Quadratic Quantum Hamiltonians revisited

Monique Combescure
IPNL, Bâtiment Paul Dirac
4 rue Enrico Fermi, Université Lyon-1
F.69622 VILLEURBANNE Cedex, France
`monique.combescure@ipnl.in2p3.fr`

Didier Robert
Département de Mathématiques
Laboratoire Jean Leray, CNRS-UMR 6629
Université de Nantes, 2 rue de la Houssinière,
F-44322 NANTES Cedex 03, France
`Didier.Robert@math.univ-nantes.fr`

Abstract

Time dependent quadratic Hamiltonians are well known as well in classical mechanics and in quantum mechanics. In particular for them the correspondance between classical and quantum mechanics is exact. But explicit formulas are non trivial (like the Mehler formula). Moreover, a good knowlege of quadratic Hamiltonians is very useful in the study of more general quantum Hamiltonians and associated Schrödinger equations in the semiclassical regime.

Our goal here is to give our own presentation of this important subject. We put emphasis on computations with Gaussian coherent states. Our main motivation to do that is application concerning revivals and Loschmidt echo.

1 Introduction

This paper is a survey concerning exact useful formulas for time dependent Schrödinger equations with quadratic Hamiltonians in the phase space. One of our motivations is to give a detailed proof for the computation of the Weyl symbol of the propagator. This formula was used recently by Melhig-Wilkinson [17] to suggest a simpler proof of the Gutzwiller trace formula [3].
There exist many papers concerning quantum quadratic Hamiltonians and exact formulas. In 1926, Schrödinger [21] has already remarked that quantification of the harmonic (or Planck) oscillator is exact.
The best known result in this field is certainly the Melher formula for the harmonic oscillator (see for example [5]).

Quadratic Hamiltonians are very important in partial differential equations on one side because they give non trivial examples of wave propagation phenomena and in quantum mechanics and on the otherside the propagation of coherent states by general classes of Hamiltonians, including $-\hbar^2\triangle + V$, can be approximate modulo $O(\hbar^\infty)$ by evolutions of quadratic time dependent Hamiltonians [1, 19, 14].

In his works on pseudodifferential calculus, A. Unterberger in [22, 23] has given several explicit formulas connecting harmonic oscillators, Gaussian functions and the symplectic group. This subject was also studied in [8, 13, 12]. More recently de Gosson [9] has given a different approch for a rigorous proof of the Melhig-Wilkinson formula for metaplectic operators, using his previous works on symplectic geometry and the metaplectic group.
Here we shall emphasis on time dependent quadratic Schrödinger equation and Gaussian Coherent States. It is well known and clear that this approach is in the heart of the subject and was more or less present in all papers on quantum quadratic Hamiltonians. In this survey, we want to give our own presentation of the subject and cover most of results appearing in particular [13, 7].
Our main motivation to revisit this subject was to prepare useful tools for applications to revivals and quantum Loschmidt echo [2]. We shall see that in our approach computations are rather natural direct and explicit.

2 Weyl quantization. Facts and Notations

Let us first recall some well known facts concerning Weyl quantization (for more details see [11, 18]).
The Planck constant $\hbar > 0$ is fixed (it is enough to assume $\hbar = 1$ in the homogeneous quadratic case).
The Weyl quantization is a continuous linear map, denoted by $\hat{\bullet}$ or by Op^w, defined on the temperate Schwartz space distribution $\mathcal{S}'(\mathbb{R}^{2n})$ into $\mathcal{L}_w(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ where $\mathcal{L}_w(E, F)$ denotes the linear space of continuous linear map from the linear topological E into the linear topological F with the weak topology.
$\mathbb{R}^{2n}$ is a symplectic linear space with the canonical symplectic form $\sigma(X, Y) = JX \cdot Y$

where J is defined by

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

and $\mathbb{1}_n$ denotes the identity $n \times n$ matrix.
Let us introduce the symplectic group $\mathrm{Sp}(2n)$: it is the set of linear transformations of $\mathbb{R}^{2n}$ preserving the 2-form σ.
For $X \in \mathbb{R}^{2n}$, we denote $X = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$.
The Weyl quantization is uniquely determined by the following conditions:

(W0) $A \mapsto \hat{A}$ is continuous.

(W1) $(\hat{q}\psi)(q) = q\psi(q)$, $(\hat{p}\psi)(q) = D_q\psi(q)$, $(D_q = \frac{\nabla_q}{i})$
for every $\psi \in \mathcal{S}(\mathbb{R}^n)$.

(W2) $\exp[i(\alpha \cdot \widehat{q + \beta} \cdot p)] = \exp[i(\alpha \cdot \hat{q} + \beta \cdot \hat{p})], \qquad \forall\, \alpha, \beta \in \mathbb{R}^n$

Let us remark that $\alpha \cdot \hat{q} + \beta \cdot \hat{p}$ is self-adjoint on $L^2(\mathbb{R}^n)$ so, $\exp[i(\alpha \cdot \hat{q} + \beta \cdot \hat{p})]$ is unitary. In particular, if $z = (x, \xi)$ then

$$\hat{T}(z) =: \exp[\frac{i}{\hbar}(\xi \cdot \hat{q} - x \cdot \hat{p})$$

is the quantized translation by z in the phase space (Weyl operators).
From (W1), (W2), using continuity and $\hbar$-Fourier transform, defined by

$$\tilde{A}(Y) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{i}{\hbar}X\cdot Y} A(X)dX$$

we have

$$\hat{A} = (2\pi\hbar)^{-2n} \int_{\mathbb{R}^{2n}} \tilde{A}(\alpha, \beta) \exp[\frac{i}{\hbar}(\alpha \cdot \hat{q} + \beta \cdot \hat{p})] d\alpha d\beta \tag{2.1}$$

In general, equality (2.1) is only defined in a weak sense i.e through the duality bracket between $\mathcal{S}'$ and $\mathcal{S}$, $< \hat{A}\varphi, \psi >$ for arbitrary $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

Definition 2.1 *A is the Weyl contravariant symbol of $\hat{A}$ if they are related through the formula (2.1).*

Using the explicit action of Weyl operators on $\mathcal{S}(\mathbb{R}^n)$ and Fourier analysis, we get the following formula (see [11, 18])

$$(\hat{A}\varphi)(x) = (2\pi\hbar)^{-n} \iint_{\mathbb{R}^{2n}} \exp[\frac{i}{\hbar}(x - y) \cdot \xi] A\left(\frac{x+y}{2}, \xi\right) \varphi(y) dy d\xi. \tag{2.2}$$

In particular, if K_A denotes the Schwartz kernel of $\hat{A}$, we have

$$A(q,p) = \int_{\mathbb{R}^n} \exp[-\frac{i}{\hbar} u \cdot p] K_A \left(q + \frac{u}{2}, q - \frac{u}{2}\right) du \tag{2.3}$$

These formulas are true in the distribution sense in general, and pointwise if $\hat{A}$ is smoothing enough (surely if for example $A \in \mathcal{S}(\mathbb{R}^n)$).
Let us remark that they are consistent and that $A \mapsto \hat{A}$ is a bijection from $\mathcal{S}'(\mathbb{R}^{2n})$ into $\mathcal{L}_w(\mathcal{S}(\mathbb{R}^{2n}), \mathcal{S}'(\mathbb{R}^{2n}))$. In particular we have the following inversion formula

Proposition 2.2 *For every $\hat{A} \in \mathcal{L}_w(\mathcal{S}(\mathbb{R}^{2n}), \mathcal{S}'(\mathbb{R}^{2n}))$, there exists a unique contravariant Weyl symbol $A \in \mathcal{S}'(\mathbb{R}^{2n})$ given by the following formula*

$$A(X) = 2^n \mathrm{Tr}[\hat{A} S_{ym}(X)] \tag{2.4}$$

where $S_{ym}(X)$ is the unitary operator in $L^2(\mathbb{R}^n)$ defined by

$$S_{ym}(X)\varphi(q) = (\pi\hbar)^{-n} \mathrm{e}^{\frac{-2i\xi}{\hbar}(x-q)} \varphi(2x - q)$$

for $X = (x, \xi)$.

Sketch of proof:

We first prove the formula for $\hat{A} \in \mathcal{L}_w(\mathcal{S}'(\mathbb{R}^{2n}), \mathcal{S}(\mathbb{R}^{2n}))$.The general case follows by duality and density.
We start with the following formula, easy to prove if $A, B \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\mathrm{Tr}[\hat{A}\hat{B}] = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A(X) B(X) dX \tag{2.5}$$

Assume now that $\hat{B}$ is a bounded operator in $L^2(\mathbb{R}^n)$. The Weyl symbol B of $\hat{B}$ satisfies:

$$\mathrm{Tr}[\hat{A}\hat{B}] = (2\pi\hbar)^{-n} < A, B > \text{[1]} \tag{2.6}$$

we have to check that the Weyl symbol of $S_{ym}(X)$ is $(\pi\hbar)^n \delta_X$ where δ_X is the Dirac mass in X.
To prove that let us consider any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and denote by W_φ the Weyl symbol of the projector $\psi \mapsto < \psi, \varphi > \varphi$. W_φ is called the Wigner function of φ. A direct computation using (2.3) gives

$$W_\varphi(q,p) = \int_{\mathbb{R}^n} \exp[-\frac{i}{\hbar} u \cdot p] \varphi(q + \frac{u}{2}) \overline{\varphi(q - \frac{u}{2})} du$$

We get the proposition applying formula (2.6) with $\hat{B} = S_{ym}(X)$ and $\hat{A} = \pi_\varphi$.$\square$

We shall see later that it may be convenient to introduce the covariant Weyl symbol for $\hat{A}$ which has a nice connection with Weyl translations.

