Drawing K_n in Three Dimensions with One Bend per Edge

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Abstract. We give a drawing of K_n in 3D in which vertices are placed at integer grid points and edges are drawn crossing-free with at most one bend per edge in a volume bounded by $O(n^{2.5})$.

1 Introduction

Drawing graphs in three dimensions has been considered by several authors in the graph-drawing field under a variety of models. One natural model is to draw vertices as points at integer-valued grid points in a 3D Cartesian coordinate system and represent edges as straight line segments between adjacent vertices with no pair of edges intersecting. The *volume* of such a drawing is typically defined in terms of a smallest bounding box containing the drawing and with sides orthogonal to one of the coordinate axes. If such a box B has width w, length l and height h, then we refer to the *dimensions* of B as $(w+1) \times (l+1) \times (h+1)$ and define the volume of B as $(w+1) \cdot (l+1) \cdot (h+1)$.

It was shown by Cohen *et al.* [3] that it is possible to draw *any* graph in this model, and indeed the complete graph K_n is drawable within a bounding box of volume $\Theta(n^3)$. Restricted classes of graphs may however be drawn in smaller asymptotic volume. For example, Calamonieri and Sterbini [2] showed that 2-, 3-, and 4-colourable graphs can be drawn in $O(n^2)$ volume. Pach *et al.* [11] showed a volume bound of $\Theta(n^2)$ for r-colourable graphs (r a constant). Dujmović *et al.* [4] investigated the connection of bounded tree-width to 3D layouts. Felsner *et al.* [8] showed that outerplanar graphs can be drawn in $O(n)$ volume. Establishing tight volume bounds for planar graphs remains an open problem. Dujmović and Wood [5] showed an upper bound of $O(n^{1.5})$ on the volume of planar graphs at Graph Drawing 2003.

In 2-dimensional graph drawing, the effect of allowing bends in edges has been well studied. For example, Kaufmann and Wiese [9] showed that all planar graphs can be drawn with only 2 bends per edge and all vertices located on a straight line.

The consequences of allowing bends in 3D has received less attention. Note that bend points must also occur at integer grid points. Bose *et al.* [1] showed

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that the number of edges in a graph provides an asymptotic lower bound on the volume regardless of the number of bends permitted, thus establishing $\Omega(n^2)$ as the lower bound on the volume for K_n. This lower bound was explicitly achieved by Dyck *et al.* [7] who presented a construction with at most 2 bends per edge. The upper bound is also a consequence of a more general result of Dujmović and Wood [6]. In [10], Morin and Wood presented a one-bend drawing of K_n that achieves $O(n^3/\log^2 n)$ volume. It is the gap between this result and the $\Omega(n^2)$ lower bound that motivates this paper; we improve the Morin and Wood result to achieve a one-bend drawing with volume $O(n^{2.5})$.

2 Preliminaries

We call the axes of our 3D Cartesian coordinate system respectively X, Y and Z.

The one-bend construction of K_n by Morin and Wood [10] considers $O(\log n)$ packets of $O(\frac{n}{\log n})$ collinear vertices. All the vertices lie in the XY-plane and edges joining vertices of different packets lie above this plane. Edges joining vertices within a packet lie below this plane and the volume of these (complete) subgraphs is a consequence of the following lemma.

Lemma 1 ([10]). *For all $q \geqslant 1$, K_m has a one-bend drawing in an axis-parallel box of size $q \times m \times \left\lceil \frac{\pi^2}{3} \frac{m^2}{q} \right\rceil$ with all the vertices on the Y-axis.*

Indeed, $\Omega(n^3)$ volume is required for a collinear one-bend drawing of K_n as shown by Morin and Wood. We present here a brief description of the construction behind Lemma 1 because we will use it in our construction. The edges are divided into $\Theta(m^2)$ chains of edges (i.e., sequences of edges). A chain connects all vertices with index equal to i modulo j such that the vertices on the chain are ordered with increasing indices. In each chain, the bends are placed on a line parallel to the Y-axis through a point of integer coordinates (x, z) in the XZ-plane. The chains thus lie in planes that contain the Y-axis (where the vertices lie). In the XZ-plane, the points (x, z) are chosen so that they are all strictly visible from the origin. The well-known fact that there are $\Theta(m^2)$ such choices in a rectangle of size $q \times \frac{m^2}{q}$ ensures that all the $\Theta(m^2)$ chains can be placed in distinct planes, and thus that the edges do not cross.

