiterung und Verallgemeinerung der Konstruktion von Huygens zu. Die Entdeckung von Hamilton basierte auf der Konzeption eines Theorems über reziproke Flächen, zu dem er mittels analytischer Methoden gelangt war. Bereits 1830 hatte sein Kollege James MacCullagh bei der Erforschung der Fresnelschen Wellenfläche mit gutem Erfolg inverse Geometrie angewandt und war auf das Theorem der reziproken Flächen gestoßen. Dieser Aufsatz betrachtet die Beiträge beider Wissenschaftler bei der Erforschung der Wellenfläche und vergleicht die jeweiligen Methoden mittels Analysis und Geometrie.

La conséquence directe des recherches de Hamilton sur la surface des ondes de Fresnel dans les cristaux bi-axes fut la découverte de la réfraction conique en 1832-33. Au coeur de ces travaux, dont l'importance à long terme réside dans l'explication de la nature et des propriétés de la surface des ondes de Fresnel et dans l'extension et la généralisation de la construction de Huygens, se trouve la conception analytique d'un théorème des surfaces réciproques. En 1830 et 1833, James MacCullagh, collèque de Hamilton travailla au même problème et, y appliquant avec grand succès les méthodes de la géométrie inverse, aboutit au même théorème. L'object du présent article est la contribution de ces deux savants à l'étude de la surface des ondes de Fresnel et leurs voies d'approche respectives du problème.

Analysis versus Geometry: William Rowan Hamilton, James MacCullagh and the Elucidation of the Fresnel Wave Surface in the Theory of Double Refraction

Introduction

The phenomenon of double refraction in crystallised minerals was discovered by Erasmus Bartholin in about the year 1669 and a few years later Christiaan Huygens explained the phenomena on the principles of the wave theory. Huygens' theory, published in his Traité de la lumière at the Hague in 1690, was subjected to the vicissitudes experienced by the wave theory during the next century and a quarter. In 1813 David Brewster discovered that the mineral topaz has two axes of no double refraction and subsequently other substances such as aragonite, borax and mica were identified as biaxial crystals. This discovery meant that Huygen's theory had lost its generality and a new and general theory of double refraction was wanting: the stage was set for the introduction of the theory of Augustin Fresnel. Fresnel's writings on double refraction, dating from the years 1821 - 1822, were published in the second volume of his collected works [Fresnel, 1866-1870, II, 261-596]. Having set out from the hypothesis that the elasticity of the vibrating medium within the crystal is unequal in three rectangular directions, he showed that the surface of the wave in biaxial crystals is a surface of the fourth order consisting of two sheets whose points of contact with tangent planes determine the direction of the two refracted rays. Taking the elasticities in the directions of the co-ordinate axes x, y and z as a^2, b^2, and c^2 respectively, Fresnel found the equation of the wave surface to be

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2) - a^2(b^2+c^2)x^2 - b^2(a^2+c^2)y^2 - c^2(a^2+b^2)z^2 + a^2b^2c^2 = 0.$$

In deriving the equation of the wave surface in biaxial crystals Fresnel applied co-ordinate or Cartesian geometry. The procedure he found for obtaining this equation was long and unwieldy and, in fact, the calculation was not given in full. It was as if the procedure was so inelegant as to be almost an embarrassment to him and that he was content to merely outline the method.

After Fresnel several authors published alternative methods for obtaining the wave equation and demonstrating its properties. The first was André Marie Ampère who published a long and rather complicated method for finding the equation in 1828 [Ampère, 1828]. During the nineteenth century about two hundred works on Fresnel's wave surface were published by some one hundred authors, who included Augustin Louis Cauchy, Arthur Cayley, Franz E. Neumann, John William Strutt (Lord Rayleigh), James Joseph Sylvester and William Rowan Hamilton, whose investigations led to the discovery of new properties of the wave surface which all previous investigators, including Fresnel himself, had misapprehended.

In an "Essay on the Theory of Systems of Rays" and in three supplements, published in the Transactions of the Royal Irish Academy between 1827 and 1833, he developed his general view of optics. In particular, in the "Third Supplement" of 1832, he set out a system of general methods for the solution of optical problems and announced the discovery of new properties of the Fresnel wave surface [Hamilton, 1837]. He had deduced the existence of four conoidal cusps on the wave surface, from which he predicted the phenomena of external and internal conical refraction in which a ray is refracted as a cone at specific angle of incidence on entering or leaving the crystal. The subsequent experimental discovery of these twin phenomena by his colleague in the University of Dublin, Humphrey Lloyd, at the end of 1832 and in early 1833 was a triumph for Hamilton's view and methodology of optics.

The prediction and discovery of conical refraction has been discussed in a number of studies by Hamilton's first biographer [Graves, 1882 - 1889, I] and others [Sarton, 1932; Hankins, 1972 & 1980; O'Hara, 1982; Spearman, 1985]. The aim of the present paper is to examine the respective contributions of Hamilton and James MacCullagh, who was also a colleague at the same university, in elucidating more generally the properties of the Fresnel wave surface. Particular attention will be paid to the methodologies adopted in investigating the wave surface, and the application of geometry by MacCullagh, in contrast to the analytical investigations of Hamilton, is seen as particularly significant.

MacCullagh published two papers on double refraction in crystalline media, one before and one following the discovery of conical refraction. In the first, entitled "On the double refraction of light in a crystallized medium, according to the principles of Fresnel", read at the Royal Irish Academy on 21st June, 1830, and published the following year in its Transactions, he set out a series of geometrical propositions which he applied with great success in an investigation of the Fresnel wave surface [MacCullagh, 1831]. In this paper he came close to a prediction of conical refraction which almost led to an unpleasant priority dispute after the announcement of the discovery of Hamilton and Lloyd. In the second paper, entitled "Geometrical Propositions Applied to the Wave Theory of Light", read at the Academy on 24th June, 1833, and subsequently published in the Transactions, he developed much more fully the ideas of the first [MacCullagh, 1837]. The exact course of events will be discussed in due course, but MacCullagh's own introduction to the first paper provides the best starting point. It may be remarked in passing that MacCullagh was at this time in his 21st year.

