Harmonic Functions on the Real Hyperbolic Ball I: Boundary Values and Atomic Decompositions of Hardy Spaces

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HARMONIC FUNCTIONS ON THE REAL HYPERBOLIC BALL I : BOUNDARY VALUES AND ATOMIC DECOMPOSITION OF HARDY SPACES

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Abstract. In this article we study harmonic functions for the Laplace-Beltrami operator on the real hyperbolic space $\mathbb{B}_n$. We obtain necessary and sufficient conditions for this functions and their normal derivatives to have a boundary distribution. In doing so, we put forward different behaviors of hyperbolic harmonic functions according to the parity of the dimension of the hyperbolic ball $\mathbb{B}_n$. We then study Hardy spaces $H^p(\mathbb{B}_n)$, $0 < p < \infty$, whose elements appear as the hyperbolic harmonic extensions of distributions belonging to the Hardy spaces of the sphere $H^p(\mathbb{S}^{n-1})$. In particular, we obtain an atomic decomposition of this spaces.

1. Introduction

In this article, we study boundary behavior of harmonic functions on the real hyperbolic ball, partly in view of establishing a theory of Hardy and Hardy-Sobolev spaces of such functions.

While studying Hardy spaces of Euclidean harmonic functions on the unit ball $\mathbb{B}_n$ of $\mathbb{R}^n$, one is often lead to consider estimates of this functions on balls with radius smaller than the distance of the center of the ball to the boundary $\mathbb{S}^{n-1}$ of $\mathbb{B}_n$. Thus hyperbolic geometry is implicitly used for the study of Euclidean harmonic functions, in particular when one considers boundary behavior. As Hardy spaces of Euclidean harmonic functions are the spaces of Euclidean harmonic extensions of distributions in the Hardy spaces on the sphere, it is tempting to study these last spaces directly through their hyperbolic harmonic extension.

The other origin of this paper is the study of Hardy and Hardy-Sobolev spaces of $\mathcal{M}$-harmonic functions related to the complex hyperbolic metric on the unit ball, as exposed in [1] and [2]. Our aim is to develop a similar theory in the case of the real hyperbolic ball. In the sequel, n will be an integer, $n \geq 3$ and p a real number, $0 < p < \infty$.

Let $SO(n,1)$ be the Lorenz group. It is well known that $SO(n,1)$ acts conformly on $\mathbb{B}_n$. The corresponding Laplace-Beltrami operator, invariant for the considered action, is given by

$$D = (1-|x|^2)^2\Delta + 2(n-2)(1-|x|^2)N$$

with Δ the Euclidean laplacian and $N = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ the normal derivation operator. Functions u that are harmonic for this laplacian will be called $\mathcal{H}$-harmonic. The "hyperbolic" Poisson kernel that solves the Dirichlet problem for D is defined for $x \in \mathbb{B}_n$ and $\xi \in \mathbb{S}^{n-1}$ by

$$\mathbb{P}_h(x,\xi) = \left(\frac{1-|x|^2}{1+|x|^2 - 2 < x,\xi >}\right)^{n-1}.$$

With help of this kernel, one can extend distributions on $\mathbb{S}^{n-1}$ to $\mathcal{H}$-harmonic functions on $\mathbb{B}_n$ in the same way as the Euclidean Poisson kernel extends distributions on $\mathbb{S}^{n-1}$ to Euclidean harmonic functions on $\mathbb{B}_n$. Our first concern is to determine which $\mathcal{H}$-harmonic functions are obtained in this way. We then study the boundary behavior of their normal derivatives. In doing so, we put forward

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that, in odd dimension, normal derivatives of $\mathcal{H}$-harmonic functions behave similarly to $\mathcal{M}$-harmonic functions whereas they behave like Euclidean harmonic functions in even dimension.

Finally, define $H^p(\mathbb{S}^{n-1})$ as $L^p(\mathbb{S}^{n-1})$ if $1 < p < \infty$ and as the real analog of Garnett-Latter's atomic H^p space if $p \leq 1$. Let $H^p(\mathbb{B}_n)$ be the space of Euclidean harmonic functions $\mathbb{B}_n$ such that $\zeta \mapsto \sup_{0<r<1} |u(r\zeta)| \in L^p(\mathbb{S}^{n-1})$. Garnett-Latter's theorem asserts that this space is the space of Euclidean harmonic extensions of distributions in $H^p(\mathbb{S}^{n-1})$. We prove here that the space $\mathcal{H}^p(\mathbb{B}_n)$ of $\mathcal{H}$-harmonic functions such that $\zeta \mapsto \sup_{0<r<1} |u(r\zeta)| \in L^p(\mathbb{S}^{n-1})$ is the space of $\mathcal{H}$-harmonic extensions of distributions in $H^p(\mathbb{S}^{n-1})$.

This article is organized as follows : in section 2 we present the setting of the problem and a few preliminary results. Section 3 is devoted to the study of boundary behavior of $\mathcal{H}$-harmonic functions and concludes with the study of the behavior of their normal derivatives. We conclude in section 4 with the atomic decomposition theorem.