3 The construction

Our construction is roughly as follows. We split the n vertices into k packets of $\frac{n}{k}$ vertices, where all vertices in one packet have the same X and Z coordinates. All edges of the complete graph contain a bend. All edges joining two vertices of one packet are placed below and right (positive X direction) of the packet, and all edges joining two vertices of different packets are placed "above". We present our construction for an arbitrary k and show later that the volume of the bounding box of the drawing is minimized for $k = n^{\frac{1}{4}}$. We assume for simplicity that $\frac{n}{k}$ is an integer.

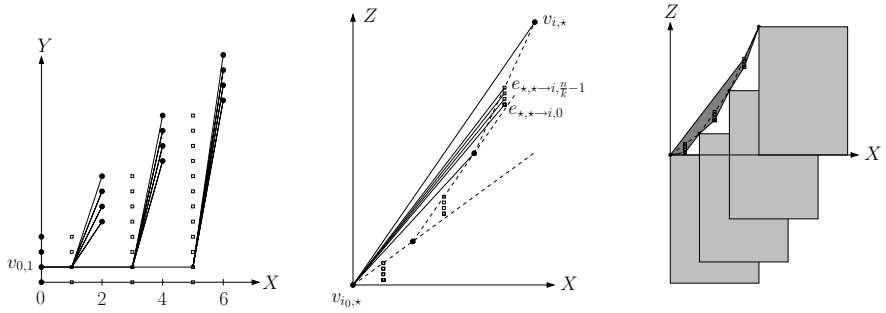

Fig. 1. (a) Projection on the XY-plane of the vertices (dots), the bends (squares), and the edges leaving vertex $v_{0,1}$. Here, $n = 16$ and $k = 4$. (b) Construction of the Z-coordinates. (c) XZ-projection of the bounding boxes of the interpacket edges.

X and Y coordinates of the vertices. We first describe the X and Y-coordinates of the vertices. Refer to Figure 1(a). All vertices have different Y-coordinates, ranging between 0 and $n-1$. We divide the set of n vertices into k packets, denoted $V_0, \ldots, V_{k-1}$, of $\frac{n}{k}$ vertices. All vertices in the same packet V_i have the same X-coordinate $2i$, the same Z-coordinate z_i (defined later), and consecutive Y-coordinates. Precisely, the j-th vertex in the i-th packet, denoted $v_{i,j}$, has coordinates $(2i, i\frac{n}{k} + j, z_i)$, with $0 \leqslant j \leqslant \frac{n}{k} - 1$ and $0 \leqslant i \leqslant k-1$.

Edges joining vertices of one packet. Since all the vertices of one packet are collinear, we can draw the complete graph on these vertices using the $q \times m \times O(m^2/q)$ volume construction of [10] described in Section 2. In that construction there are m collinear vertices; here we have $m = \frac{n}{k}$ vertices. We choose $q = k$ and draw the edges so that the bounding box of this complete subgraph is below (negative Z) and to the right (positive X) of the vertices. Notice that we have chosen $q = k$ so that these complete subgraphs do not asymptotically increase the width of the final drawing. *In the sequel of the construction, we only consider edges that join vertices of distinct packets, and their bends.*

X and Y coordinates of the bends. Refer to Figure 1(a). The bend of an edge joining vertex v_{i_1,j_1} to vertex v_{i_2,j_2}, with $i_1 < i_2$, is denoted $e_{i_1,j_1\to i_2,j_2}$. It separates the edge into two distinct segments, the *outgoing* segment which starts at v_{i_1,j_1} and ends at $e_{i_1,j_1\to i_2,j_2}$, and the *incoming* segment which starts with $e_{i_1,j_1\to i_2,j_2}$ and ends at v_{i_2,j_2}.

A bend $e_{i_1,j_1\to i_2,j_2}$ has coordinates $(2i_2 - 1, i_1\frac{n}{k} + j_1, z_{i_2,j_2})$, that is, its X-coordinate is one less than the X-coordinate of v_{i_2,j_2}, its Y-coordinate is the same as for v_{i_1,j_1}, and its Z-coordinate, which only depends on v_{i_2,j_2}, will be defined later.

Z-coordinates of the vertices and bends. We will assign values to z_i and $z_{i,j}$ so that edges do not cross. In fact, our construction is designed to verify the following lemma. In the following we consider the projection on the XZ-

plane of the vertices $v_{i,\star}$ and bends $e_{\star,\star\to i,j}$ where $\star$ can take any value since the projected points are identical.