> The mathematical difficulties under which the beautiful and interesting theory of Fresnel has hitherto laboured are well known, and have been regarded as almost insuperable. He tells us in his Memoire (see Memoires of the Royal Academy of Sciences of Paris, tom. vii. p. 136), that the calculations, by which he assured himself of the truth of his construction for finding the surface of the wave, were so tedious and embarrassing that he was obliged to omit them altogether. A direct demonstration has since been supplied by M. Ampère (Annales de Chimie et de Physique, tome xxxix. p. 113); but his solution is excessively complicated and difficult.
>
> Judging from the simplicity and elegance of the results that there must be some simple method of arriving at them, I have been led to consider the subject with the attention which it

> merits, and have succeeded in discovering a method by which the whole may be explained with that simplicity which is characteristic of every theory that is founded in nature.
>
> In the following paper I shall give a brief view of this method, sufficient to enable those two are acquainted with the mechanical principles laid down in the original Memoir of Fresnel, to trace at a glance, the connection between the several parts of his theory [MacCullagh, 1831; Haughton & Jellett, eds., 1880, 1].

MacCullagh's method in this paper, as in the second enlarged paper, is to set out a series of propositions, and then to apply them to physical phenomena. His work is concise, rigorous and well illustrated with diagrams. The system of geometry, which MacCullagh developed, is that now known as inversive geometry; it was in fact discovered long before and it is necessary first to briefly review its history [Coolidge, 1963; Boyer, 1968; Court, 1961, 1962 ; Patterson, 1933] .

1. Inversive Geometry

The geometry in question is a geometrical transformation which follows from a simple definition and a basic theorem and which may be stated as follows [Court, 1962].

> 1. Definition. Given a point 0 and a constant k, two points P,P', collinear with 0 and such that OP,OP' = k, are said to be inverse with respect to the circle (0) having 0 for center and the square of its radius equal to k. The circle (0) is called the circle of inversion, and the point 0 the center of inversion.
>
> 2. Basic Theorem. (1) If a point P describes a circle not passing through 0, the point P' describes a circle not passing through 0. (2) If a point P describes a circle passing through 0, the point P' describes a straight line not passing through 0, and conversely.

The basic proposition may be traced back to Apollonius of Perga (floruit c. 225 B.C.). In the Plane Loci of Apollonius, restored by Robert Simson, the following proposition is stated.

> If from any two points A,B, line segments $AC_1 = r_1$, $BC_2 = r_2$ are drawn making a constant angle with one another, if $r_1 r_2 = k^2$, a constant, and if the locus of C_1 is a circle, so also in general is the locus of C_2 [Court, 1962].

The proposition was also given by Pappus of Alexandria (fl. c. 320 A.D.) and by François Viète. Pierre de Fermat extended the two propositions to space in order to solve the problem of drawing a sphere tangent to four given spheres [Court, 1961 & 1962].

There were no further developments for nearly two centuries until inversion was again discovered independently by several mathematicians. In a Traité des propriétés projectives des figures, published at Paris in 1822, Jean Victor Poncelet came across inverse points with respect to a circle, which he called "points réciproques" [Court, 1962, 655]. Jacob Steiner discovered circular inversion in about 1824, but the only evidence of this is an unpublished manuscript as he published nothing on the subject [Coolidge, 1963, 279]. Another German professor Julius Plücker raised inversion from the level of a simple proposition to that of a geometrical transformation. In a paper dated October 1831 and entitled "Über ein neues Übertragungs-Princip", the fifth in a series "Analytisch-geometrische Aphorismen", which was published three years later in A.L. Crelle's Journal für die reine und angewandte Mathematik, Plücker proved analytically a number of propositions, showing among other things that the transformation is a conformal one, viz. that angles are preserved in magnitude, but not in sign, by the transformation [Plücker, 1834].

Among other discoverers of circular inversion were two Dublin mathematicians, pupils of MacCullagh, John William Stubbs [1843] and John Kells Ingram [1844]. Furthermore, the first elementary synthetic exposition is that found in the textbook, Chapters on the Modern Geometry of the Point, Line and Circle, of the Dublin mathematician Richard Townsend [1863, I, 198 - 215]. Among more famous discoverers of inversion must be counted the German August Ferdinand Möbius. He published two papers, "Über eine neue Verwandtschaft zwischen ebenen Figuren" [Möbius, 1853] and "Die Theorie der Kreisverwandtschaft in rein geometrischer Darstellung" [Möbius, 1855].

In circular inversion corresponding points are collinear with a fixed centre, while the product of their distances therefrom is a positive constant. It is usually expressed analytically by writing:

$$OP.OP' = r^2, \quad x' = \frac{r^2 x}{x^2 + y^2}, \quad y' = \frac{r^2 y}{x^2 + y^2}$$

(x',y') being the inverse of (x,y). The transformation is used for simplifying figures and its leading features may be briefly outlined as follows [Coolidge, 1940 & 1963, 279].

1. It is one-to-one except for the centre of inversion O.
2. Circles not through O go into other such circles.
3. Circles through O go into lines not through there, and vice versa.
4. Angles are preserved in absolute magnitude, though reversed in sign, that is to say, the transformation is inversely conformal.
5. Corresponding distances are connected by the simple relation

$$P'Q' = \frac{r^2\, PQ}{OP.OQ} \, .$$

Solid inversive geometry is of course also possible; in this case one defines the inverse of a point in three demensional space with respect to a sphere. The transformation equations above (for x', y') suggested to the Italian Luigi Cremona, the study of the much more general transformation, which now bears his name, $x' = R_1\,(x,y)$, $y' = R_2\,(x,y)$, where R_1 and R_2 are rational algebraic functions. Cremona published an account of these transformations in 1863 and later generalized them for three dimensions [Boyer, 1968, 576].

2. Introduction of Inversive Geometry in Mathematical Physics: Thomson and Plücker

Whatever the importance of the transformations of circular inversion in the development of pure mathematics, they were certainly applied with great success in the elucidation of some physical problems such as that of the description of the wave surface of Fresnel. Among those who arrived at inversive geometry through physics or applied it to physical problems we must include Julius Plücker of Bonn, the British physicist William Thomson (Lord Kelvin), and James MacCullagh.

In 1845, while investigating electrostatic problems, Thomson discovered the "Principle of electric images", which he also applied in potential theory and in the theory of the transmission of heat [Court, 1962, 656]. It appears that Kelvin was not acquainted with the papers of MacCullagh on inversion, published fifteen years earlier.