2. Setting

2.1. $SO(n,1)$ **and its action on** $\mathbb{B}_n$**.** Let $SO(n,1) \subset GL_{n+1}(\mathbb{R})$, $(n \geq 3)$ be the identity component of the group of matrices $g = (g_{ij})_{0\leq i,j\leq n}$ such that $g_{00} \geq 1$, $\det g = 1$ and that leaves invariant the quadratic form $-x_0^2 + x_1^2 + \ldots + x_n^2$.

Let $|.|$ be the Euclidean norm on $\mathbb{R}^n$, $\mathbb{B}_n = \{x \in \mathbb{R}^n \ : \ |x| < 1\}$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}_n = \{x \in \mathbb{R}^n \ : \ |x| = 1\}$. It is well known (*cf.* [12]) that $SO(n,1)$ acts conformaly on $\mathbb{B}_n$. The action is given by $y = g.x$ with

$$y_p = \frac{\frac{1+|x|^2}{2} g_{p0} + \sum_{l=1}^n g_{pl} x_l}{\frac{1-|x|^2}{2} + \frac{1+|x|^2}{2} g_{00} + \sum_{l=1}^n g_{0l} x_l} \qquad \text{for } p = 1, \ldots, n.$$

The invariant measure on $\mathbb{B}_n$ is given by

$$d\mu = \frac{dx}{(1-|x|^2)^{n-1}} = \frac{r^{n-1} dr d\sigma}{(1-r^2)^{n-1}}$$

where dx is the Lebesgue measure on $\mathbb{B}_n$ and $d\sigma$ is the surface measure on $\mathbb{S}^{n-1}$.

We will need the following fact about this action (see [7]):

Fact 1 *Let $g \in SO(n,1)$ and let $x_0 = g.0$. If $0 < \varepsilon < \frac{1}{6}$, then*

$$B\big(x_0, \frac{\sqrt{2}}{8}(1-|x_0|^2)\varepsilon\big) \subset g.B(0,\varepsilon) \subset B\big(x_0, 6(1-|x_0|^2)\varepsilon\big).$$

2.2. **The invariant laplacian on $\mathbb{B}_n$ and the associated Poisson kernel.** From [12] we know that the invariant laplacian on $\mathbb{B}_n$ for the considered action can be written as

$$D = (1-r^2)^2\Delta + 2(n-2)(1-r^2)\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and Δ is the Euclidean laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Note that D is given in radial-tangential coordinates by

$$D = \frac{1-r^2}{r^2}\left[(1-r^2)N^2 + (n-2)(1+r^2)N + (1-r^2)\Delta_\sigma\right]$$

with $N = r\frac{d}{dr} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ and Δ_σ the tangential part of the Euclidean laplacian.

Definition *A function u on $\mathbb{B}_n$ is $\mathcal{H}$-harmonic if $Du = 0$ on $\mathbb{B}_n$.*

The Poisson kernel that solves the Dirichlet problem associated to D is

$$\mathbb{P}_h(r\eta,\xi)=\left(\frac{1-r^2}{1+r^2-2r<\eta,\xi>}\right)^{n-1}$$

for $0\leq r<1$, $\eta,\xi\in\mathbb{S}^{n-1}$ *i.e.* for $r\eta\in\mathbb{B}_n$ and $\xi\in\mathbb{S}^{n-1}$.

Recall that the Euclidean Poisson kernel on the ball is given by

$$\mathbb{P}_e(r\eta,\xi)=\frac{1-r^2}{(1+r^2-2r<\eta,\xi>)^{\frac{n}{2}}}$$

Notation : For a distribution φ on $\mathbb{S}^{n-1}$, we define $\mathbb{P}_e[\varphi]:\mathbb{B}_n\mapsto\mathbb{R}$ and $\mathbb{P}_h[\varphi]:\mathbb{B}_n\mapsto\mathbb{R}$ by

$$\mathbb{P}_e[\varphi](r\eta)=<\varphi,\mathbb{P}_e(r\eta,.)>$$
$$\mathbb{P}_h[\varphi](r\eta)=<\varphi,\mathbb{P}_h(r\eta,.)>$$

$\mathbb{P}_e[\varphi]$ is the *Poisson integral* of φ, and $\mathbb{P}_h[\varphi]$ will be called the $\mathcal{H}$*-Poisson integral* of φ.

Finally, $\mathcal{H}$-harmonic functions satisfy mean value equalities : *let $a\in\mathbb{B}_n$ and $g\in SO(n,1)$ such that $g.0=a$. Then, for every $\mathcal{H}$-harmonic function u,*

$$u(a)=\frac{1}{\mu\big(B(0,r)\big)}\int_{g.B(0,r)}u(x)d\mu(x).$$

Thus, with fact 1 and $d\mu=\frac{dx}{(1-|x|^2)^{n-1}}$, we see that

$$|u(a)|\leq\frac{C}{(1-|a|^2)^n}\int_{B\left(a,6(1-|a|^2)\varepsilon\right)}|u(x)|dx \tag{2.1}$$

2.3. **Expansion of $\mathcal{H}$-harmonic functions in spherical harmonics.**

Notation : For $a\in\mathbb{R}$, write $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$ thus $(a)_0=1$ and $(a)_k=a(a+1)\dots(a+k-1)$ if $k=1,2,\dots$. For a,b,c three real parameters, ${}_2F_1$ denotes Gauss' *hyper-geometric function* defined by

$${}_2F_1(a,b,c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k.$$

Let $F_l(x)={}_2F_1(l,1-\frac{n}{2},l+\frac{n}{2};x)$ and $f_l(x)=\frac{F_l(x)}{F_l(1)}$. (See [4] for properties of ${}_2F_1$).