[1] the bracket $<,>$ denotes the usual bilinear form (integral or distribution pairing). We shall denote $\langle \bullet | \bullet \rangle$ the Hermitean sesquilinear form on Hilbert spaces, linear in the second argument.

Proposition 2.3 *For every $\hat{A} \in \mathcal{L}_w(\mathcal{S}(\mathbb{R}^{2n}), \mathcal{S}'(\mathbb{R}^{2n}))$ there exists a unique temperate distribution $A^{\#}$ on $\mathbb{R}^{2n}$, named covariant Weyl symbol of $\hat{A}$, such that*

$$\hat{A} = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^{\#}(X)\hat{T}(X)dX \tag{2.7}$$

Moreover we have the inverse formula

$$A^{\#}(X) = \mathrm{Tr}[\hat{A}\hat{T}(-X)] \tag{2.8}$$

(As above, if $\hat{A}$ is not trace-class, this formula has to be interpreted in a weak distribution sense).
The covariant and contravariant Weyl symbols are related with the following formula

$$A^{\#}(X) = (2\pi\hbar)^{-n}\tilde{A}(JX). \tag{2.9}$$

$\tilde{A} \circ J$ is named the symplectic Fourier transform of A

Proof
These properties are not difficult to prove following for example [20].□

We define the (usual) Gaussian coherent states φ_z as follows:

$$\varphi_z = \hat{T}(z)\varphi_0, \qquad \forall\ z \in \mathbb{R}^{2n} \tag{2.10}$$

where

$$\varphi_0(x) := (\pi\hbar)^{-n/4}e^{-x^2/2\hbar} \tag{2.11}$$

We get the following useful formula for the mean value of observables:

Corollary 2.4 *With the above notations, for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\langle\varphi|\hat{A}\psi\rangle = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^{\#}(X)\langle\varphi|\hat{T}(X)\psi\rangle dX \tag{2.12}$$

In particular for Gaussian Coherent States, we have

$$\langle\varphi_z|\hat{A}\varphi_0\rangle = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^{\#}(X)\exp\left(\frac{-|X-z|^2}{4\hbar} - \frac{i}{2\hbar}\sigma(X,z)\right)dX \tag{2.13}$$

Proof:
The first formula is a direct consequence of definition for covariant symbols.
The second formula is a consequence of the first and the following easy to prove equalities

$$\begin{aligned} \hat{T}(z)\hat{T}(z') &= \exp\left(\frac{i\sigma}{2\hbar}(z,z')\right)\hat{T}(z+z') & (2.14)\\ \langle\varphi_z|\varphi_0\rangle &= \mathrm{e}^{-\frac{|z|^2}{4\hbar}} & (2.15) \end{aligned}$$

□

For later use let us recall the following

Definition 2.5 *Let be $\hat{A}, \hat{B} \in \mathcal{L}_w(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ such that the operator composition $\hat{A}\hat{B}$ is well defined. Then the Moyal product of A and B is the unique $A\#B \in \mathcal{S}'(\mathbb{R}^{2n})$ such that*

$$\hat{A}\hat{B} = \widehat{A\#B} \tag{2.16}$$

For details concerning computations rules and properties of Moyal products see [11, 18].

3 Time evolution of Quadratic Hamiltonians

In this section we consider a quadratic time-dependent Hamiltonian, $H_t(z) = \sum_{1\leq j,k\leq 2n} c_{j,k}(t) z_j z_k$, with real and continuous coefficients $c_{j,k}(t)$, defined on the whole real line for simplicity. It is convenient to consider the symplectic splitting $z = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and to write down $H_t(z)$ as

$$H_t(q,p) = \frac{1}{2}\left(G_t q\cdot q + 2L_t q\cdot p + K_t p\cdot p\right)$$

where K_t, L_t, G_t are real $n\times n$ matrices, K_t and G_t being symmetric.
The classical motion in the phase space is given by the linear equation

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J.\begin{pmatrix} G_t & L_t^T \\ L_t & K_t \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}, \tag{3.17}$$

where L^T is the transposed matrix of L. This equation defines a flow, F_t (linear symplectic transformations) such that $F_0 = \mathbb{1}$. On the quantum side, $\widehat{H_t}$ is a family of self-adjoint operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ (this will be proved later). The quantum evolution follows the Schrödinger equation, starting with an initial state $\varphi \in \mathcal{H}$.

$$i\hbar\frac{\partial\psi_t}{\partial t} = \widehat{H_t}\psi_t, \quad \psi_{t_0} = \varphi \tag{3.18}$$

Suppose that we have proved existence and uniqueness for solution of (3.18), we write $\psi_t = \widehat{U}_t\varphi$. The correspondence between the classical evolution and quantum evolution is exact. For every $A \in \mathcal{S}(\mathbb{R}^{2n})$, we have

Proposition 3.1

$$\widehat{U}_t.\widehat{A}.\widehat{U_t^{-1}} = \widehat{A.F_t} \tag{3.19}$$

Sketch of proof
For any quadratic Weyl symbol B we have the exact formula

$$\frac{i}{\hbar}[\widehat{B}, \widehat{A}] = \widehat{\{B, A\}} \tag{3.20}$$

where $\{B,A\} = \nabla B\cdot J\nabla A$ denote the Poisson bracket, and $[\hat{B},\hat{A}] = \hat{B}.\hat{A} - \hat{A}.\hat{B}$ is the Moyal bracket. So we can prove Proposition (3.1) by taking derivative in time and using (3.20) (see [18] for more details). ∎

Now we want to compute explicitly the quantum propagator $U_{t_0,t}$ in terms of classical evolution of H_t.
One approach is to compute the time evolution of Gaussian coherent sta(es, $\widehat{U}_{t_0,t}\varphi_z$, or in other word to solve the Schrödinger equation (3.18) with $\varphi = \varphi_z$, the Gaussian coherent state in $z \in \mathbb{R}^n$. Let us recall that $\varphi_z = \hat{T}(z)\varphi_0$ and $\varphi_0(x) = (\pi\hbar)^{-n/4}\exp\left(\frac{-|x|^2}{2\hbar}\right)$.

4 Time evolution of Coherent States

The coherent states system $\{\varphi_z\}_{z\in\mathbb{R}^{2n}}$ introduced before is a very convenient tool to analyze properties of operators in $L^2(\mathbb{R}^n)$ and their Schwartz distribution kernel. To understand that let us underline the following consequence of the Plancherel Formula for the Fourier transform. In all this section we assume $\hbar = 1$. For every $u \in L^2(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |u(x)|^2 dx = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} |\langle u, \varphi_z\rangle|^2 dz \tag{4.21}$$

Let $\hat{R}$ be some continuous linear opertor from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ and K_R its Schwartz distribution kernel. By an easy computation, we get the following representation formula

$$K_R(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (\hat{R}\varphi_z)(x)\overline{\varphi_z(y)}dz. \tag{4.22}$$

In other words we have the following continuous resolution of the Schwartz distribution kernel of the identity

$$\delta(x-y) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \varphi_z(x)\overline{\varphi_z(y)}dz.$$

This formula explains why the Gaussian coherent system may be an efficient tool for analysis of operators on the Euclidean space $\mathbb{R}^n$.
Let us consider first the harmonic oscillator

$$\hat{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \tag{4.23}$$

It is well known that for $t \neq k\pi,\ k \in \mathbb{Z}$ the quantum propagator $\mathrm{e}^{-it\hat{H}}$ has an explicit Schwartz kernel $K(t; x, y)$ (Mehler formula).
It is easier to compute with the coherent states φ_z. φ_0 is an eigenstate of $\hat{H}$, so we have

$$\mathrm{e}^{-it\hat{H}}\varphi_0 = \mathrm{e}^{-it/2}\varphi_0 \tag{4.24}$$

Let us compute $\mathrm{e}^{-it\hat{H}}\varphi_z$, $\forall z \in \mathbb{R}^2$, with the following ansatz

$$\mathrm{e}^{-it\hat{H}}\varphi_z = \mathrm{e}^{i\delta_t(z)}\hat{T}(z_t)e^{-it/2}\varphi_0 \tag{4.25}$$

where $z_t = (q_t, p_t)$ is the generic point on the classical trajectory (a circle here), coming from z at time $t = 0$. Let be $\psi_{t,z}$ the state equal to the r.h.s in (4.25), and let us compute $\delta_t(z)$ such that $\psi_{t,z}$ satisfies the equation $i\frac{d}{dt}\varphi = \hat{H}\varphi \quad \varphi|_{t=0} = \psi_{0,z}$. We have

$$\hat{T}(z_t)u(x) = e^{i(p_t x - q_t p_t/2)}u(x - q_t)$$

and

$$\psi_{t,z}(x) = \mathrm{e}^{i(\delta_t(z) - t/2 + p_t x - q_t p_t/2)}\varphi_0(x - q_t) \tag{4.26}$$