Lemma 2. *Projected onto the XZ-plane, the polar ordering $\prec_{i_0}$ viewed from a vertex $v_{i_0,\star}$ of the vertices $v_{i,\star}$ and bends $e_{\star,\star\to i,j}$ with $i_0 < i < k$ and $0 \leqslant j < \frac{n}{k}$ satisfies $v_{i-1,\star} \prec_{i_0} e_{\star,\star\to i,0} \prec_{i_0} \ldots \prec_{i_0} e_{\star,\star\to i,j} \prec_{i_0} \ldots \prec_{i_0} e_{\star,\star\to i,\frac{n}{k}-1} \preceq_{i_0} v_{i,\star}$.*

Our construction is as follows. First, let $z_0 = 0$ and $z_{1,j} = j + 1$, then z_1 is chosen such that $v_{1,\star}$ is at the same polar angle about $v_{0,\star}$ as $e_{0,\star\to 1,\frac{n}{k}-1}$, which gives $z_1 = 2z_{1,\frac{n}{k}-1} = 2\frac{n}{k}$ (see Figure 1(b)).

Assume now that we have placed vertices and bends up to index i. To get a correct polar ordering around $v_{i-1,\star}$ we need to have the next bends above the line through $v_{i-1,\star}$ and $v_{i,\star}$ thus we place the next bend at $z_{i+1,0} = z_i + \frac{1}{2}(z_i - z_{i-1}) + 1$ and the following bends on edges going to $v_{i+1,\star}$ at $z_{i+1,j} = z_i + \frac{1}{2}(z_i - z_{i-1}) + 1 + j$.

The vertex $v_{i+1,\star}$ is placed at the same polar angle about $v_{i,\star}$ as $e_{\star,\star\to i+1,\frac{n}{k}-1}$ which gives $z_{i+1} = z_i + 2(z_{i+1,\frac{n}{k}-1} - z_i) = z_i + 2(\frac{z_i - z_{i-1}}{2} + \frac{n}{k}) = 2z_i - z_{i-1} + 2\frac{n}{k}$; solving this recurrence[1] yields $z_i = i(i+1)\frac{n}{k}$. Then we obtain $z_{i,j} = z_{i-1} + \frac{1}{2}(z_{i-1} - z_{i-2}) + 1 + j = (i-1)(i+1)\frac{n}{k} + 1 + j$. To summarize, the coordinates of the vertices and bends are

$$v_{i,j} = \left(2i,\ i\frac{n}{k} + j,\ i(i+1)\frac{n}{k} \right)$$
$$e_{i_1,j_1\to i_2,j_2} = \left(2i_2 - 1,\ i_1\frac{n}{k} + j_1,\ (i_2^2 - 1)\frac{n}{k} + 1 + j_2 \right)$$

Proof of Lemma 2: The correct polar ordering of the $v_{i,\star}$ viewed from $v_{i_0,\star}$ is guaranteed since all these points are ordered on a convex curve (i.e. a parabola). Let L_i be the line through $v_{i,\star}$ and $v_{i+1,\star}$. The correct polar ordering of $v_{i,\star}$, the $e_{\star,\star\to i+1,j}$ and $v_{i+1,\star}$, viewed from $v_{i,\star}$, comes directly from the construction; moreover, this ordering is the same for all viewpoints $v_{i_0,\star}$, $i_0 < i$, since these viewpoints lie above L_{i-1} (see Figure 1(b)). ■

4 Proof of correctness

We say that two edges cross if their relative interiors intersect. We prove in this section that no two edges of our construction cross. We first show that the edges joining vertices within the same packet induce no crossing. Then, we show that there is no crossing between two outgoing segments, two incoming segments, and finally one outgoing and one incoming segment.

Edges joining vertices within packets. We use the same technique as in the Morin-Wood construction [10] to ensure that no two edges joining vertices within a given packet cross. An edge joining vertices within a packet crosses no other

[1] An inductive verification is easy since with this formula we have:
$2z_i - z_{i-1} + 2\frac{n}{k} = \frac{n}{k}[2i(i+1) - i(i-1) + 2] = \frac{n}{k}[i^2 + 3i + 2] = (i+1)(i+2)\frac{n}{k} = z_{i+1}$.

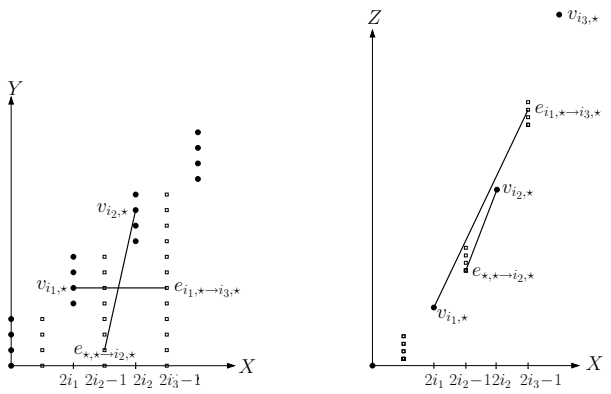

Fig. 2. Incoming and outgoing segments in XY and XZ-projections.

edge joining vertices within another packet since the projection onto the Y-axis of the bounding boxes of the Morin-Wood constructions do not intersect. Finally, the bounding box of the Morin-Wood construction properly intersects no edge joining distinct packets since they do not properly intersect in XZ-projection (see Fig. 1(c)). Hence, edges joining vertices within a packet cross no other edge.