Remark : If $n>2$ is even, $1-\frac{n}{2}$ is a negative integer thus ${}_2F_1(l,1-\frac{n}{2},l+\frac{n}{2},r^2)$ is a polynomial in r of degree n.

In [10], [11] and [12], the spherical harmonic expansion of $\mathcal{H}$-harmonic functions has been obtained. An other proof based on [1] can be found in [7]. We have the following :

Theorem 1 *Let u be an $\mathcal{H}$-harmonic function of class $\mathcal{C}^2$ on $\mathbb{B}_n$. Then the spherical harmonic expansion of u is given by*

$$u(r\zeta)=\sum_l F_l(r^2)u_l(r\zeta),$$

where this series is absolutely convergent and uniformly convergent on every compact subset of $\mathbb{B}_n$.

Moreover if $\varphi\in\mathcal{C}(\mathbb{S}^{n-1})$, the Dirichlet problem $\begin{cases}Du=0 & \text{in } \mathbb{B}_n\\ u=\varphi & \text{on } \mathbb{S}^{n-1}\end{cases}$ *has a unique solution $u\in\mathcal{C}(\overline{\mathbb{B}_n})$ given by*

$$u(z) = \int_{\mathbb{S}^{n-1}} \varphi(\zeta)\mathbb{P}_h(z,\zeta)d\sigma(\zeta) = \mathbb{P}_h[\varphi](z)$$

also given by

$$u(r\zeta) = \sum_l f_l(r^2)r^l\varphi_l(\zeta)$$

where $\varphi = \sum_l \varphi_l$ is the spherical harmonic expansion of φ.

3. Boundary values of $\mathcal{H}$-harmonic functions

In this chapter we prove results about the behavior on the boundary of $\mathcal{H}$-harmonic functions and their normal derivatives. For $\mathcal{H}$-harmonic functions, the results are similar to the results for Euclidean harmonic functions. On the opposite, for the normal derivatives of $\mathcal{H}$-harmonic functions, the boundary behavior depends on the dimension of the space.

3.1. **Definition of Hardy spaces.**

Notation : For u a function defined on $\mathbb{B}_n$, define the *radial maximal function* $\mathcal{M}[u] : \mathbb{S}^{n-1} \mapsto \mathbb{R}_+$ by

$$\mathcal{M}[u](\zeta) = \sup_{0<t<1} |u(t\zeta)|.$$

We will now study $\mathcal{H}^p$ spaces of $\mathcal{H}$-harmonic functions defined as follows :

Definition *Let $0 < p < \infty$. Let $\mathcal{H}^p$ be the space of $\mathcal{H}$-harmonic functions u such that $\mathcal{M}[u] \in L^p(\mathbb{S}^{n-1})$, endowed with the "norm"*

$$\|u\|_{\mathcal{H}^p} = \|\mathcal{M}u\|_{L^p(\mathbb{S}^{n-1})} = \left\| \sup_{0<t<1} |u(t.)| \right\|_{L^p(\mathbb{S}^{n-1})}.$$

We will call $\mathcal{H}^p$ the Hardy space of $\mathcal{H}$-harmonic functions.

Remark : If $0 < p < 1$, the application $u \mapsto \|u\|_{\mathcal{H}^p}$ is not a norm, however the application $u, v \mapsto \|u - v\|_{\mathcal{H}^p}$ defines a metric on $\mathcal{H}^p$. In the sequel, we will often use the abuse of language to call $\|.\|_{\mathcal{H}^p}$ a norm whatever p might be.

Definition *A function u on $\mathbb{B}_n$ is said to have a distribution boundary value if for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, the limit*

$$\lim_{r\to 1} \int_{\mathbb{S}^{n-1}} u(r\zeta)\Phi(\zeta)d\sigma(\zeta)$$

exists. In case u is $\mathcal{H}$-harmonic, this is equivalent to the existence of a distribution f such that $u = \mathbb{P}_h[f]$.

3.2. **Boundary distributions of functions in $\mathcal{H}^p$.** In this section, we are going to characterize boundary values of functions in $\mathcal{H}^p$. The characterizations we obtain are similar to those obtained for harmonic functions on $\mathbb{R}^{n+1}_+$ or for $\mathcal{M}$-harmonic functions. The proofs are inspired by [1] and [5].

The first result concerns functions in $\mathcal{H}^p$, $p \geq 1$.

Proposition 2 *Let u be an $\mathcal{H}$-harmonic function.*

(1) *If* $1 < p < \infty$, *then*

$$\sup_{0<r<1} \int_{\mathbb{S}^{n-1}} |u(r\zeta)|^p d\sigma(\zeta) < +\infty$$

if and only if there exists $f \in L^p(\mathbb{S}^{n-1})$ *such that* $u = \mathbb{P}_h[f]$.

(2) *For* $p = 1$,

$$\sup_{0<r<1} \int_{\mathbb{S}^{n-1}} |u(r\zeta)| d\sigma(\zeta) < +\infty$$

if and only if there exists a measure μ *on* $\mathbb{S}^{n-1}$ *such that* $u = \mathbb{P}_h[\mu]$.