So, after some computations left to the reader, using properties of the classical trajectories

$$\dot{q}_t = p_t \quad \dot{p}_t = -q_t, \quad p_t^2 + q_t^2 = p^2 + q^2,$$

the equation

$$i\frac{d}{dt}\psi_{t,z}(x) = \frac{1}{2}(D_x^2 + x^2)\psi_{t,z}(x) \tag{4.27}$$

is satisfied if and only if

$$\delta_t(z) = \frac{1}{2}(p_t q_t - pq) \tag{4.28}$$

□

Let us now introduce the following general notations for later use.
F_t is the classical flow with initial time $t_0 = 0$ and final time t. It is represented as a $2n \times 2n$ matrix which can be written as four $n \times n$ blocks :

$$F_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}. \tag{4.29}$$

Let us introduce the following squeezed states. φ^Γ is defined as follows.

$$\varphi^\Gamma(x) = a_\Gamma \exp\left(\frac{i}{2\hbar}\Gamma x \cdot x\right) \tag{4.30}$$

where $\Gamma \in \Sigma_n$, Σ_n is the Siegel space of complex, symmetric matrices Γ such that $\Im(\Gamma)$ is positive and non degenerate and $a_\Gamma \in \mathbb{C}$ is such that the L^2-norm of φ^Γ is one.
We also denote $\varphi_z^\Gamma = \widehat{T}(z)\varphi^\Gamma$.
For $\Gamma = i\mathbb{1}$, we denote $\varphi = \varphi^{i\mathbb{1}}$.

Theorem 4.1 *We have the following formulae, for every $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^{2n}$,*

$$\widehat{U}_t\varphi^\Gamma(x) = \varphi^{\Gamma_t}(x) \tag{4.31}$$

$$\widehat{U}_t\varphi_z^\Gamma(x) = \widehat{T}(F_t z)\varphi^{\Gamma_t}(x) \tag{4.32}$$

where $\Gamma_t = (C_t + iD_t\Gamma)(A_t + iB_t\Gamma)^{-1}$ and $a_{\Gamma_t} = a_\Gamma\left(\det(A_t + iB_t\Gamma)\right)^{-1/2}$.

Beginning of the proof
The first formula can be proven by the ansatz

$$\widehat{U}_t\varphi_0(x) = a(t)\exp\left(\frac{i}{2\hbar}\Gamma_t x\cdot x\right)$$

where $\Gamma_t \in \Sigma_n$ and $a(t)$ is a complex values time dependent function. We get first a Riccati equation to compute Γ_t and a linear equation to compute $a(t)$.
The second formula is easy to prove from the first, using the Weyl translation operators and the following known property

$$\widehat{U}_t\widehat{T}(z)\widehat{U}_t^* = \widehat{T}(F_t z).$$

Let us now give the details of the proof for $z = 0$.
We begin by computing the action of a quadratic Hamiltonian on a Gaussian ($\hbar = 1$).

Lemma 4.2

$$Lx\cdot D_x \mathrm{e}^{\frac{i}{2}\Gamma x\cdot x} = (L^T x\cdot \Gamma x - \frac{i}{2}\mathrm{Tr}L)\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x}$$

Proof
This is a straightforward computation, using

$$Lx\cdot D_x = \frac{1}{i}\sum_{1\le j,k\le n} L_{jk}\frac{x_j D_k + D_k x_j}{2}$$

and, for $\omega \in \mathbb{R}^n$,

$$(\omega . D_x)\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x} = (\Gamma x\cdot\omega)\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x}$$

□

Lemma 4.3

$$(GD_x\cdot D_x)\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x} = (G\Gamma x\cdot\Gamma x - i\mathrm{Tr}(G\Gamma))\,\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x}$$

Proof
As above, we get

$$\widehat{H}\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x} = \left(\frac{1}{2}Kx\cdot x + x\cdot L\Gamma x + \frac{1}{2}G\Gamma x\cdot\Gamma x - \frac{i}{2}\mathrm{Tr}(L + G\Gamma)\right)\mathrm{e}^{\frac{i}{2}\Gamma x\cdot x} \tag{4.33}$$

We are now ready to solve the equation

$$i\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{4.34}$$

with

$$\psi|_{t=0}(x) = g(x) := (2\pi)^{-n/2}e^{-x^2/2}.$$

We try the ansatz

$$\psi(t,x) = a(t)e^{\frac{i}{2}\Gamma_t x\cdot x} \tag{4.35}$$

which gives the equations

$$\begin{aligned} \dot{\Gamma}_t &= -K - 2\Gamma_t^T L - \Gamma_t G \Gamma_t & (4.36)\\ \dot{f}(t) &= -\frac{1}{2}\left(Tr(L+G\Gamma_t)\right) f(t) & (4.37) \end{aligned}$$

with the initial conditions

$$\Gamma_0 = i\mathbb{1}, \quad a(0) = (2\pi)^{-n/2}$$

Remark: $\Gamma^T L$ et $L\Gamma$ determine the same quadratic forms. So the first equation is a Ricatti equation and can be written as

$$\dot{\Gamma}_t = -K - \Gamma_t L^T - L\Gamma_t - \Gamma_t G\Gamma_t, \tag{4.38}$$

where L^T denotes the transposed matrix for L. We shall now see that equation (4.38) can be solved using Hamilton equation

$$\begin{aligned} \dot{F}_t &= J\begin{pmatrix} K & L \\ L^T & G \end{pmatrix} F_t & (4.39)\\ F_0 &= \mathbb{1} & (4.40) \end{aligned}$$

We know that

$$F_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$$

is a symplectic matrix $\forall t$. So we have $\det(A_t + iB_t) \neq 0 \quad \forall t$ (see below). Let us denote

$$M_t = A_t + iB_t, \quad N_t = C_t + iD_t \tag{4.41}$$

We shall prove that $\Gamma_t = N_t M_t^{-1}$. By an easy computation, we get

$$\begin{aligned} \dot{M}_t &= L^T M_t + GN_t \\ \dot{N}_t &= -KM_t - LN_t & (4.42) \end{aligned}$$

Now, compute

$$\begin{aligned} \frac{d}{dt}(N_t M_t^{-1}) &= \dot{N}M^{-1} - NM^{-1}\dot{M}M^{-1} \\ &= -K - LNM^{-1} - NM^{-1}(L^T M + GN)M^{-1} \\ &= -K - LNM^{-1} - NM^{-1}L^T - NM^{-1}GNM^{-1} & (4.43) \end{aligned}$$

which is exactly equation (4.38).
Now we compute $a(t)$, using the following equality,

$$\mathrm{Tr}\left(L^T + G(C+iD)(A+iB)^{-1}\right) = \mathrm{Tr}(\dot{M})M^{-1} = \mathrm{Tr}\left(L + G\Gamma_t\right)$$

using $\mathrm{Tr} L = \mathrm{Tr} L^T$. Let us recall the Liouville formula

$$\frac{d}{dt}\log(\det M_t) = \mathrm{Tr}(\dot{M}_t M_t^{-1}) \tag{4.44}$$

which give directly

$$a(t) = (2\pi)^{-n/2}\left(\det(A_t + iB_t)\right)^{-1/2} \tag{4.45}$$

To complete the proof, we need to prove the following

Lemma 4.4 *Let S be a symplectic matrix.*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then $\det(A+iB) \neq 0$ *and* $\Im(C+iD)(A+iB)^{-1}$ *is positive definite.*

We shall prove a more general result concerning the Siegel space Σ_n.

Lemma 4.5 *If*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a symplectic matrix and $Z \in \Sigma_n$ then A+BZ et C+DZ are non singular and $(C+DZ)(A+BZ)^{-1} \in \Sigma_n$

Proof

Let us denote $E := A+BZ, \quad F := C+DZ$. F is symplectic, so we have $F^T J F = J$. Using

$$\begin{pmatrix} E \\ F \end{pmatrix} = S \begin{pmatrix} I \\ Z \end{pmatrix}$$

we get

$$(E^T, F^T) J \begin{pmatrix} E \\ F \end{pmatrix} = (I, Z) J \begin{pmatrix} I \\ Z \end{pmatrix} = 0 \tag{4.46}$$

which gives

$$E^T F = F^T E$$

In the same way, we have

$$\frac{1}{2i}(E^T, F^T) J \begin{pmatrix} \bar{E} \\ \bar{F} \end{pmatrix} = \frac{1}{2i}(I, Z) F^T J F \begin{pmatrix} I \\ \bar{Z} \end{pmatrix} \tag{4.47}$$

$$= \frac{1}{2i}(I, Z) J \begin{pmatrix} I \\ \bar{Z} \end{pmatrix} = \frac{1}{2i}(\bar{Z} - Z) = -\Im Z$$

We get the following equation

$$F^T \bar{E} - E^T \bar{F} = 2i \Im Z \tag{4.48}$$

Because $\Im Z$ is non degenerate, from (5.68), we get that E and F are injective. If $x \in \mathbb{C}^n$, $Ex = 0$, we have

$$\bar{E}\bar{x} = x^T E^T = 0$$

hence

$$x^T \Im Z \bar{x} = 0$$

then $x = 0$.
So, we can define,

$$\alpha(S)Z = (C + DZ)(A + BZ)^{-1} \tag{4.49}$$

Let us prove that $\alpha(S) \in \Sigma_n$. We have:

$$\alpha(M)Z = FE^{-1} \Rightarrow (\alpha(M)Z)^T = (E^{-1})^T F^T = (E^{-1})^T E^T F E^{-1} = FE^{-1} = \alpha(M)Z.$$

We have also:

$$E^T \frac{FE^{-1} - \bar{F}\bar{E}^{-1}}{2i} \bar{E} = \frac{F^T \bar{E} - E^T \bar{F}}{2i} = \Im Z$$

and this proves that $\Im(\alpha(M))$ is positive and non degenerate.
This finishes the proof of the Theorem for $z = 0$. $\Box$

Remark 4.6 *For a different proof of formula (4.31), using the usual approach of the metaplectic group, see the book [8].*
The family $\{\varphi_z\}_{z\in\mathbb{R}^{2n}}$ spans all of $L^2(\mathbb{R}^n)$ (see for exemple [19] for properties of the Fourier-Bargmann transform) so formula (4.31) wholly determines the unitary group $\widehat{U}_t$. In particular it results that $\widehat{U}_t$ is a unitary operator and that $\hat{H}_t$ has a unique self-adjoint extension in $L^2(\mathbb{R}^n)$. This is left as exercises for the reader.