Outgoing-outgoing segments. If two outgoing segments start from different vertices, they lie in two different planes parallel to the XZ-plane. Otherwise, by Lemma 2, the two segments only share their starting point. Hence no two outgoing segments cross.

Incoming-incoming segments. Note that an incoming segment joining $e_{\star,\star\to i,j}$ to $v_{i,j}$ lies in the plane $P_{i,j}$ through the two lines parallel to the Y-axis and containing, respectively, all the $e_{\star,\star\to i,j}$ and all the $v_{i,\star}$. For a pair of incoming segments, we consider three cases according to whether both segments finish at the same vertex, at distinct vertices of the same packet, or at vertices of different packets. In the first case, the segments live in a plane $P_{i,j}$; they start at different bends and end at the same vertex, hence they do not cross. In the second case, the two segments live in two planes $P_{i,j}$ and $P_{i,j'}$ whose intersection is the line $v_{i,j}v_{i,j'}$. The segments end there and thus cannot cross. In the third case, the segments do not overlap in the X-direction, thus they do not cross.

Incoming-outgoing segments. Consider an outgoing segment joining vertex $v_{i_1,\star}$ to bend $e_{i_1,\star\to i_3,\star}$ and an incoming segment joining bend $e_{\star,\star\to i_2,\star}$ to vertex $v_{i_2,\star}$, where $\star$ can be any value (see Figure 2). The ranges over the X-axis of the two segments are $[2i_1, 2i_3-1]$ and $[2i_2-1, 2i_2]$. They overlap only if $i_1 < i_2 < i_3$, and, in such a case, Lemma 2 yields that, viewed from $v_{i_1,\star}$, the points satisfy the polar ordering $e_{\star,\star\to i_2,\star} \preceq_{i_1} v_{i_2,\star} \prec_{i_1} e_{i_1,\star\to i_3,\star}$. This implies that, in projection onto the XZ-plane, points $e_{\star,\star\to i_2,\star}$ and $v_{i_2,\star}$ are below the line segment joining $v_{i_1,\star}$ and $e_{i_1,\star\to i_3,\star}$. Hence the two segments do not cross.

5 Volume analysis

The dimension of the bounding box of our construction for edges between packets has size smaller than $2k \times n \times kn$ since the highest vertex has Z-coordinate $z_{k-1} = k(k-1)\frac{n}{k}$. The complete subgraphs within packets have size $k \times \frac{n}{k} \times \left\lceil \frac{\pi^2}{3} \frac{n^2}{k^3} \right\rceil$ and thus our complete construction fits in a box of size $O(k) \times n \times O(\frac{n^2}{k^3} + kn)$. To balance the increasing and decreasing terms of the Z-dimension we choose k such that $\frac{n^2}{k^3} = kn$ that is $k = n^{\frac{1}{4}}$. Recall that we assumed for simplicity that k and $\frac{n}{k}$ were integers; for any n we can apply our construction with $\lceil n^{\frac{1}{4}} \rceil^4$ vertices and then remove the extra vertices and edges. We thus have the following result.

Theorem 1. *Every complete graph K_n has a one-bend drawing in an axis-parallel box of dimensions $O(n^{\frac{1}{4}}) \times n \times O(n^{\frac{5}{4}})$ and volume $O(n^{2.5})$.*

Remark 1. An alternative for applying Lemma 1 with boxes that match the X-dimension of our construction (choosing $q = k$ with $m = \frac{n}{k}$) is to take boxes whose size matches the Z-dimension of our construction (choosing $q = kn$ with $m = \frac{n}{k}$). Then the dimension of the bounding box of the Morin-Wood construction for interpacket edges is $O(\frac{n^2}{k^2} \cdot \frac{1}{kn}) \times \frac{n}{k} \times kn$ which gives a total size for our construction of $O(k + \frac{n}{k^3}) \times n \times 2kn$. This is still optimal for $k = n^{\frac{1}{4}}$ but it offers a trade-off between volume and aspect ratio of the box for $k \in [1, n^{\frac{1}{4}}]$.

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