Proof. Assume that $u = \mathbb{P}_h[f]$ with $f \in L^p(\mathbb{S}^{n-1})$. As

$$\|\mathbb{P}_h(r\zeta, .)\|_{L^1(\mathbb{S}^{n-1})} = 1,$$

Hölder's inequality gives

$$|u(r\zeta)|^p \leq \int_{\mathbb{S}^{n-1}} \mathbb{P}_h(r\zeta,\xi)|f(\xi)|^p d\sigma(\xi) = \int_{\mathbb{S}^{n-1}} \mathbb{P}_h(\zeta,r\xi)|f(\xi)|^p d\sigma(\xi)$$

an integration in ζ and Fubini leads to the desired result.

Conversely, if the $L^p(\mathbb{S}^{n-1})$ norms of $\zeta \mapsto u(r\zeta)$ are uniformly bounded, there exists a sequence $r_m \to 1$ and a function $\varphi \in L^p$ such that $u(r_m\zeta) \to \varphi(\zeta)$ $*$-weakly thus weakly in $L^p(\mathbb{S}^{n-1})$. But then, for $r\zeta \in \mathbb{B}_n$ fixed,

$$\begin{aligned}
\mathbb{P}_h[\varphi](r\zeta) &= \lim_{m\to+\infty} \int_{\mathbb{S}^{n-1}} \mathbb{P}_h(r\zeta,\xi)u(r_m\xi)d\sigma(\xi) \\
&= \lim_{m\to+\infty} \sum_{l\geq 0} \frac{F_l(r_m^2)}{F_l(1)} r_m^l \int_{\mathbb{S}^{n-1}} \mathbb{P}_h(r\zeta,\xi)u_l(\xi)d\sigma(\xi) \\
&= \lim_{m\to+\infty} \sum_{l\geq 0} \frac{F_l(r_m^2)}{F_l(1)} r_m^l f_l(r)r^l u_l(\zeta) \\
&= \sum_{l\geq 0} f_l(r)r^l u_l(\zeta) = u(r\zeta).
\end{aligned}$$

The proof in the case $p = 1$ is obtained in a similar fashion using the duality $\left(L^1, \mathcal{M}(\mathbb{S}^{n-1})\right)$. □

We are now going to prove that $\mathcal{H}$-harmonic functions have a boundary distribution if and only if they satisfy a given growth condition. For this, we will need the folowing lemma ([1], lemma 10).

Lemma 3 *Let* $F \in \mathcal{C}^2\left(\left[\frac{1}{2},1\right]\right)$ *and* $h \in \mathcal{C}^1\left(\left[\frac{1}{2},1\right]\right)$. *Assume that*

$$F''(x) + \frac{h(x)}{1-x}F'(x) = O(1-x)^{-\alpha}$$

when $x \to 1$. *Then*

(1) *If* $\alpha > 2$ *then* $F(x) = O(1-x)^{-\alpha+1}$.

(2) *If* $1 < \alpha < 2$ *Then* $\lim_{x\to 1} F(x)$ *exists.*

We are now in position to prove

Theorem 4 *Let* u *be an* $\mathcal{H}$-*harmonic function. Then* u *admits a boundary value in the sense of distributions if and only if there exists a constant* A *such that*

$$u(r\zeta) = O(1-r)^{-A}.$$

Proof. Recall that

$$D=\frac{1-r^2}{r^2}\big[(1-r^2)N^2+(n-2)(1+r^2)N+(1-r^2)\Delta_\sigma\big] \tag{3.1}$$

Assume that $Du=0$ and that $u(r\zeta)=O\big((1-r)^{-A}\big)$. Let $\varphi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ and let

$$F(r)=\int_{\mathbb{S}^{n-1}}u(r\zeta)\varphi(\zeta)d\sigma(\zeta).$$

Formula (3.1) with $Du=0$ tells us that

$$(1-r^2)N^2F+(n-2)(1+r^2)NF+(1-r^2)\Delta_\sigma F=0.$$

where $\Delta_\sigma F$ stands for

$$\Delta_\sigma F(r)=\int_{\mathbb{S}^{n-1}}\Delta_\sigma u(r\zeta)\varphi(\zeta)d\sigma(\zeta)=\int_{\mathbb{S}^{n-1}}u(r\zeta)\Delta_\sigma^*\varphi(\zeta)d\sigma(\zeta)$$

with Δ_σ^* the adjoint operator to Δ_σ. Recall that $N=r\frac{d}{dr}$ thus

$$r^2F''(r)+\frac{(n-1)+(n-3)r^2}{1-r^2}rF'(r)+\Delta_\sigma F=0 \tag{3.2}$$

Write $\psi=-\Delta_\sigma^*\varphi$ and T the differential operator

$$T=r^2\frac{d^2}{dr^2}+\frac{(n-1)+(n-3)r^2}{1-r^2}r\frac{d}{dr}$$

so that equation (3.2) reads

$$TF(r)=\int_{\mathbb{S}^{n-1}}u(r\zeta)\psi(\zeta)d\sigma(\zeta).$$

One then immediately deduces the existence for $k=1,2,\ldots$ of a function $\psi_k\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ such that

$$T^kF(r)=\int_{\mathbb{S}^{n-1}}u(r\zeta)\psi_k(\zeta)d\sigma(\zeta).$$

But we assumed that $u(r\zeta)=O(1-r)^{-A}$. We thus have

$$T^kF(r)=O(1-r)^{-A}$$

and applying lemma 3 we obtain

$$T^{k-1}F(r)=O(1-r)^{-A+1}.$$

Therefore, starting from T^k with $k=[A]+1$ and iterating the process k times, one gets that $\lim_{r\to1}F(r)$ exists.