Remark 4.7 *The map $S \mapsto \alpha(S)$ defines a representation of the symplectic group* $\mathrm{Sp}(2n)$ *in the Siegel space Σ_n. It is easy to prove that $\alpha(S_1S_2) = \alpha(S_1)\alpha(S_2)$. This representation is transitive. Many other properties of this representation are studied in [16].*

5 The metaplectic group and Weyl symbols computation

A metaplectic transformation associated with a linear symplectic tranformation $F \in \mathrm{Sp}(2n)$ in $\mathbb{R}^{2n}$, is a unitary operator $\widehat{R}(F)$ in $L^2(\mathbb{R}^n)$ satisfying one of the following equivalent conditions

$$\widehat{R}(F)^*\widehat{A}\widehat{R}(F) = \widehat{A \circ F}, \quad \forall A \in \mathcal{S}(\mathbb{R}^{2n}) \tag{5.50}$$

$$\widehat{R}(F)^*\widehat{T}(X)\widehat{R}(F) = \widehat{T}[F^{-1}(X)], \quad \forall X \in \mathbb{R}^{2n} \tag{5.51}$$

$$\widehat{R}(F)^*\widehat{A}\widehat{R}(F) = \widehat{A \circ F},$$

$$\text{for } A(q,p) = q_j, 1 \leq j \leq n \quad \text{and} \quad A(q,p) = p_k, 1 \leq k \leq n. \tag{5.52}$$

We shall see below that for every $F \in \mathrm{Sp}(2n)$ there exists a metaplectic transformation $\widehat{R}(F)$.
Let us remark that if $\widehat{R_1}(F)$ and $\widehat{R_2}(F)$ are two metaplectic operators associated to the same symplectic map F then there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $\widehat{R_1}(F) = \lambda\widehat{R_2}(F)$. It is also required that $F \mapsto \widehat{R}(F)$ defines a projective representation of the real symplectic group $\mathrm{Sp}(2n)$ with sign indetermination only. More precisely, let us denote by $\mathrm{Mp}(n)$ the group of metaplectic transformations and π_p the natural projection: $\mathrm{M_p} \to \mathrm{Sp}(2n)$ then the metaplectic representation is a group homomorphism $F \mapsto \hat{R}(F)$, from $\mathrm{Sp}(2n)$ onto $\mathrm{Mp}(n)/\{\mathbb{1}, -\mathbb{1}\}$, such that $\pi_p[\widehat{R}(F)] = F$, $\forall F \in \mathrm{Sp}(2n)$ (for more details concerning the metaplectic representation see [15]). We shall show here that this can be achieved straightforward using Theorem 4.1.

For every $F \in \mathrm{Sp}(2n)$ we can find a C^1- smooth curve F_t, $t \in [0,1]$, in $\mathrm{Sp}(2n)$, such that $F_0 = \mathbb{1}$ and $F_1 = F$. An explicit way to do that is to use the polar decomposition of F, $F = V|F|$ where V is a symplectic orthogonal matrix and $|F| = \sqrt{F^T F}$ is positive symplectic matrix. Each of these matrices have a logarithm, so $F = \mathrm{e}^K \mathrm{e}^L$ with K, L Hamiltonian matrices, and we can choose $F_t = \mathrm{e}^{tK}\mathrm{e}^{tL}$. Any way, F_t is clearly the linear flow defined by the quadratic Hamiltonian $H_t(z) = \frac{1}{2} S_t z \cdot z$ where $S_t = -J\dot{F}_t F_t^{-1}$. So using above results, **we define** $\widehat{R}(F) = \widehat{U_1}$. From this definition and Theorem 4.1 we can easily recover the usual properties of the metaplectic representation.

Proposition 5.1 *Let us consider two symplectic paths F_t and F'_t joining $\mathbb{1}$ $(t = 0)$ to F $(t = 1)$. Then we have $\widehat{U_1} = \pm\widehat{U'_1}$ (with obvious notations).*
Moreover, if $F^1, F^2 \in \mathrm{Sp}(2n)$ then we have

$$\widehat{R}(F^1)\hat{R}(F^2) = \pm\widehat{R}(F^1 F^2).$$

Proof
Using (4.31) we see that the phase shift between the two paths comes from variation of argument between 0 and 1 of the complex numbers $b(t) = \det(A_t + iB_t)$ and $b'(t) = \det(A'_t + iB'_t)$.
We have $\arg[b(t)] = \Im\left(\int_0^t \frac{\dot{b}(s)}{b(s)} ds\right)$ and it is well known (see Lemma (5.4) below and its proof) that

$$\Im\left(\int_0^1 \frac{\dot{b}(s)}{b(s)} ds\right) = \Im\left(\int_0^1 \frac{\dot{b}'(s)}{b'(s)} ds\right) + 2\pi N$$

with $N \in \mathbb{Z}$. So we get

$$b(1)^{-1/2} = \mathrm{e}^{iN\pi} b'(1)^{-1/2}$$

The second part of the proposition is an easy consequence of Theorem 4.1 concerning propagation of squeezed coherent states with little computations. □

In a recent paper [17] the authors use a nice explicit formula for the Weyl symbol of metaplectic operators $\hat{R}(F)$. In what follows we detail a rigorous proof of this formula including computation of the phase factor. In principle we could use Theorem 4.1 to compute the Weyl symbol of the propagator $U_{t_0,t}$. But in this approach it seems difficult to compute phase factors (Maslov- Conley-Zehnder index).
For technical reason, It is easier for us to compute first the contravariant Weyl symbol, U_t, for the propagator $\widehat{U}_t$ defined by $\widehat{H}_t$. In any case, U_t is a Schwartz temperate distribution on the phase space $\mathbb{R}^{2n}$.
We follow the approach used in Fedosov [7].
It is enough to assume $\hbar = 1$. In a first step we shall solve the following problem

$$\begin{aligned} i\frac{\partial}{\partial t}\widehat{U_t^\varepsilon} &= \widehat{H}_t\widehat{U_t^\varepsilon} \\ \widehat{U_0^\varepsilon} &= \widehat{1\!\!1}^\varepsilon \end{aligned} \tag{5.53}$$

where $1\!\!1^\varepsilon$ is a smoothing family of operators such that $\lim_{\varepsilon\to 0} 1\!\!1^\varepsilon = 1\!\!1$. It will be convenient to take $1\!\!1^\varepsilon(X) = \exp(-\varepsilon|X|^2)$.

Let us recall that $\#$ denotes the Moyal product for Weyl symbols. So for the contravariant symbol $U_t(X)$ of U_t we have

$$i\frac{\partial}{\partial t}U_t(X) = (H_t \# U_t)(X) \tag{5.54}$$

Because H_t is a quadratic polynomial we have

$$\begin{aligned} (H_t\#U_t)(X) = H_t(X)U_t(X) + \frac{1}{2i}\{H_t, U_t\}(X) \\ -\frac{1}{8}(\partial_\xi\partial_y - \partial_x\partial_\eta)^2 H_t(X)U_t(Y)|_{X=Y} \end{aligned} \tag{5.55}$$

where $\partial_x = \frac{\partial}{\partial_x}$, $X = (x,\xi)$, $Y = (y,\eta)$.
It seems natural to make the following ansatz

$$\begin{aligned} U_t(X) &= \alpha(t)E_t(X), \text{ where} \\ E_t(X) &= \exp\left(iM_tX\cdot X\right). \end{aligned} \tag{5.56}$$

$\alpha(t)$ is a complex time dependent function, M_t is a time dependent $2n\times 2n$ complex, symmetric matrix such that $\Im M_t$ is positive and non degenerate.
A, B being two classical observables, we have:

$$\{A\ ,\ B\} = \nabla A\ .\ J\nabla B$$

and

$$\begin{aligned} (\partial_x\partial_\eta - \partial_y\partial_\xi)^2 A(x,\xi)\ B(y,\eta)|_{X=Y} &= \partial^2_{x^2}A\ \partial^2_{\xi^2}B + \partial^2_{x^2}B\ \partial^2_{\xi^2}A - 2\partial^2_{x\xi}A\ \partial^2_{x\xi}B \\ &= -\mathrm{Tr}(JA''JB'') \end{aligned} \tag{5.57}$$

where A'' is the Hessian of A (and similarly for B). Applying this with

$$A(X) = \frac{1}{2} S_t X.X, \quad B(X) = \exp(iM_t X.X)$$

we get:

$$H_t \# E_t(X) = H_t(X)\ E_t(X) + JS_t X\ .\ M_t X\ E_t(X) + \frac{1}{8}\mathrm{Tr}(JS_t JB'')$$