Thomson published his discovery in the form of a letter to Joseph Liouville, editor of the Journal de Mathématique pures et appliques, written on 8th October, 1845. The following brief extract shows that Thomson was aware of the angle-preserving property of the transformation [Thomson, 1845 , 364-5].

> Soient C le centre d'une sphère S; Q,Q' deux points pris sur un même rayon CA et sur son prolongement, de telle manière que
>
> $$CQ.CQ' = CA^2;$$
>
> et P un point quelconque sur la surface S. On a, comme on sait,
>
> $$\frac{PQ}{PQ'} = \frac{AQ}{AQ'} .$$
>
> On peut, à cause de ce théorème, appeler Q et Q' points réciproques relatifs à la sphère S, dont chacun est l'image de l'autre dans la sphère. Suivant cette définition, l'image d'une ligne ou surface sera le lieu des images de points pris sur cette ligne ou surface. Ainsi, on trouve que l'image d'un planou d'une sphère est toujours une sphère (le plan étant compris sous cette désignation). Les images de deux sphères se coupent sous le même angle, réel ou imaginaire, que les surfaces données.

At the close of the letter to Liouville, Thomson announced his intention of publishing a fuller account of his discovery [Thomson, 1845, 367].

> Je vais publier ces recherches dans le Cambridge and Dublin Mathematical Journal aussitôt que possible, mais vous pouvez faire ce que vous voudrez du résumé précédent.

In fact Thomson continued the development of his analytical theory of inversion in two further letters to Liouville, written on the 26th June and 16th September, 1846, extracts of which Liouville published in the following year [Thomson, 1847]. In a note on the subject of Thomson's letters Liouville himself introduced and elaborated his own analytic theory of inversion, which he called "transformation by reciprocal radii" [Liouville, 1847]. The following is an extract from Liouville's memoire.

> Mais il est plus commode, je crois, d'introduire dans nos recherches une de ces transmutations de figures si familières aux géomètres, et qui ont tant contribué aux progès de la science dans ces derniers temps. La transformation dont il s'agit est bien connue, du reste, et des plus simples; c'est celle que M. Thomson lui-même a jadis employée sous le nom de principe des images. Considérez x,y,z comme les co-ordonnées d'un point quelconque m d'une figure rapportée à trois axes rectangulaires Ox,Oy,Oz, et ξ, η, ζ, comme celles d'un point μ d'une autre figure rapportée à trois axes $O\xi$, $O\eta$, $O\zeta$, rectangulaires aussi, et auxquels nous donnons la même origine O, et respectivement les mêmes directions, une de ces figures dérivant de l'autre, et le point μ, en particulier, correspondant au point m, en vertu des relations par lesquelles ξ, η, ζ s'experiment en x,y,z, ou x,y,z en ξ, η, ζ. Il est évident que les deux points correspondants m, μ sont en ligne droite avec l'origine O, et que le produit $Om.O\mu$ des rayons vecteurs Om, $O\mu$ est constant et = n. Une des figures se déduit donc de l'autre en prenant sur chacun des rayons vecteurs menés du point O à un point quelconque de la première figure d'autres rayons vecteurs en raison inverse des premiers; les extrémités de ces nouveaux rayons vecteurs déterminent la seconde figure. Nous donnerons à cette transformation le nom de transformation par rayons

> vecteurs réciproques, relativement à l'origine 0. Si, pour un point m, on a $Om = \sqrt{n}$, on aura aussi $O\mu = \sqrt{n}$, et les points m et μ qui se correspondent ainsi dans les deux figures coïncideront. En disposant de n, on peut faire en sorte qu'un point donné m reste fixe dans la transformation; il suffit de prendre $n = \overline{Om}^2$, et alors tous les points situés sur la sphère dont 0 est le centre, et Om le rayon, resteront fixes aussi, mais tous les autres seront déplacés [Liouville, 1847, 275-6].

Julius Plücker, in a paper "Discussion de la forme générale des ondes lumineuses", dated April and May 1838 and published in the Journal für die reine und angewandte Mathematik in the following year, applied inversion to illustrate the form of the Fresnel wave surface [Plücker, 1839]. Plücker firstly defines reciprocal polar surfaces ("surfaces polaires réciproques"): two surfaces such that the points of one are the poles of the tangent planes of the other, with respect to some second order surface called the "directrice", are called reciprocal polars [Plücker, 1839, 6]. For example two concentric ellipsoids are reciprocal polars with respect to a sphere having the same centre, if the product of the corresponding semi-axes is equal to the sphere of the radius of the sphere. The two ellipsoids, whose equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad a^2x^2 + b^2y^2 + c^2z^2 = 1,$$

are reciprocal polars, with respect to a sphere of radius unity [Plücker, 1839, 7]. Then he defines conjugate poles ("pôles conjuqués") with respect to a second order surface as two points, such that one of them is the point where the straight line which passes through the centre of the surface and the other point meets the polar plane of the latter point. With respect to a sphere, two points situated on the same diameter, such that the product of their distances from the centre is equal to the square of the radius, are two such conjugate points. One is obtained by dropping a perdendicular from the centre on the polar plane of the other [Plücker, 1839, 8]. Plücker's conjugate points are of course inverse points. Now in Fresnel's theory of double refraction the elasticity of the ether in different directions, in the interior of a biaxial crystal, is determined by the elasticity along the three mutually perpendicular fixed axes of elasticity; a surface of elasticity may be constructed with reference to them by taking vector rays proportional to the squares of the elasticities in the direction of the rays. The equation of the surface of elasticity, as given by Plücker, is $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2 = r^4$, $c^2 > b^2 > a^2$ assuming the elasticity along the z axis to be a maximum and that along the x a minimum [Plücker, 1839, 10].

The form of the equation of the surface of elasticity, given by Fresnel, viz. $v^2 = a^2\cos^2X + b^2\cos^2Y + c^2\cos^2Z$, where v is the velocity of propagation in the direction of the radius vector r and X,Y,Z, the direction cosines of r, is equivalent as the radius vector r must be inversely proportional to the velocity v [see Preston, 1895, 322-323]. Plücker showed that the surface of elasticity is related to the two ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad a^2x^2 + b^2y^2 + c^2z^2 = 1.$$

The axes of the first are those of the surface of elasticity, while the axes of the second are the inverses. Secondly, he showed that the lengths of two vectors rays of the surface of elasticity and of the second ellipsoid, whose direction coincides with the same straight line, are inverse to each other, so that their product is equal to unity $(r^2r_1^{\,2} = 1)$.