Conversely, if u admits a boundary distribution f, then $u=\mathbb{P}_h[f]$ *i.e.* $u(r\zeta)=< f,\mathbb{P}_h(r\zeta,.)>$. But then f being a compactly supported distribution, it is of finite order, thus there exists $k\geq0$ such that

$$|u(r\zeta)|=|< f,\mathbb{P}_h(r\zeta,.)>|\leq C\big\|\nabla_\xi^k\mathbb{P}_h(r\zeta,.)\big\|_{L^\infty}\leq\frac{C}{(1-r)^{n-1+k}}$$

which gives the desired estimate. □

Proposition 5 *Let $0<p<+\infty$ and u be an $\mathcal{H}$-harmonic function. Assume that*

$$\sup_{0<r<1}\int_{\mathbb{S}^{n-1}}|u(r\zeta)|^pd\sigma(\zeta)<\infty.$$

Then, there exists a constant C such that for every $a\in\mathbb{B}_n$,

$$|u(a)| \leq \frac{C}{(1-|a|)^{\frac{n-1}{p}}}.$$

In particular, u has a boundary distribution f i.e. $u = \mathbb{P}_h[f]$.

Proof. The mean value inequality implies that

$$|u(a)|^p \leq \frac{C}{(1-|a|)^n} \int_{B\big(a,(1-|a|)\varepsilon\big)} |u(x)|^p dx$$

for ε small enough. But $B\big(a,(1-|a|)\varepsilon\big) \subset \{r\zeta \ : \ (1-\varepsilon)(1-|a|) \leq 1-r \leq (1+\varepsilon)(1-|a|)\}$ thus

$$|u(a)|^p \leq \frac{C}{(1-|a|)^n} \int_{1-(1+\varepsilon)(1-|a|)}^{1-(1-\varepsilon)(1-|a|)} \int_{\mathbb{S}^{n-1}} |u(r\zeta)|^p d\sigma(\zeta) r^{n-1} dr \leq \frac{C}{(1-|a|)^{n-1}}. \qquad \square$$

Remark : Theorem 4 is well known. It has been proved by J.B. Lewis [9] in the case of symmetric spaces of rank 1 and eigenvectors of the Laplace-Beltrami operator (for arbitrary eigenvalues) and further generalized by E.P. van den Ban and H. Schlichtkrull [14].

3.3. **Distribution boundary values of $\mathcal{H}$-harmonic functions.**

Notation : For $1 \leq i,j \leq n$, $i \neq j$, let $\mathcal{L}_{i,j} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$. Then the $\mathcal{L}_{i,j}$'s commute and commute with N. Further, if u is $\mathcal{H}$-harmonic, then $\mathcal{L}_{i,j}$ is also $\mathcal{H}$-harmonic. Finally, N and $\{\mathcal{L}_{i,j}\}_{1\leq i\neq j\leq n}$ generate ∇^k outside a neighbourhood of the origin.

Recall that $Du = 0$ if and only if

$$(1-r^2)N^2u + (n-2)(1+r^2)Nu + (1-r^2)\Delta_\sigma u = 0. \tag{3.3}$$

Apply N^{k-1} on both sides of this equality and isolate terms of order $k+1$ and k :

$$\begin{aligned}(1-r^2)N^{k+1}u - 2(k-1)r^2N^ku &+ (n-2)(1+r^2)N^ku \\ &= r^2\sum_{j=0}^{k-3}\binom{k-1}{j}2^{k-j-1}N^{j+2}u + r^2\sum_{j=0}^{k-2}\binom{k-1}{j}2^{k-j-1}N^j\Delta_\sigma u \\ &\quad -(n-2)r^2\sum_{j=0}^{k-2}\binom{k-1}{j}2^{k-j-1}N^{j+1}u - (1-r^2)N^{k-1}\Delta_\sigma u\end{aligned} \tag{3.4}$$

We are now in position to prove the following lemma :

Lemma 6 *Let u be an $\mathcal{H}$-harmonic function with a boundary distribution. Let $\mathbb{Y}$ be a product of operators of the form $\mathcal{L}_{i,j}$ and let $\mathbb{X} = N^k\mathbb{Y}$. Then if $k \leq n-2$, $\mathbb{X}u$ has a distribution boundary value in the sense that*

$$\lim_{r\to 1}\int_{\mathbb{S}^{n-1}} \mathbb{X}u(r\zeta)\Phi(\zeta)d\sigma(\zeta)$$

exists for every function $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$.