However,

$$\nabla B = 2iB(X)\ M_t X$$
$$(B'')_{jk} = 2i\left((M_t)_{jk} + 2i(M_t X)_j\ (M_t X)_k\right)\ B(X)$$

so that:

$$\frac{1}{8}\mathrm{Tr}(JS_t JB'') = \left(\frac{i}{4}\mathrm{Tr}(JS_t JM_t) - \frac{1}{2} M_t X\ .\ JS_t JM_t X\right) B(X)$$

Therefore the Ansatz (5.56) leads to the equation:

$$\begin{aligned} i\dot{\alpha}(t) - \alpha(t)\dot{M}_t X\ .\ X = \frac{1}{2}\left(S_t X\ .\ X + M_t JS_t X\ .\ X - S_t JM_t X\ .\ X\right)\alpha(t) \\ + \frac{i}{4}\alpha(t)\mathrm{Tr}(\mathcal{M}_t \mathcal{S}_t) - \frac{1}{2}\alpha(t) M_t X\ .\ \mathcal{S}_t \mathcal{M}_t X \end{aligned} \quad (5.58)$$

where we have introduced the Hamiltonian matrices

$$\mathcal{M}_t := JM_t, \quad \mathcal{S}_t := JS_t$$

Then equation (5.58) is equivalent to

$$\dot{\mathcal{M}}_t = \frac{1}{2}(\mathcal{M}_t + 1)\mathcal{S}_t(\mathcal{M}_t - 1) \quad (5.59)$$
$$\dot{\alpha}_t = \frac{1}{4}\mathrm{Tr}\left(\mathcal{M}_t \mathcal{S}_t\right)\alpha_t \quad (5.60)$$

The first equation is a Riccati equation and can be solved with a Cayley transform:

$$\mathcal{M}_t = (\mathbb{1} - \mathcal{N}_t)(\mathbb{1} + \mathcal{N}_t)^{-1}$$

which gives the linear equation

$$\dot{\mathcal{N}}_t = \mathcal{S}_t \mathcal{N}_t$$

so we have, recalling that $\mathcal{S}_t = \dot{F}_t F_t^{-1}$,

$$\mathcal{N}_t = F_t \mathcal{N}_0$$

Coming back to $\mathcal{M}$, we get

$$\mathcal{M}_t = \left(\mathbb{1} + \mathcal{M}_0 - F_t(\mathbb{1} - \mathcal{M}_0)\right)\left(\mathbb{1} + \mathcal{M}_0 + F_t(\mathbb{1} - \mathcal{M}_0)\right)^{-1} \quad (5.61)$$

Let us now compute the phase term. We introduce

$$\chi_t^{\pm} = \mathbb{1} + \mathcal{M}_0 \pm F_t(\mathbb{1} - \mathcal{M}_0) \tag{5.62}$$

Using the following properties

$$\begin{aligned} \chi_t^- &= \chi_t^+ - 2F_t(\mathbb{1} - \mathcal{M}_0) && (5.63)\\ \mathrm{Tr}\mathcal{S}_t &= 0 && (5.64)\\ \mathcal{S}_t &= \dot{F}_t F_t^{-1} && (5.65)\end{aligned}$$

we have

$$\mathrm{Tr}(\mathcal{M}_t\mathcal{S}_t) = -2\mathrm{Tr}\left(\dot{\chi}_t^+(\chi_t^+)^{-1})\right) \tag{5.66}$$

so we get

$$\alpha_t = \alpha_0 \exp\left(-\frac{1}{2}[\log \det \chi_\bullet^+]_0^t\right), \quad \alpha_0 = 1. \tag{5.67}$$

In formula (5.67) the log is defined by continuity, because we shall see that χ_t^+ is always non singular.

Until now we just compute at the formal level. To make the argument rigorous we state some lemmas.
It is convenient here to introduce the following notations.
$\mathrm{sp}_+(2n,\mathbb{C})$ is the set of complex, $2n\times 2n$ matrices $\mathcal{M}$ such that $\mathcal{M} = JM$ where M is symmetric (that means that $\mathcal{M}$ is a complex Hamiltonian matrix) and such that $\Im M$ is positive non degenerate.

$\mathrm{Sp}_+(2n,\mathbb{C})$ is the set of complex, symplectic, $2n\times 2n$ matrices $\mathcal{N}$ such that the quadratic form

$$z \mapsto \Im\left(\mathbb{1} - \overline{\mathcal{N}}^{-1}\mathcal{N}\right) z\cdot J\bar{z}$$

is positive and non degenerate on $\mathbb{C}^{2n}$.

Lemma 5.2 *If F is a real symplectic matrix and $\mathcal{M} \in \mathrm{sp}_+(2n,\mathbb{C})$, then $\mathbb{1} + \mathcal{M} + F(\mathbb{1} - \mathcal{M})$ is invertible.*

Proof
Write $\mathcal{M} = JM$. It is enough to prove that the adjoint $\mathbb{1} + F^T + \mathcal{M}^*(\mathbb{1} - F^T)$ is injective. But $\mathcal{M}^* = -\overline{M}J$. So if $z \in \mathbb{C}^{2n}$ is such that $(\mathbb{1} + F^T + \mathcal{M}^*(\mathbb{1} - F^T))z = 0$ then we get

$$((\mathbb{1} + F^T)z - \overline{M}J(\mathbb{1} - F^T)z \cdot J(\mathbb{1} - F^T)\bar{z}) = 0 \tag{5.68}$$

But, using that F is symplectic, we have that $(1+F)J(1-F^T) = FJ - JF^T$ is symmetric so taking the imaginary part in (5.68), we have

$$\Im\left(\overline{M}J(\mathbb{1} - F^T)z \cdot J(\mathbb{1} - F^T)\bar{z}\right) = 0.$$

Then using that $\Im M$ is non degenerate, we get successively $(\mathbb{1} - F^T)z = 0$, $(\mathbb{1} + F^T)z = 0$ and $z = 0$. $\square$

Lemma 5.3 *Assume that -1 is not an eigenvalue of $\mathcal{N}$. Then $\mathcal{N} \in \mathrm{Sp}_+(2n,\mathbb{C})$ if and only if $\mathcal{M} \in \mathrm{sp}_+(2n,\mathbb{C})$ where $\mathcal{N}$ and $\mathcal{M}$ are linked by the formula*

$$\mathcal{M} = (\mathbb{1} - \mathcal{N})(\mathbb{1} + \mathcal{N})^{-1}.$$

Proof
Assuming that $\mathcal{N} + 1$ is invertible, and

$$\mathcal{M} = (\mathbb{1} - \mathcal{N})(\mathbb{1} + \mathcal{N})^{-1}.$$

then we can easily see that $\mathcal{N}$ is symplectic if and only if $\mathcal{M}$ is Hamiltonian.
Now, using $\overline{\mathcal{N}} = \mathcal{N}^{*,T}$, we get

$$\mathbb{1} - \overline{\mathcal{N}}^{-1}\mathcal{N} = 2(1 - \overline{\mathcal{M}})^{-1}(\mathcal{M} - \overline{\mathcal{M}})(1 + \mathcal{M})^{-1}$$

If $z = (\mathbb{1} + \mathcal{M})z'$ we have, using $(J(\mathbb{1} + \overline{\mathcal{M}}))^T = -J(\mathbb{1} - \overline{\mathcal{M}})$,

$$(\mathbb{1} - \overline{\mathcal{N}}^{-1}\mathcal{N})z \cdot J\overline{z} = 2(M - \overline{M})z' \cdot \overline{z'}.$$

So the conclusion of the lemma follows easily from the last equality $\square$

The last lemma has the following useful consequence.
Let us start with some $\mathcal{M}_0 \in \mathrm{Sp}_+(2n,\mathbb{C})$ without the eigenvalue -1. It not difficult to see that $\mathcal{M}_t z = -z$ if and only if $\mathcal{M}_0 u = -u$, where $u = (\chi_t^+)^{-1}z$. In particular for every time t, -1 is not an eigenvalue for $\mathcal{M}_t$.
Furthermore, using that $\mathcal{N}_t = F_t\mathcal{N}_0$, we have $\overline{\mathcal{N}}_t^{-1}\mathcal{N}_t = \overline{\mathcal{N}}_0^{-1}\mathcal{N}_0$. So we get that the matrix $\mathcal{M}_t \in \mathrm{sp}_+(2n,\mathbb{C})$ at every time t.
If $\mathcal{M}_0$ has the eigenvalue -1 it is no more possible to use the Cayley transform but we see that $\mathcal{M}_t$ is still defined by equation (5.61)(from lemma (5.2)) and solves the Riccati equation (5.59).