The points of the surface of elasticity are the conjugate poles of the points of the second ellipsoid and reciprocally, with respect to a sphere whose radius is unity [Plücker, 1839, 10-11]. According to Fresnel's theory any vibratory movement, in any plane diametrical section of the surface of elasticity, may be decomposed into two rectilinear vibrations, represented by the two axes of the curve of intersection. The movement of propagation is at right angles to that of vibrations and the velocities of propagation are inversely proportional to the two semiaxes, which in turn are proportional to the squares of the elasticities. From this Fresnel had concluded that the planes parallel to the diametrical section, at distances from it equal to the axes of the curve of intersection, are tangents to the luminous wave, and he had determined the equation of the wave surface by a complicated analytical method, where the wave is regarded as the envelope of the tangent planes. According to Plücker's method the wave surface is obtained in the following manner [Plücker, 1839, 12-17]. Take any diametrical plane through the centre of the second ellipsoid ($a^2x^2 + b^2y^2 + c^2z^2 = 1$), and erect on this plane perpendiculars equal to the inverse values of the two semiaxes of the ellipse of intersection; the planes parallel to the diametrical section and passing through the extremities of these perpendiculars are tangents to the luminous wave. The luminous wave so obtained is the reciprocal polar surface, with respect to a sphere of radius unity, of that which one obtains on putting in, in this construction, the first ellipsoid in place of the second. Alternatively, the wave may be constructed by cutting the first ellipsoid by a diametrical plane and erecting at its centre two perpendiculars of lengths equal to the greatest and least semi-diameters of the ellipse of intersection, respectively; the locus of the extremities of these perpendiculars will be the surface of the luminous wave.

Plücker obtained analytical expressions for the wave surface and for the plane of vibrations before considering in considerable detail the cusps on the wave surface and the circles of contact formed by the tangent planes around the cusps which Hamilton had discovered and from which he had deduced the phenomenon of conical refraction. Analytical expressions were obtained for the luminous external cone, for the planes of vibrations of the emergent rays, for the singular planes and for the circles of contact, before returning again to purely geometrical considerations in respect of the singularities of the wave surface [Plücker, 1839, 30-32]. He outlined the following method, by which one may arrive simply, by purely geometrical reasoning, at a complete determination of the two sorts of singularities of the wave surface. To any point m of the first ellipsoid there always corresponds a single point p of the surface of elasticity which is the foot of the perpendicular dropped from the centre O on the tangent plane to the ellipsoid at the first point m.

The diametrical plane passing through the points m and p is the plane of vibration; when one draws, in this plane, two right lines OM and OV perpendicular and equal to Om and Op respectively, the corresponding luminous ray will be OM, whereas the wave front is perpendicular to OV at V. Plücker went on to examine the theory of double refraction according to the principle of Huygens and demonstrated the inverse relationship between the wave surface and Hamilton's "Surface of Wave Slowness" [Plücker, 1839, 38-44]. The equation of this latter surface in rectangular co-ordinates is the following:

$$(b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2)(x^2 + y^2 + z^2) - [(b^2 + c^2)x^2 + (a^2 + c^2)y^2 + (a^2 + b^2)z^2] + 1 = 0.$$

It is the reciprocal polar surface of the wave surface, with respect to a sphere of radius unity, and may be obtained by replacing in the equation of the wave surface the three constants a^2, b^2, c^2 by $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$. This is a consequence of the general theorem that the two surfaces, with common origin and axes inversely related, are reciprocal polars with respect to a sphere of radius unity. The vector rays of one of the two surfaces represent the velocities of different luminous rays while those of the other are the inverses of the velocities of the corresponding plane waves. Moreover perpendiculars dropped from the centre or origin on the tangent planes of one surface represent the velocities of the plane waves, whereas those dropped on the tangent planes of the other are the reciprocals of the velocities of the corresponding rays. The relations are completely reciprocal.

Plücker added the following note with reference to Hamilton's Third Supplement ; it shows in particular that, although not acquainted with the latter's memoir, he had understood the essence of Hamilton's method. Plücker's paper provided in fact the first exposition of Hamilton's work in understandable terms and in an international journal.

> Je dois conclure des indications données par Mr. Lloyd dans son excellent Report on the progress and present state of physical optics (London 1835) que la surface polaire réciproque de l'onde lumineuse, par rapport à une sphère, dont le rayon est égal à l'unité, soit précisement celle, que Mr. Hamilton a nommé "surface of wave slowness". N'ayant aucune connaissance du Mémoire de M. Hamilton, je me borne à emprunter du Report la construction suivante dont je n'y trouve que le simple énoncé.
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> > ... they lead to a very elegant construction for the reflected or refracted ray, which is, in most cases, more convenient than that of Huygens. That when a ray proceeds from air into a crystal, we have only to construct the surfaces of wave slowness belonging to the two media and having their common centre at the point of incidence. Let the incident ray be then produced to meet the sphere, which represents the normal slowness of the wave in air; and from the point of intersection let a perpendicular be drawn to the reflecting or refracting surface. This will cut the surface of slowness of the reflected or refracted waves in general in two points. The lines connecting these points with the centre, will represent the direction and normal slowness of the waves; while the perpendiculars from the centre on the tangent planes, at the same points, will represent the direction and slowness of the rays themselves.
>
> L'on verra que cette construction revient à celle du numéro suivant, si l'on entend par slowness l'unité divisée par la vitesse. [Plücker, 1839, footnote p. 41].

To this was added an outline of the "Construction de M. Hamilton" for obtaining the directions and velocities of the refracted rays in the crystal. Perpendiculars are dropped from the centre on the tangent planes to the polar surface of the wave, which give the direction of the two refracted rays; the velocities are the inverses of these perpendiculars. Plücker observed that the wave surface has its own reciprocal polar; the wave surface is in fact its own reciprocal polar, with respect to an ellipsoid whose three axes are mean proportionals between the products of the three axes of the first ellipsoid, taken in pairs, viz.

$$\frac{x^2}{bc} + \frac{y^2}{ac} + \frac{z^2}{ab} = 1.$$

As to the significance of this he writes:

> Ce théorème constitue, quant à l'optique des cristaux, une espèce de duplicité, ...; les phénomènes se présentent par couples, de sorte que l'un de chaque couple se déduit de l'autre au moyen de l'ellipsoide directeur [Plücker, 1839, 42-43].