If $k = n-1$, the previous integral is a $O\left(\log\frac{1}{1-r}\right)$, in particular

$$\lim_{r\to 1}(1-r^2)\int_{\mathbb{S}^{n-1}} \mathbb{X}u(r\zeta)\Phi(\zeta)d\sigma(\zeta) = 0.$$

Remark 1 : If u has a boundary distribution, then $\mathcal{L}_{i,j}u$ has a boundary distribution.

Remark 2 : As ∇^k is generated outside a neighbourhood of the origin by operators of the form $N^l\mathbb{Y}$ where $\mathbb{Y}$ is a product of at most $k-l$ operators of the form $\mathcal{L}_{i,j}$, We deduce from the lemma that if $k\leq n-2$, ∇^k has a boundary distribution, whereas

$$\int_{\mathbb{S}^{n-1}} \nabla^{n-1}u(r\zeta)\Phi(\zeta)d\sigma(\zeta)$$

has *a priori* logarithmic growth.

Proof. Proceed by induction on k. Fix $\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ and let $\mathbb{Y}$ be a product of operators of the form $\mathcal{L}_{i,j}$. Let

$$\psi_k(r)=\int_{\mathbb{S}^{n-1}} N^k\mathbb{Y}u(r\zeta)\Phi(\zeta)d\sigma(\zeta);\qquad 0<r<1.$$

Applying $\mathbb{Y}$ to formula (3.4) and noticing that $\mathbb{Y}$ and N commute, the induction hypothesis implies that the function

$$g(r)=(1-r^2)N\psi_k(r)-2(k-1)r^2\psi_k(r)+(n-2)(1+r^2)\psi_k(r) \tag{3.5}$$

has a limit L when $r\to 1$.

But, solving the differential equation (3.5), ($N=r\frac{d}{dr}$), we get

$$\psi_k(r)=\lambda\frac{(1-r^2)^{n-k-1}}{r^{n-2}}+\frac{1}{r^{n-2}}(1-r^2)^{n-k-1}\int_0^r\frac{g(s)s^{n-3}}{(1+s)^{n-k}}(1-s)^{-(n-k-1)-1}ds.$$

Thus, if $k<n-1$, we obtain that $\psi_k(r)$ has limit $\frac{L}{2(n-k-1)}$ whereas if $k=n-1$, $\psi_k(r)$ has logarithmic growth. □

Remark : We will show at the end of this section that if n is even, $N^{n-1}u$ can have a better than logarithmic growth, whereas if n is odd, only constant functions have a better than logarithmic growth.

Corollary 7 *Let P_k be the sequence of polynomials defined by $P_0=2(n-1)$, $P_1=0$ and for $2\leq k\leq n$,*

$$\begin{aligned}P_k(X)=&2^{k-1}(k-1)!\sum_{j=2}^{k-2}\frac{n(j-1)-(n-2)k}{2^j(n-j-1)(k-j+1)!(j-1)!}P_j(X)\\&+2^{k-2}(k-1)!\sum_{j=2}^{k-3}\frac{1}{2^j(n-j-1)(k-j-1)!j!}XP_j(X)+2^{k-1}X\end{aligned}$$

Then, for every $\mathcal{H}$-harmonic function u having a distribution boundary value, and for every $1\leq k\leq n-2$, $N^ku=\frac{1}{2(n-k-1)}P_k(\Delta_\sigma)u$ as boundary distributions, i.e. for every $\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$,

$$\lim_{r\to1}\int_{\mathbb{S}^{n-1}}\left(N^ku(r\zeta)-\frac{1}{2(n-k-1)}P_k(\Delta_\sigma)u(r\zeta)\right)\Phi(\zeta)d\sigma(\zeta)=0.$$

Proof. For convenience, write $Q_k=\frac{1}{2(n-k-1)}P_k$. As $n\geq 3$, for u $\mathcal{H}$-harmonic having a boundary distribution, formula (3.3) and lemma 6 impliy that $Nu=0$ on the boundary, thus the result for $k=1$.

Next, notice that $N^k u = Q_k(\Delta_\sigma)u$ on the boundary implies $\Delta_\sigma N^k u = \Delta_\sigma Q_k(\Delta_\sigma)u$ on the boundary.

Assume now that $N^j u = Q_j(\Delta_\sigma)u$ on the boundary for $j \leq k-1$. If $k \leq n-2$, lemma 6 tells us that $(1-r^2)N^{k+1}u = 0$ on the boundary and that $(1-r^2)N^{k-1}\Delta_\sigma u = 0$ on the boundary. Formula (3.4) gives then, when $r \to 1$,

$$\begin{aligned}\big(-2(k-1)+2(n-2)\big)N^k u =& \sum_{j=0}^{k-3}\binom{k-1}{j}2^{k-j-1}N^{j+2}u + \sum_{j=0}^{k-3}\binom{k-1}{j}2^{k-j-1}N^j\Delta_\sigma u \\ &-(n-2)\sum_{j=0}^{k-2}\binom{k-1}{j}2^{k-j-1}N^{j+1}u.\end{aligned}$$

But, by the induction hypothesis, $N^j u = Q_j(\Delta_\sigma)u$ and with the previous remark $N^j\Delta_\sigma u = \Delta_\sigma N^j u = \Delta_\sigma Q_j(\Delta_\sigma)u$, therefore

$$\begin{aligned}\big(-2(k-1)+2(n-2)\big)N^k u =& \sum_{j=0}^{k-3}\binom{k-1}{j}2^{k-j-1}Q_{j+2}(\Delta_\sigma)u + \sum_{j=0}^{k-3}\binom{k-1}{j}2^{k-j-1}\Delta_\sigma Q_j(\Delta_\sigma)u \\ &-(n-2)\sum_{j=0}^{k-2}\binom{k-1}{j}2^{k-j-1}Q_{j+1}(\Delta_\sigma)u.\end{aligned}$$

finally, using $Q_0 = 1$ and $Q_1 = 0$ and grouping terms, we get the desired result. □

Remark 1 : One easily sees that P_k is a polynomial of degree $\left[\frac{k}{2}\right]$ and that for $k \geq 2$, P_k has no constant term.