Now we want to discuss in more details the phase factor included in the term α_t and to consider the limiting case $\mathcal{M}_0 = 0$ to compute the Weyl symbol of the propagator $\widehat{U_t^0}$. So doing we shall recover the Mehlig-Wilkinson formula, including the phase correction Maslov-Conley-Zehnder index).
Let us denote

$$\delta(F_t, \mathcal{M}_0) = \det\left(\frac{\mathbb{1} + \mathcal{M}_0 + F_t(\mathbb{1} - \mathcal{M}_0)}{2}\right).$$

Hence we have

$$\alpha_t = \exp\left(-\frac{1}{2}\int_0^t \frac{\dot{\delta}(F_t, \mathcal{M}_0)}{\delta(F_t, \mathcal{M}_0)}\right) ds.$$

Lemma 5.4 *Let us consider $t \mapsto F_t$ a path in $\mathrm{Sp}(n,\mathbb{R})$. Then for every $\mathcal{M}_0 \in \mathrm{sp}_+(n,\mathbb{C})$ we have, for the real part:*

$$\Re\left[\int_0^t \frac{\dot{\delta}(F_s, \mathcal{M}_0)}{\delta(F_s, \mathcal{M}_0)} ds\right] = \log\left(\frac{|\delta(F_t, \mathcal{M}_0)|}{|\delta(F_0, \mathcal{M}_0)|}\right),$$

If F_t is τ-periodic, then

$$\int_0^\tau \frac{\dot{\delta}(F_t, \mathcal{M}_0)}{\delta(F_t, \mathcal{M}_0)} ds = 2i\pi\nu$$

with $\nu \in \mathbb{Z}$. Furthermore, ν is independent on $\mathcal{M}_0 \in \mathrm{sp}_+(2n, \mathbb{C})$ and depends only on the homotopy class of the closed path $t \mapsto F_t$ in $\mathrm{Sp}(2n)$.

Proof

For simplicity, let us denote $\delta(s) = \delta(F_s, \mathcal{M}_0)$ and

$$h(t) = \int_0^t \dot{\delta}(s)\delta(s)^{-1} ds, \quad g(t) = \mathrm{e}^{-h(t)}\delta(t)$$

g is clearly constant in *time* and $g(0) = \delta(0) = 1$. Then we get $\Re(h(t)) = \log|\delta(t)|$. In the periodic case we have $\mathrm{e}^{h(\tau)} = 1$ so we have

$$\frac{1}{2\pi} \int_0^\tau \frac{\dot{\delta}(F_t, \mathcal{M}_0)}{\delta(F_t, \mathcal{M}_0)} ds = \nu, \quad \nu \in \mathbb{Z}.$$

By a simple continuity argument, we see that ν is invariant by continuous deformation on $\mathcal{M}_0$ and F_t. $\square$

We can now compute the Weyl symbol of $\widehat{R}(F)$ when $\det(\mathbb{1} + F) \neq 0$. Let us consider first the case $\det(\mathbb{1} + F) > 0$. The case $\det(\mathbb{1} + F) < 0$ is a little bit more complicated because the identity $\mathbb{1}$ is not in this component.

We start with an arbitrary C^1, path $t \mapsto F_t$ going from $\mathbb{1}$ ($t = 0$) to F ($t = 1$). It is known that $\mathrm{Sp}^+(2n) = \{F \in \mathrm{Sp}(2n), \text{ such that } \det(\mathbb{1} + F) > 0\}$ is an open connected subset of $\mathrm{Sp}(2n)$. So, we can choose a piecewise C^1 path F'_t in $\mathrm{Sp}^+(2n)$ going from $\mathbb{1}$ to F and $\mathcal{M}_0 = i\varepsilon J$. We have, using Lemma(5.4),

$$\Im\left(\int_0^1 \frac{\dot{\delta}(F_t, i\varepsilon J)}{\delta(F_t, i\varepsilon J)} dt\right) = 2\pi\nu + \Im\left(\int_0^1 \frac{\dot{\delta}(F'_t, i\varepsilon J)}{\delta(F'_t, i\varepsilon J)} dt\right) \tag{5.69}$$

But $\det(\mathbb{1} + F'_t)$ is never 0 on $[0, 1]$ and is real; so if $\varepsilon > 0$ is going to zero, the last term in r.h.s goes to 0 and we get

$$\lim_{\varepsilon \searrow 0} \Im\left(\int_0^1 \frac{\dot{\delta}(F_t, i\varepsilon J)}{\delta(F_t, i\varepsilon J)} dt\right) = 2\pi\nu$$

So we have proved for the Weyl symbol $R(F, X)$ of $\widehat{R}(F)$ the following Melhig-Wilkinson formula

$$R(F, X) = \mathrm{e}^{i\pi\nu} |\det(\mathbb{1} + F)|^{-1/2} \exp\left(-iJ(\mathbb{1} - F)(\mathbb{1} + F)^{-1} X \cdot X\right) \tag{5.70}$$

Let us now consider $F \in \mathrm{Sp}_{-}(2n)$ where $\mathrm{Sp}_{-}(2n) = \{F \in \mathrm{Sp}(2n),\ \text{such that}\ \det(\mathbb{1}+F) < 0\}$. Here we shall replace the identity matrix by

$$F_2^0 = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

for $n = 1$ and $F^0 = F_2^0 \otimes \mathbb{1}_{2n-2}$ for $n \geq 2$ where $\mathbb{1}_{2n-2}$ is the identity in $\mathbb{R}^{2n-2}$.
Let us consider a path connecting $\mathbb{1}$ to F^0 then F^0 to $F_1 = F$. Because $\mathrm{Sp}(2n)$ is open and connected we can find a path in $\mathrm{Sp}(2n)$ going from F^0 to $F_1 = F$ and that part does not contribute to the phase by the same argument as above.
Let us consider the model case $n = 1$. The following formula gives an explicit path in $\mathrm{Sp}(2)$.

$$F'_t = \begin{pmatrix} \cos t\pi & -\sin t\pi \\ \sin t\pi & \cos t\pi \end{pmatrix} \begin{pmatrix} \eta(t) & 0 \\ 0 & \frac{1}{\eta(t)} \end{pmatrix},$$

where $\eta(t)$ is analytic on a complex neighborhood of $[0,1]$, $\eta(0) = 1$, $\eta(1) = 2$, $1 \leq \eta(t) \leq 2$ for $t \in [0,1]$. A simple example is $\eta(t) = 1 + t$. Then we can compute:

Lemma 5.5

$$\lim_{\varepsilon \to 0} \Im \left(\int_0^1 \frac{\dot{\delta}(F_t, i\varepsilon J)}{\delta(F_t, i\varepsilon J)} dt \right) = \pi \tag{5.71}$$

Proof
Let us denote $F'_t = R(t)B(t)$, where $R(t)$ is the rotation matrix of angle $t\pi$ and for $\varepsilon \in [0, 1[$,

$$f_\varepsilon(t) = \det[(\mathbb{1} - i\varepsilon J) + F'_t(\mathbb{1} - i\varepsilon J)]$$

We have

$$f_0(t) = \det(R(-t) + B(t)) = 2 + \cos(t\pi)\left(1 + t + \frac{1}{1+t}\right)$$

f_0 has exactly one simple zero t_1 on $[0,1]$, $f_0(t_1) = 0$, $\dot{f}_0(t_1) \neq 0$).
This is easy to see by solving the equation $\cos t\pi = h(t)$, for a suitable h, with a geometric argument .
Then by a standard complex analysis argument (contour deformation) we get the equality (5.71). $\square$

So in this case we have the formula (5.70) for the contravariant Weyl symbol of $\hat{R}(F)$, with index $\nu \in \mathbb{Z} + 1/2$. Summing up the discussion of this section we have proved:

Theorem 5.6 *We can realize the metaplectic representation $F \mapsto \hat{R}(F)$ of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ into the unitary group of $L^2(\mathbb{R}^n)$ by taking for every F a C^1-path γ going from $\mathbb{1}(t = 0)$ to $F(t = 1)$ and solving explicitly the corresponding quadratic Schrödinger equation for the Hamiltonian $H_t(z) = -1/2 J\dot{F}_t F_t^{-1} z \cdot z$. So let us define $\widehat{R_\gamma}(F)$, the propagator at time 1 obtained this way.*

If γ' is another path going from $\mathbb{1}$ $(t=0)$ to F $(t=1)$ then there exists an index $N(\gamma,\gamma')\in\mathbb{Z}$ such that

$$\widehat{R_\gamma}(F) = \mathrm{e}^{i\pi N(\gamma,\gamma')}\widehat{R_{\gamma'}}(F)$$

The metaplectic operator $\hat{R}(F)$ is the two valued unitary operator $\pm\widehat{R_\gamma}(F)$.
Moreover if $\det(\mathbb{1}+F)\neq 0$, $\hat{R}(F)$ has a smooth contravariant Weyl symbol $R(F,X)$, given by formula (5.70), where $\nu\in\mathbb{Z}$ if $\det(\mathbb{1}+F)>0$ and $\nu\in\mathbb{Z}+1/2$ if $\det(\mathbb{1}+F)<0$.