This principle of duality is illustrated by the correspondence of four singular points on the wave surface and four singular planes, and of course the phenomenon of external and internal conical refraction. Plücker concluded his memoir by formulating a construction of his own for obtaining the refracted rays.

> Construisez, par rapport à l'ellipsoide directeur, la ligne droite polaire de celle qui est perpendiculaire au plan d'incidence en O'. Elle coupera la surface de l'onde, décrite autour du point O, en deux points. Les deux lignes qui vont du point O aboutir à ces points, seront les deux rayons réfractés; tandis, que les deux plans, qui, contenant la perpendiculaire en O', passent par ces deux mêmes points, seront les fronts des deux ondes planes correspondantes. Enfin il a été démontré dans ce qui précède, que les deux plans de vibration sont ceux qu'on obtient en conduisant par les rayons lumineux des plans perpendiculaires aux fronts des ondes correspondantes [Plücker, 1839, 43].

Plücker seems to have been aware of the work of MacCullagh only through the reference to his memoirs given by Humphrey Lloyd in his Report on Optics, presented to the British Association in 1834, as we can judge from the very last sentence of his "discussion de la forme générale des ondes lumineuses":

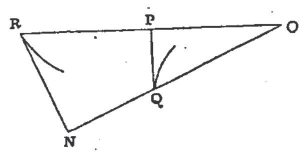
> D'après le Report, cité plus haut, outre M. Hamilton, un autre géomètre irlandais s'en est également occupé; je n'ai pu me procurer le travail ne de l'un ni de l'autre de ces géomètres. J'ai cite tout ce qui étoit a ma connoissance [Plücker, 1839, 44].

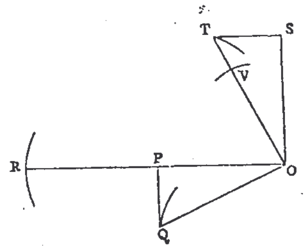
The second geometer referred to here is of course James MacCullagh. It may be noted also that Plücker was actually a correspondent of Humphrey Lloyd and had paid him a visit in Dublin in 1835. Thus on November 3, 1835, in a letter to George Biddell Airy on the subject of the establishment of a magnetic observatory, Lloyd remarks:

> A friend of Gauss - M. Plücker - professor of mathematics at Bonn has been with me this morning and gave me a very full account of Gauss' operations ... M. Plücker will not remain here many days .. . [1].

3. Contribution of James MacCullagh

We now turn to MacCullagh's first memoir, "On the double refraction of light in a crystallized medium, according to the principles of Fresnel" in which his system of inversive geometry was first published [MacCullagh, 1831]. In this paper he proved a series of propositions or lemmas concerning two ellipsoids which are "reciprocally proportional"; these ellipsoids are concentric, and the semiaxes a,b,c, and a', b', c', coincide so that $aa' = bb' = cc' = K^2$. He showed that, if a semidiameter OR of one be cut perpendicularly in P by a plane which touches the other (here fig. 1), then OR is inversely as OP, so that OP.OR will always be equal to k^2 [MacCullagh, 1831; Haughton & Jellett, eds., 1880, 3].





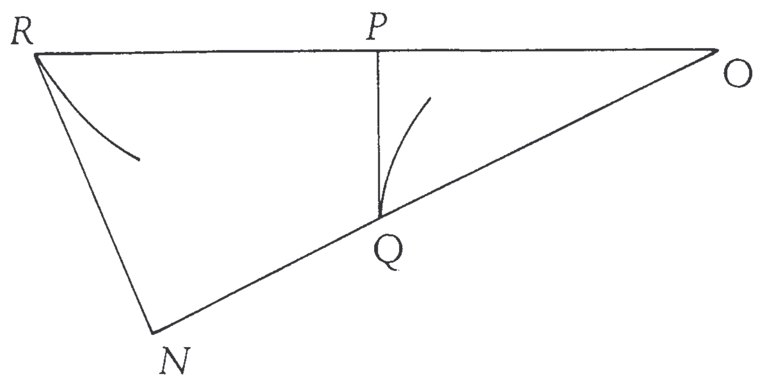


Fig. 1.

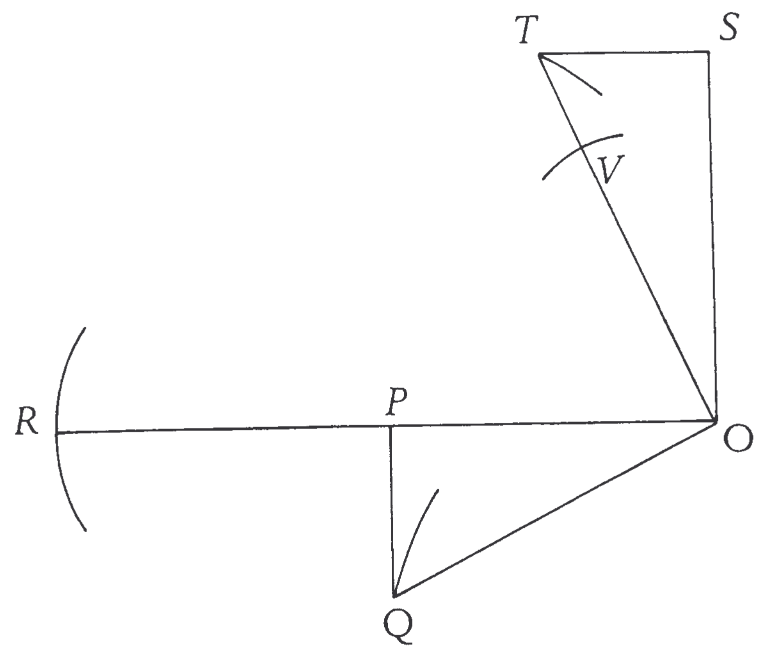


Fig. 2.