Remark 2 : According to corollary 7, $Nu = 0$ on the boundary. On the other hand, an easy computation leads to $DNu = -4(n-2)Nu$ *i.e.* Nu is an eigenvector of D for an eigenvalue of the form $(s^2-1)(n-1)^2$ (with $s = \frac{n-3}{n-1}$) thus $(s+1)\frac{n-1}{2} = n-2 \in \mathbb{N}^*$. This is precisely the case where it is impossible to reconstruct Nu with help of a convolution by a power of the Poisson kernel (see [12]).

Remark 3 : The fact that for every $\mathcal{H}$-harmonic function u, $Nu = 0$ on the boundary is in strong contrast with Euclidean harmonic functions. Actually, if v is an Euclidean harmonic function on $\mathbb{B}_n$, and if $Nv = 0$ on the boundary, then v is a constant.

3.4. **Boundary distribution of the $n-1^{th}$ derivative.** *In this section we prove that, in odd dimension, normal derivatives of $\mathcal{H}$-harmonic functions have a boundary behavior similar to the complex case of $\mathcal{M}$-harmonic functions as exhibited in* [2] *(with pluriharmonic functions playing the role of constant functions) whereas, in even dimension, the behavior is similar to the Euclidean harmonic case.*

Theorem 8 ⋄ *Assume n is odd.*

Let u be an $\mathcal{H}$-harmonic function having a boundary distribution. The following assertions are equivalent :

(1) *u is a constant,*
(2) *$N^{n-1}u$ has a boundary distribution,*

(3) $\int_{\mathbb{S}^{n-1}} N^{n-1}u(r\zeta)\Phi(\zeta)d\sigma(\zeta) = o\left(\log\frac{1}{1-r}\right)$ *for every* $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$.

$\diamond$ *Assume n is even, then if $\varphi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\mathbb{P}_h[\varphi] \in \mathcal{C}^\infty(\overline{\mathbb{B}_n})$. In particular, if u is $\mathcal{H}$-harmonic with a boundary distribution, then for every $k \geq 0$, $N^k u$ has a boundary distribution.*

Proof. $\diamond$ Assume first n is odd. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) being obvious, let us prove (3) $\Rightarrow$ (1). Theorem 1 tells us that an $\mathcal{H}$-harmonic function u admits an expansion in spherical harmonics

$$u(r\zeta) = \sum_{l\geq 0} f_l(r^2)r^l u_l(\zeta) \tag{3.6}$$

where u_l is a spherical harmonic of degree l and f_l is the hypergeometric function

$$f_l(x) = \frac{{}_2F_l(l, 1-\frac{n}{2}, l+\frac{n}{2}, x)}{{}_2F_l(l, 1-\frac{n}{2}, l+\frac{n}{2}, 1)} = \sum_{k=0}^{\infty} \frac{\Gamma(l+k)\Gamma(1-\frac{n}{2}+k)\Gamma(l+\frac{n}{2})\Gamma(1)}{\Gamma(l)\Gamma(1-\frac{n}{2})\Gamma(l+\frac{n}{2}+k)\Gamma(1+k)} x^k.$$

Moreover the sum (3.6) converges uniformly on compact subsets of $\mathbb{B}_n$, in particular

$$\|u_l\|_{L^2(\mathbb{S}n-1)} f_l(r^2)r^l = \int_{\mathbb{S}^{n-1}} u(r\zeta)u_l(\zeta)d\sigma(\zeta).$$

On the other hand, if $l \neq 0$ as n is odd,

$$\frac{\Gamma(l+k)\Gamma(1-\frac{n}{2}+k)\Gamma(l+\frac{n}{2})\Gamma(1)}{\Gamma(l)\Gamma(1-\frac{n}{2})\Gamma(l+\frac{n}{2}+k)\Gamma(1+k)} = \frac{\Gamma(l+\frac{n}{2})\Gamma(1)}{\Gamma(l)\Gamma(1-\frac{n}{2})} \frac{1}{k^n}\left[1 + O\left(\frac{1}{k}\right)\right],$$

thus the $n-2$ first derivatives of F_l have a limit when $x \to 1$, whereas the $n-1$-st derivative grows like $\log(1-x)$ when $x \to 1$, thus (3) implies that $u_l = 0$ for $l \neq 0$, that is u is constant. $\diamond$