It will be useful to translate the above theorem for the covariant Weyl symbol. Before that we start to discuss the general case, including $\det[F\pm\mathbb{1}]=0$. The Weyl symbol of $\hat{R}(F)$ may be singular, so it is easier to analyse it using coherent states (for our application it is exactly what we need).
Let us recall that

$$R^{\#}(F,X) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}}\langle\varphi_{z+X}|\hat{R}(F)\varphi_z\rangle\mathrm{e}^{-\frac{i}{2}\sigma(X,z)}dz \tag{5.72}$$

Let us denote $\hat{U}_1=\hat{R}(F)$ and U_1^ε the contravariant Weyl symbol at time 1 constructed above, such that $U_0^\varepsilon(X)=\mathrm{e}^{-\varepsilon|X|^2}$.
For every $\varepsilon>0$ we have computed the following formula for the contravariant Weyl symbol:

$$\begin{aligned}
U^\varepsilon(X) &= \alpha^\varepsilon\exp\left(iM^\varepsilon X\cdot X\right),\ \text{where}\\
M^\varepsilon(X) &= = -J(\mathbb{1}+i\varepsilon J-F(\mathbb{1}-i\varepsilon J))(\mathbb{1}+i\varepsilon J+F(1-i\varepsilon J))^{-1} \quad (5.73)\\
\alpha^\varepsilon &= \det(\mathbb{1}+i\varepsilon J+F(\mathbb{1}-i\varepsilon J))^{-1/2} \quad (5.74)
\end{aligned}$$

At the end we get the result by taking the limit:

$$\lim_{\varepsilon\to+\infty}\langle\varphi_{z+X}|\hat{U}_1^\varepsilon\varphi_z\rangle = \langle\varphi_{z+X}|\hat{R}(F)\varphi_z\rangle \tag{5.75}$$

The computation uses the following formula for the Wigner function, $W_{z,z+X}(Y)$, of the pair $(\varphi_z,\varphi_{z+X})$.

$$W_{z,z+X}(Y) = 2^{2n}\exp\left(-\left|Y-z-\frac{X}{2}\right|^2-i\sigma(X,Y-\frac{z}{2})\right) \tag{5.76}$$

So, we have to compute the following Fourier-Gauss integral

$$\langle\varphi_{z+X}|\hat{U}_1^\varepsilon\varphi_z\rangle = 2^{2n}(2\pi)^{-n}\alpha^\varepsilon\int_{\mathbb{R}^{2n}}dY\exp\left(iM^\varepsilon Y\cdot Y-\left|Y-z-\frac{X}{2}\right|^2-i\sigma(X,Y-\frac{z}{2})\right) \tag{5.77}$$

So we get

$$\langle\varphi_{z+X}|\hat{U}_1^\varepsilon\varphi_z\rangle = 2^n\left(\det(\mathbb{1}-iM^\varepsilon)\right)^{-1/2}\alpha^\varepsilon. \tag{5.78}$$

$$\times \exp\left(-\left|z+\frac{X}{2}\right|^2+\frac{1}{2}i\sigma(X,z)+(\mathbb{1}-iM^\varepsilon)^{-1}(z+\frac{X-iJX}{2})\cdot(z+\frac{X-iJX}{2})\right)$$

Now we can compute the limit when $\varepsilon\searrow 0$. We have

$$\mathbb{1}-iM^\varepsilon=\mathbb{1}+iJ(\mathbb{1}+i\varepsilon J-F(\mathbb{1}-i\varepsilon J))(\mathbb{1}+i\varepsilon J-F(\mathbb{1}-i\varepsilon J))^{-1} \tag{5.79}$$

Then we get, using that $(\mathbb{1}+F+iJ(\mathbb{1}-F))$ is invertible (see Lemma 5.2),

$$\lim_{\varepsilon\to 0}(\mathbb{1}-iM^\varepsilon)^{-1}=(\mathbb{1}+F)(\mathbb{1}+F+iJ(\mathbb{1}-F))^{-1} \tag{5.80}$$

and

$$\lim_{\varepsilon\to 0}\det(\mathbb{1}-iM^\varepsilon)^{-1/2}\alpha^\varepsilon=(\det(\mathbb{1}+F+iJ(\mathbb{1}-F))^{-1/2} \tag{5.81}$$

So, finally, we have proved the following

Proposition 5.7 *The matrix elements of $\widehat{R}(F)$ on coherent states φ_z, are given by the following formula:*

$$\langle\varphi_{z+X}|\hat{R}(F)\varphi_z\rangle=2^n\,(\det(\mathbb{1}+F+iJ(\mathbb{1}-F))^{-1/2}\,.$$
$$\times \exp\left(-\left|z+\frac{X}{2}\right|^2+\frac{1}{2}i\sigma(X,z)+K_F(z+\frac{X-iJX}{2})\cdot(z+\frac{X-iJX}{2})\right) \tag{5.82}$$

where

$$K_F=(\mathbb{1}+F)(\mathbb{1}+F+iJ(\mathbb{1}-F))^{-1} \tag{5.83}$$

Now we can compute the distribution covariant symbol of $\hat{R}(F)$ by plugging formula (5.82) in formula (5.72).
Let us begin with the regular case $\det(\mathbb{1}-F)\neq 0$.

Corollary 5.8 *If $\det(\mathbb{1}-F)\neq 0$, the covariant Weyl symbol of $\widehat{R_\gamma}(F)$ is computed by the formula:*

$$R^{\#}(F,z)=\mathrm{e}^{i\pi\mu}|\det(\mathbb{1}-F)|^{-1/2}\exp\left(-\frac{i}{4}J(\mathbb{1}+F)(\mathbb{1}-F)^{-1}z\cdot z\right) \tag{5.84}$$

where $\mu=\bar{\nu}+\frac{n}{2}$, $\bar{\nu}\in\mathbb{Z}$ is an index computed below in formulas (5.88), (5.89).

Proof
Using Proposition (5.7) and formula (5.72), we have to compute a Gaussian integral with a complex, quadratic, non degenerate covariance matrix (see [11]).
This covariance matrix is $K_F-\mathbb{1}$ and we have clearly

$$K_F-\mathbb{1}=-iJ(\mathbb{1}-F)(\mathbb{1}+F+iJ(\mathbb{1}-F))^{-1}=-(\mathbb{1}-i\Lambda)^{-1}$$

where $\Lambda=(\mathbb{1}+F)(\mathbb{1}-F)^{-1}J$ is a real symmetric matrix. So we have

$$\Re(K_F-\mathbb{1})=-(\mathbb{1}+\Lambda^2)^{-1},\ \ \Im(K_F-\mathbb{1})=-\Lambda(\mathbb{1}+\Lambda^2)^{-1} \tag{5.85}$$

So that $\mathbb{1}-K_F$ is in the Siegel space Σ_{2n} and Theorem (7.6.1) of [11] can be applied. The only serious problem is to compute the index μ.
Let us define a path of $2n \times 2n$ symplectic matrices as follows: $G_t = \mathrm{e}^{t\pi J_{2n}}$ if $\det(\mathbb{1} - F) > 0$ and:
$G_t = G_t^2 \otimes \mathrm{e}^{t\pi J_{2n-2}}$ if $\det(\mathbb{1} - F) < 0$, where

$$G^2 = \begin{pmatrix} \eta(t) & 0 \\ 0 & \frac{1}{\eta(t)} \end{pmatrix}$$

where η is a smooth function on $[0,1]$ such that $\eta(0) = 1$, $\eta(t) > 1$ on $]0,1]$ and where J_{2n} is the $2n \times 2n$ matrix defining the symplectic matrix on the Euclidean space $\mathbb{R}^{2n}$.
G_1 and F are in the same connected component of $\mathrm{Sp}_\star(2n)$ where $\mathrm{Sp}_\star(2n) = \{F \in \mathrm{Sp(2n)},\ \det(\mathbb{1} - F) \neq 0\}$. So we can consider a path $s \mapsto F'_s$ in $\mathrm{Sp}_\star(2\mathrm{n})$ such that $F'_0 = G_1$ and $F'_1 = F$.
Let us consider the following "argument of determinant" functions for families of complex matrices.

$$\begin{aligned} \theta[F_t] &= \arg_c[\det(\mathbb{1} + F_t + iJ(\mathbb{1} - F_t)] && (5.86)\\ \beta[F] &= \arg_+[\det(\mathbb{1} - K_F)^{-1}] && (5.87) \end{aligned}$$

where $\arg_c$ means that $t \mapsto \theta[F_t]$ is continuous in t and $\theta[\mathbb{1}] = 0$ ($F_0 = \mathbb{1}$), and $S \mapsto \arg_+[\det(S)]$ is the analytic determination defined on the Siegel space Σ_{2n} such that $\arg_+[\det(S)] = 0$ if S is real (see [11], vol.1, section (3.4)).
With these notations we have

$$\mu = \frac{\beta[F] - \theta[F]}{2\pi}. \tag{5.88}$$

Let us consider first the case $\det(\mathbb{1} - F) > 0$.
Using that J has the spectrum $\pm i$, we get: $\det(\mathbb{1} + G_t + iJ(\mathbb{1} - G_t)) = 4^n \mathrm{e}^{nt\pi i}$ and $\mathbb{1} - K_{G_1} = \mathbb{1}$.
Let us remark that $\det(\mathbb{1} - K_F)^{-1} = \det(\mathbb{1} - F)^{-1}\det(\mathbb{1} - F + iJ(\mathbb{1} + F))$. Let us introduce $\triangle(E, \mathcal{M}) = \det(\mathbb{1} - E + \mathcal{M}(\mathbb{1} + E))$ for $E \in \mathrm{Sp}(2n)$ and $\mathcal{M} \in \mathrm{sp}_+(2n, \mathbb{C})$.
Let consider the closed path $\mathcal{C}$ in $\mathrm{Sp}(2n)$ defined by adding $\{G_t\}_{0\leq t\leq 1}$ and $\{F'_s\}_{0\leq s\leq 1}$.
We denote by $2\pi\bar{\nu}$ the variation of the argument for $\triangle(\bullet, \mathcal{M})$ along $\mathcal{C}$. Then we get easily

$$\beta(F) = \theta[F] + 2\pi\bar{\nu} + n\pi,\ n \in \mathbb{Z}. \tag{5.89}$$

When $\det(\mathbb{1} - F) < 0$, by an explicit computation, we find $\arg_+[\det(\mathbb{1} - K_{G_1})] = 0$.
So we can conclude as above. □