He then demonstrated a geometrical construction for finding the elastic force, in magnitude and direction, from which, with the help of the lemmas, he arrived at all the conclusions obtained by Fresnel concerning the axes of elasticity, and surface of elasticity which is an ellipsoid in a doubly refracting medium [MacCullagh, 1831; Haughton & Jellett, eds., 1880, 7 ff]. In order to find the surface of the double wave in the crystal, the ellipsoid of elasticity is cut by any diametrical plane and then two other planes are taken parallel to this section at distances from it which are third proportionals to the semiaxes and any given line k. The surface of the wave is the double surface which touches these planes in all their positions [MacCullagh, 1831; Haughton & Jellett, eds., 1880, 11].

Let ROr (here fig. 2) be the ellipsoid in question, or being perpendicular to the plane of the page. Another concentric ellipsoid is now formed about the first, with the semiaxes in the same directions but with lengths inversely proportional to those of the former, so that the rectangle under any coinciding pair of semiaxes is equal to k^2. OT is drawn perpendicular to QOq and equal to OQ, oq being perpendicular to the plane of the page, and the surface which is the locus of OT is described. The following are the principal results following from the series of geometrical lemmas proved by MacCullagh [MacCullagh, 1831; Haughton & Jellett, 1880, 11-13]: 1°, if a plane touching the second ellipsoid in Q cuts OR perpendicularly in P, then OP will be a third proportional to OR and k, i.e. $OP.OR = k^2$; 2°, if Or intersect the second ellipsoid in q, the semiaxes of the section QOq will be OQ and Oq; 3°, OS is perpendicular to the tangent plane at T and also perpendicular to the plane ROr and equal to OP.

Moreover, if OV is drawn perpendicular to QOq and is made equal to the other semiaxes Oq, and the locus of OV is then described, similar results obtain. Therefore, to find the wave surface by MacCullagh's method, one describes an ellipsoid whose axes are in the directions of the axes of elasticity, so that the squares of their lengths are directly as the elasticities in their respective directions; this ellipsoid is then cut by a plane through its centre (such as

QOq, fig. 2), and perpendicular to that plane two lengths OT and OV, equal to the semiaxes OQ and Oq of the section, respectively, are drawn. The surface of the wave is the double surface which is the locus of the points T and V.

The planes of polarisation of the rays belonging to the two parts of the wave may be also found by means of the same ellipsoid which serves to find the surface of the wave. TS, which is parallel to OR, is the direction of the vibrations of the ray OT, and the tangent plane at Q is perpendicular to OR, and therefore parallel to the plane of polarisation of OT. Similarly the plane of polarisation of the ray OV is parallel to the tangent plane at q. Hence the planes of polarisation of two different rays, having a common direction, are parallel to the planes touching the ellipsoid at the extremities of the semiaxes of the diametrical section perpendicular to their common direction [MacCullagh, 1831; Haughton & Jellett, eds., 1880, 12]. OT and OV, which are equal to QO and Oq , represent the velocities of the two such rays having a common direction in the crystal; the optic axes are the two right lines which lie in the plane of the greatest and least axes of the ellipsoid, and are perpendicular to its two circular sections. According to one of the lemmas (the sixth), the difference of the squares of the reciprocals of the velocities of the two rays, having a common direction in the crystal, is proportional to the product of the sines of the angles which that direction makes with the optic axes [2].

In his second paper of 1833 MacCullagh developed much more completely his system of reciprocal or inversive geometry; here he introduced a body of theorems concerning reciprocal points, planes and surfaces, and the ellipsoid considered above is regarded as that which "generates" the wave surface or the "biaxial surface" as he calls it [MacCullagh, 1837].

In this second paper MacCullagh introduces, as the third of a series of theorems, that which is precisely lemma 5 of the former memoir. He writes:

> This theorem is taken from a former communication to the Academy. The surface to which it relates, being the wave surface of Fresnel, is one of frequent occurence in optical inquiries, and it is therefore desirable to give it a distinctive name not derived from any physical hypothesis. I shall call it a biaxial surface, from the circumstance implied in its construction, and adopted as the definition on which the preceding theorem is founded - namely, that any pair of its coincident diameters are equal to the two axes of a central section made in the generating ellipsoid abc, by a plane perpendicular to the common direction of the two diameters. The name, perhaps may appear the more appropriate, as it reminds us of the place which the surface holds in the optical theory of biaxial crystals [MacCullagh, 1837; Haughton & Jellett, ebs., 1880, 24].

Few of MacCullagh's contemporaries seem to have been interested in or even aware of his geometrical theorems. They may have after his death been made known to a greater public when included, in the discussion on the wave surface, in the popular textbook of the Dublin Mathematician George Salmon, A Treatise on the Analytic Geometry of Three Dimensions, which also contained an account of the general theory of inversion [Salmon, 1865, 387 & 406].

MacCullagh was motivated to publish the second of his two papers by the announcement of the discovery of conical refraction early in 1833. When the discovery was first reported in the Philosophical Magazine MacCullagh wrote a "Note on the Subject of conical refraction" to the Editors, which when published caused offence to his colleague Hamilton [Haughton & Jellett, eds., 1880, 17-19]. MacCullagh writes:

> When Professor Hamilton announced his discovery of Conical Refraction, he did not seem to have been aware that it is an obvious and immediate consequence of the theorems published by me, three years ago, in the Transactions of the Royal Irish Academy, vol. xvi.,pt.ii., p. 65 & c. The indeterminate cases of my own theorems, which, optically interpreted, mean conical refraction, of course occured to me at the time; but they had nothing to do with the subject of that Paper; and the full examination of them,along with the experiments they might suggest, was reserved for a subsequent essay, which I expressed my intention of writing. Business of a different nature, however, prevented

me from following up the inquiry.

Then referring the reader to the discussion of the two concentric ellipsoids (fig. 2 here) in his first paper, he continues:

> He will see that when the section of either of the two ellipsoids employed there is a circle, the semiaxes - answering to OR, Or, and to OQ, Oq, in the general statement - are infinite in number, giving of course an infinite number of corresponding rays. And this is conical refraction.

To this he added a further explanation of the two species of conical refraction. When ROr is a circle, then any two of its rectangular radii may be taken for OR and Or. The line OS and the tangent plane perpendicular to it at S are fixed. However, the point of contact T varies because the plane in which it lies (ROS) varies with OR. The result is a curve of contact on the tangent plane of the wave surface, and a cone of rays OT arising from the same incident ray.

The three lines OQ, Or, OT, are at right angles to each other and the first two are confined to given planes; Or is always in the plane of the circle ROr, and the point Q must be in a given plane because the line OP, perpendicular to the plane that touches the ellipsoid in Q, is in a given plane ROr. In the second case, when QOq is a circle, T and V coincide in a so-called "nodal point" n, where the two sheets of the wave cross each other. At this point there are an infinite number of tangent planes, for OQ and Oq are now indeterminate. The refracted ray 'On' may be derived from any one of an infinite number of incident rays, and its polarisation will differ accordingly; for the vibrations are in the line nS from the node to the foot of the perpendicular OS on the tangent plane. The lines OP, Oq, OS are at right angles to each other and as before, the first two are confined to given planes; Oq is in the plane of the circle QOq, and OP, being perpendicular to the tangent plane at Q, must lie in a given plane. These given planes are parallel to two principle tangent planes passing through n, and touching the circle and ellipse that compose the wave section in the plane of the nodes. Finally, MacCullagh completed the examination of the two cases of conical refraction by giving a statement of the following theorem, the application of which yields conical refraction, and which he subsequently introduced and proved in his second memoir [MacCullagh, 1837; Haughton & Jellett, eds., 1880, 22].

> When three right lines at right angles to each other pass through a fixed point, in such a manner that two of them are confined to given planes, the plane of these two, in all its positions, touches the surface of a cone whose sections parallel to the given planes are parabolas; while the third right line describes another cone, whose sections parallel to the same planes are circles.

In the first case, which represents internal conical refractions, the curve of contact is a circle. In the second case, representing external conical refraction, the points S are by the theorem above also in a circle.

MacCullagh's note on conical refraction was published in the August 1833 number of the Philosphical Magazine; it almost provoked a letter in reply from Hamilton and possibly an unpleasant priority dispute; however, thanks to the mediation of Humphrey Lloyd this was avoided and MacCullagh wrote the following note, qualifiying his earlier claim, which appeared in the September number of the Philosphical Magazine:

> The introductory part of my Note which appeared in your last Number was written in haste, and I have reason to think it may not be rightly understood. You will therefore allow me to add a few observations that seem to be wanting.
>
> The principal thing pointed out in the paper published some time ago in the Transactions of the Royal Irish Academy is a very simple relation between the tangent planes of Fresnel's Wave Surface and the sections of the two reciprocal ellipsoids. Now this relation depends upon the axes of the sections, and therefore naturally suggested to me the peculiar case of circular sections in which every

> diameter is an axis. Thus a new inquiry was opened to my mind.
>
> And accordingly, without caring just then to obtain final results, which seems to be an easy matter at the time, I expressed in conversation my intention of returning to the subject of Fresnel's theory in a supplementary Paper. The design was interrupted, and I was prevented from attending to it again, until I was told that Professor Hamilton had discovered cusps and circles of contact on the wave surface. This reminded me of the cases of circular section, and the details given in my last note were immediately deduced [Haughton & Jellett, eds., 1880, 19].

The events recounted here show clearly that MacCullagh could have arrived at the prediction of conical refraction before Hamilton; however, it was certainly not clear to anybody but himself that it was "an obvious and immediate consequence" of his geometrical theorems. There can be little doubt that the physical reality of these phenomena was first comprehended by Hamilton. Perhaps, as both were working simultaneously on problems concerning the Fresnel wave surface, a little suspicion and rivalry had arisen between them. Hankins has pointed out that Hamilton not only knew of MacCullagh's 1830 paper, as was to be expected since he attended regularly at the meetings of the Royal Irish Academy where such papers were read, but that he had actually written a review of this and another paper by MacCullagh in a Dublin magazine [Hankins, 1980, 93]. The correspondence between Hamilton, Lloyd and MacCullagh regarding the priority dispute about the prediction of conical refraction has been published as an appendix in Graves' biography [Graves, 1882 - 1889, I, 685 - 692].

As far as the prediction of conical refraction is concerned we must conclude like Graves, from the evidence in the correspondenee, that Hamilton made his theoretical discovery independently and foresaw the corresponding physical facts [Graves, 1882-1889, I, 692]. Equally there can be little **doubt** that the reciprocal property of the two ellipsoids in the theory was discovered and published by MacCullagh before Hamilton's investigations had matured, and that MacCullagh's geometrical methods were simpler and clearer and more easily applied than Hamilton's involved analytical approach. It is also clear that Hamilton knew of MacCullagh' s 1830 paper and his interest in investigating the Fresnel wave surface may have been stimulated by MacCullagh's work. MacCullagh advanced far in the direction of providing a complete picture of the wave surface, but even if he comprehended all the features of the wave surface he failed to connect them with the physical phenomena.

4. Geometrical versus Analytical Methods: Hamilton and MacCullagh

Whereas Hamilton proceeded in the Third Supplement mainly by analytical reasoning his discovery of the cusps and circles of contact on the wave were essentially results of geometry, and he concluded the work with some geometrical considerations. To illustrate the theory of the wave propagated from a point in any uniform medium, he drew a comparison between this wave surface and a certain other surface described by Augustin Cauchy in the "Exercises de Mathématique" in 1830 [Cauchy, Oeuvres, 2, 9, 405-410]. Cauchy had considered the propagation of plane waves in a system of mutually attracting or repelling particles, and arrived at a relation between the normal velocity of propagation, denoted by s, and the direction cosines, the cosines of its inclination to the positive semiaxes, denoted by a,b,c.

The relation given by Cauchy is expressed as a homogeneous function (of the sixth dimension) of s, a, b, and c, where s itself is a homogeneous function of the first dimension of the direction cosines a,b,c. In Hamilton' s Third Supplement the normal velocity w is treated as a homogeneous function of dimension zero of its direction cosines. The difference in method does not affect the results, because the relation existing between the direction cosines (namely, that the sum of the squares is unity) permits the transformation, in an infinite variety of ways, of any equation into which they enter [Hamilton, 1837, 142]. Cauchy deduced an equation of the form

$F(\frac{a}{s}, \frac{b}{s}, \frac{c}{s}) = 0$, representing a surface having $\frac{a}{s}$, $\frac{b}{s}$, $\frac{c}{s}$, for its co-ordinates. According to Hamilton's method the same surface is described by the equation $\Omega(\sigma, \tau, \upsilon) = 0$; σ, τ, υ, being the "components of normal slowness". This surface ($\Omega = 0$) is designated the "surface of components of normal slowness", or simply the "surface of components". Cauchy had shown that this surface is connected with the curved wave propagated from the origin of coordinates in the unit of time (viz. V = 1, in Hamilton's notation, V being the "Characteristic Function") by the two following relations:

> 1° the product of any two corresponding radii multiplied by the cosine of the inclined angle is unity.
>
> 2° the wave is the envelope of the planes which cut perpendicularly the radii of the surface of components at distances from the centre equal to the reciprocals of these radii [Cauchy, Oeuvres, (2), 9, 410].