Assume now that F has the eigenvalue 1 with some multiplicity $2d$. We want to compute the temperate distribution $R^\sharp(F)$ as a limit of $R^\sharp(F^\varepsilon)$ where $\det(\mathbb{1} - F^\varepsilon) \neq 0$, $\forall\varepsilon > 0$.
Let us introduce the generalized eigenspace $\mathcal{E}' = \bigcup_{j\geq 1} \ker(\mathbb{1} - F)^j$ (dim[$\mathcal{E}'$]=2d). and

$\mathcal{E}''$ its symplectic orthogonal in $\mathbb{R}^{2n}$. We denote F' the restriction of F to $\mathcal{E}'$ and F'' the restriction to $\mathcal{E}''$. We also denote by J' and J'' the symplectic applications defined by the restrictions of the symplectic form σ: $\sigma(u,v) = J'u\cdot v$, $\forall u,v\in\mathcal{E}'$ and the same for J''. Let us introduce the Hamiltonian maps $L' = (\mathbb{1}-F')(\mathbb{1}+F')^{-1}$ and $L'^{,\varepsilon} = L' - \varepsilon J'$.

It is clear that $\det(L'^{,\varepsilon}-\mathbb{1}) \neq 0$ for $0<\varepsilon$ small enough, so we can define $F'^{,\varepsilon} = (\mathbb{1}+L'^{,\varepsilon})(\mathbb{1}-L'^{,\varepsilon})^{-1}$. Let us remark that

$$Q'^{,\varepsilon} := J'(F'^{,\varepsilon}-\mathbb{1})^{-1}(F'^{,\varepsilon}+\mathbb{1}) = -(L'J'+\varepsilon)^{-1} \tag{5.90}$$

is a symmetric non degenerate matrix in $\mathcal{E}'$ defined for every $\varepsilon>0$.

Lemma 5.9 *We have the following properties.*
1) $F'^{,\varepsilon}$ *is symplectic.*
2) $\lim\limits_{\varepsilon\to 0} F'^{,\varepsilon} = F'$.
3) For $\varepsilon\neq 0$*, small enough,* $\det(F'^{,\varepsilon}-\mathbb{1})\neq 0$.

Proof.
1) comes from the fact that $L'^{,\varepsilon}$ is Hamiltonian.
2) is clear.
For 3), let us assume that $F'^{,\varepsilon}u = u$. Then we have $L'^{,\varepsilon}u = 0$ hence $J'L'u = \varepsilon u$. Now, choose $0<\varepsilon<\operatorname{dist}\{0,\operatorname{spec}(J'L')\backslash\{0\}\}$ then we have $u=0$.
Finally we define $F^{\varepsilon} = F'^{,\varepsilon}\otimes F''^{,\varepsilon}$. It is clear that F^{ε} satisfies also properties 1), 2), 3) of the above lemma with F^{ε} in place of $F'^{,\varepsilon}$. □

Proposition 5.10 *Under the above assumption, the covariant symbol of* $\hat{R}(F)$ *has the following form:*

$$\begin{aligned} R^{\#}(F,z_1,z_2,z'') &= e^{i\pi\mu_1}|\det(\mathbb{1}-F'')|^{-1/2}\delta(z_1) \\ &\times\exp\left(\frac{i}{4}J(\mathbb{1}+F)(F-\mathbb{1})^{-1}(z_2+z'')\cdot(z_2+z'')\right) \end{aligned} \tag{5.91}$$

where $z := ((z_1,z_2),z'')$ *is the decomposition of the phase-space for which* $F = F'\otimes F''$ *and* $z' = z_1+z_2$*, with* $z_2\in\operatorname{Im}(F'-\mathbb{1})$*,* $z_1\in\operatorname{Im}(F'-\mathbb{1})^{\perp}$*, the orthogonal complement in* $\mathcal{E}'$ *for the Euclidean scalar product.* $\delta(z_1)$ *denotes the Dirac mass at point* $z_1=0$. $\mu_1\in\mathbb{Z}+1/2$ *is given as follows:* $\mu_1 = \mu''+\frac{\mathrm{sg}^+Q'}{4}$ *where* μ'' *is the limit of the* μ *index for* F^{ε}*, computed in (5.84), and* sg^+Q' *is the limit for* $\varepsilon\to 0$ *of the signature of* $Q'^{,\varepsilon}$*, defined in (5.90).*

Proof
We use the same kind of computation as for Corollary (5.8). The new factor comes from the contribution of $\mathcal{E}'$. So we can forget $\mathcal{E}''$. So we assume that $\mathcal{E}' = \mathbb{R}^{2n}$ and and we forget the superscripts '. We have to compute the limit in the distribution

sense of $R^\sharp(F^\varepsilon)$ which was computed in Corollary (5.8).
Let us consider a test function $f \in \mathcal{S}(\mathbb{R}^{2n})$ and its Fourier transform $\tilde{f}$. Using Plancherel formula, we have

$$\int_{\mathbb{R}^{2n}} \exp\left(-\frac{i}{4}J(\mathbb{1}+F^\varepsilon)(\mathbb{1}-F^\varepsilon)^{-1}z\cdot z\right) f(z)dz =$$
$$2^n(\pi)^{2n}|\det(Q^\varepsilon)|^{-1/2}\mathrm{e}^{\mathrm{sg}(Q^\varepsilon)\frac{i\pi}{4}}\int_{\mathbb{R}^{2n}} \exp\left(-i(\mathbb{1}+F^\varepsilon)^{-1}(\mathbb{1}-F^\varepsilon)J\zeta\cdot\zeta\right)\tilde{f}(\zeta)d\zeta \quad (5.92)$$

So we get the result by taking the limit for $\varepsilon \to 0$ in (5.92).
Let us remark that we have used that $J'(\mathbb{1}+F)(\mathbb{1}-F)^{-1}$ is an isomorphism from $\ker(\mathbb{1}-F')^\perp$ onto itself. $\square$

In our paper [2] the leading term for the return probability and for fidelity on coherent states is computed with a Gaussian exponential defined by the quadratic form which was defined in Proposition 5.7.

$$\gamma_F(X) := \frac{1}{4}\left(K_F(\mathbb{1}-iJ)X\cdot(\mathbb{1}-iJ)X - |X|^2\right) \quad (5.93)$$

In our application we shall have $X = z_t - z$ where $t \mapsto z_t$ is the classical path starting from z. So we need to estimate the argument in the exponent of formula (5.82) for $z = 0$.

Lemma 5.11 *we have,* $\forall X \in \mathbb{R}^{2n}$,

$$\Re(\gamma_F(X)) \le -\frac{|X|^2}{2(1+s_F)}, \quad (5.94)$$

where s_F *is the largest eigenvalue of* FF^T *(*F^T *is the transposed matrix of* F*) .*

Proof
Let us begin by assuming that $\det(\mathbb{1}+F) \neq 0$. Then we have

$$K_F = (\mathbb{1}+iN)^{-1}, \text{ where } N = J(\mathbb{1}-F)(\mathbb{1}+F)^{-1}.$$

So we can compute

$$\Re(K_F) = (\mathbb{1}+N^2)^{-1} = K_FK_F^\star \text{ and } \Im(K_F) = -N(\mathbb{1}+N^2)^{-1}.$$

So, we get,

$$\gamma_F(X) = \frac{1}{4}\left((\mathbb{1}+JN)K_FK_F^\star(\mathbb{1}-NJ)X\cdot X - 2|X|^2\right) \quad (5.95)$$

By definition of K_F, we have

$$(\mathbb{1}+JN)K_F = 2\left((\mathbb{1}+iJ)F^{-1} + \mathbb{1} - iJ\right)^{-1} \quad (5.96)$$

Le us denote $T_F = ((\mathbb{1} + iJ)F^{-1} + \mathbb{1} - iJ)^{-1}$. We have, using that F is symplectic,

$$T_F^{\star,-1} T_F^{-1} = 2(F^{-1,T} F^{-1} + \mathbb{1}).$$

hence we get

$$T_F T_F^{\star} = (2(F^{-1,T} F^{-1} + \mathbb{1}))^{-1}$$

and the conclusion of the lemma follows for $\det(\mathbb{1} - F) \neq 0$ hence for every symplectic matrix F by continuity.□

Acknowledgement

We begun this work by discussions around the Mehlig-Wilkinson formula and application to the Loschmidt echo at Mittag-Leffler Institute, in fall 2002. We thank the institute and the organizers of the semester on Partial Differential Equations and Spectral Theory, A. Laptev, V. Guillemin and B. Helffer for their hospitality and exceptionnal working surroundings.

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