To these relations of Cauchy, Hamilton added the following:

> 3° the surface of components is the envelope of the planes which cut perpendicularly the radii of the wave at distances from its centre equal to the reciprocals of these radii, i.e. equal to the slowness of the rays [Hamilton, 1837, 142-132].

This third relation in Hamilton's mind is a consequence of a "general theorem of reciprocity" between surfaces, which may be stated as follows: if a surface B be deduced from another A by drawing vector rays to the latter from an arbitrary origin O, and making the lengths of these radii equal to their reciprocals without changing their directions, and seeking the envelope B of the planes perpendicular at the extremities to these altered radii of A, then, reciprocally, A may be deduced from B by a repetition of the same construction, employing the same origin O, and the same arbitrary unit of length. This is simply the theory of reciprocal surfaces outlined by MacCullagh for the case of two concentric and coaxial ellipsoids, referred to their centre as origin, and having the semi-axes of the one equal to the reciprocals of those of the other, a fact which

Hamilton duly acknowledged [Hamilton, 1837, 143]. When applied to the unit wave and the surface of components the theorem would give a new construction for the unit wave in any uniform medium, and for any law of velocity. The construction is namely, that the wave is the locus of the points obtained by letting fall all perpendiculars from the centre on the tangent planes of the surface of components and then altering the lengths of these perpendiculars to their reciprocals, without altering their directions. Hamilton explains that it was from this reciprocal theorem that he came to deduce the four circles of contact of the wave surface having once deduced the existence of the four cusps on the surface of components

> It follows from this general theory of reciprocal surfaces, that a conoidal cusp on any surface A corresponds in general to a curve of plane contact on the reciprocal B, and reciprocally; and, accordingly the cusps and circles on Fresnel's wave are connected with circles and cusps on the corresponding surface of components, which latter surface is indeed deducible from the former by merely changing the semiaxes of elasticity a b c to their reciprocals. And it was in fact by this general theorem that I was led to discover the four circles of contact on Fresnel's wave, by concluding that that wave must touch four planes in curves instead of points of contact, as soon as I had perceived the existence of four conoidal cusps on the surface of components, by obtaining ... the formula ..., which is the approximate equation of such a cusp. I easily found also that there were only four such cusps on each of the two reciprocal surfaces, and therefore concluded that there were only four curves of plane contact on each. I may mention that though I have taken care to attribute to Mr. Cauchy the discovery of the surface of components, yet before I met the *Exercises de Mathématique*, I was familiar, in my own investigations, with the existence and with the foregoing properties of this surface: it is indeed suggested by the first principles of my view of optics, since it constructs the fundamental partial differential equation
>
> $$\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z} \right) = 0$$
>
> which my characteristic function V must satisfy in a final uniform medium [Hamilton, 1837, 144].

This Third Supplement is concluded with the enunciation of a new geometrical construction for the determination of a reflected or refracted ray, which follows from the general equations of reflection or refraction given in the Supplement, viz.$\Delta\sigma = 0$,$\Delta\tau = 0$,

> the corresponding points (σ, τ, υ, and $\sigma + \Delta\sigma$, $\tau + \Delta\tau$, $\upsilon + \Delta\upsilon$) upon the surface or surfaces of components ($0 = \Omega$, $0 = \Omega + \Delta\Omega$) before and after any reflection or refraction ordinary or extraordinary, are situated on one common perpendicular to the plane which touches the reflecting or refracting surface at the point of reflection or refraction; a new geometrical relation, which gives a new and general construction to determine a reflected or refracted ray, simpler in many cases than the construction proposed by Huyghens [Hamilton, 1837, 144].

This is of course the construction which Julius Plücker called "Hamilton's construction". It comes, together with the constructions suggested by MacCullagh and Plücker, as the final refinement of the construction conceived by Huyghens 140 years earlier. The crowning achievement of the method was the discovery of the cusps and the circles of contact on the wave surface from which Hamilton predicted the two species of conical refraction.

The depth, elegance and rigour of Hamilton's analytical methods in the investigation of the Fresnel wave surface can be contrasted with the similar qualities of MacCullagh's approach through geometry. Hamilton was one of the most formidable figures in algebra in the mid-nineteenth century, whereas MacCullagh is remembered mainly for his contributions to physical optics. Samuel Haughton and John Hewitt Jellett, two of MacCullagh's pupils who edited his Collected Works after his death, have stressed the geometrical content of his optical investigations. They have also commented on the apparent discrepancy between the volume of his constributions in physical optics and geometry; Mac Cullagh published only four papers in the domain of pure geometry whereas his output in physical optics and rational mechanics made up twenty six of the contributions included in his Collected Works. Haughton and Jellett were at pains to point out, however, that the discrepancy was not as great as might appear at first sight. A considerable part of his optical researches, especially those from the early 1830's, really could be classed as geometry and thus the inequality was not as great as would appear from an examination of the titles of his papers [Haughton & Jellett, eds., 1880, i-ix]. Mac Cullagh's early geometrical papers were undoubtedly inspired by his studies of the theory of double refraction and especially of the Fresnel wave surface. But whatever the source of inspiration of MacCullagh's geometry, and this forms the principal conclusion of the present study, its ultimate success lay in its application in physics, in particular in the elucidation of the Fresnel wave surface.

NOTES

1. This letter is preserved among the Airy-Lloyd correspondence relating to the establishment of magnetic observatories at Greenwich and in the colonies, which were in part modelled on Gauss' observatory at Göttingen [Archives of the Royal Greenwich Observatory, Airy Manuscripts, No. 1250].

2. This law, MacCullagh maintained, had been discovered by Jean Baptiste Biot and David Brewster.

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