$\diamond$ Assume now n is even and write $n = 2p$. Then if $\varphi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, φ admits a decomposition into spherical harmonics $\varphi = \sum_{l=0}^{+\infty} \varphi_l$ with $\|\varphi_l\|_\infty = O(l^{-\alpha})$ for every $\alpha > 0$ ([13] appendix C). But then

$$\mathbb{P}_h[\varphi](r\zeta) = \sum_{l=0}^{+\infty} f_l(r)r^l\varphi_l(\zeta)$$

with

$$f_l(r)r^l = \frac{{}_2F_1(l, 1-p, l+p, r^2)}{{}_2F_1(l, 1-p, l+p, 1)} r^l = \frac{\Gamma(l+2p-1)\Gamma(p)}{\Gamma(l+p)\Gamma(2p-1)} \sum_{j=0}^{p} \frac{(l)_j(1-p)_j}{(l+p)_j j!} r^{2j+l}.$$

But, for every $k \geq 0$,

$$N^k\left(\sum_{j=0}^{p} \frac{(l)_j(1-p)_j}{(l+p)_j j!} r^{2j+l}\right) = \sum_{j=0}^{p} \frac{(l)_j(1-p)_j}{(l+p)_j j!}(2j+l)^k 2^k r^{2j+l}.$$

Therefore $N^k(f_l r^l)(1) = O(l^{k+p-1})$. But $\|\varphi_l\|_\infty = O(l^{-(k+p+1)})$ thus $\sum_{l=0}^{+\infty} N^k f_l(r)\varphi_l(\zeta)$ converges uniformly on $\overline{\mathbb{B}_n}$ and $\mathbb{P}_h[\varphi] \in \mathcal{C}^\infty(\overline{\mathbb{B}_n})$.

The fact that for u $\mathcal{H}$-harmonic with a boundary distribution, $N^k u$ has also a boundary distribution then results from the symmetry of the Poisson kernel : $\mathbb{P}_h(r\zeta, \xi) = \mathbb{P}_h(r\xi, \zeta)$. $\square$

Remark 1 : Normal derivatives of $\mathcal{H}$-harmonic functions have two opposite behaviors depending on the dimension of $\mathbb{B}_n$. In odd dimension, the behavior is similar to the complex case (see [2], in this case, the analog of constant functions are pluriharmonic functions).

In opposite, in even dimension, the behavior is similar to that of Euclidean harmonic functions.

Remark 2 : The similarity with the Euclidean case can be seen in a different way. In [12], the following link between Euclidean harmonic functions and $\mathcal{H}$-harmonic functions has been proved :

Lemma 9 *For every $\mathcal{H}$-harmonic function u, there exists a unique Euclidean harmonic function v such that $v(0) = 0$ and :*

$$u(r\zeta) = u(0) + \int_0^1 v(rt\zeta)\big[(1-t)(1-tr^2)\big]^{\frac{n}{2}-1}\frac{dt}{t}$$

for every $0 \leq r < 1$ and every $\zeta \in \mathbb{S}^{n-1}$.

Moreover, let $f = \sum_l u_l$ is the spherical harmonics expansion of $f \in L^2(\mathbb{S}^{n-1})$ and if $g = \sum_l \frac{\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l)} u_l$, then lemma 9 links $u = \mathbb{P}_h[f]$ to $v = \mathbb{P}_e[g]$.

But, if $f = \sum_l u_l \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $g = \sum_l \frac{\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l)} u_l$. Then, as $\|u_l\|_\infty = O(l^{-\alpha})$ for every $\alpha > 0$, $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ thus $v = \mathbb{P}_e[g] \in \mathcal{C}^\infty(\overline{\mathbb{B}_n})$.

Moreover, if n is even $(1-tr^2)^{\frac{n}{2}-1}$ is a polynomial and is therefore $\mathcal{C}^\infty$, we then find again that $u \in \mathcal{C}^\infty(\overline{\mathbb{B}_n})$.

In opposite, if n is odd, we find again the $n-1$ obstacle since the highest order term of $(1-t)^{\frac{n}{2}-1}N^k(1-tr^2)^{\frac{n}{2}-1}$ is

$$(1-t)^{\frac{n}{2}-1}(1-tr^2)^{\frac{n}{2}-1-k} \simeq (1-t)^{n-2-k}$$

when $r \to 1$, and since $(1-t)^{n-2-k}$ is not integrable for $k \geq n-1$.

4. Atomic decomposition of $\mathcal{H}^p$ spaces

In this section we prove that $\mathcal{H}^p$ spaces admit an atomic decomposition. In 4.2 we define $\mathcal{H}^p_{at}$ and show that this space is included in $\mathcal{H}^p$. Conversely, we have seen in the previous chapter that $\mathcal{H}$-harmonic functions in $\mathcal{H}^p$ are obtained by $\mathcal{H}$-Poisson integration of distributions on $\mathbb{S}^{n-1}$, thus they are extensions of distributions from $\mathbb{S}^{n-1}$ to $\mathbb{B}_n$. An other mean to extend a distribution on $\mathbb{S}^{n-1}$ to $\mathbb{B}_n$ is integration with respect to the Euclidean Poisson kernel. In 4.1 we study the links between this two extensions, which allows us in 4.3 to obtain the inclusion $\mathcal{H}^p \subset \mathcal{H}^p_{at}$ from the atomic decomposition of H^p spaces of Euclidean harmonic functions.

4.1. **Links between Euclidean harmonic functions and $\mathcal{H}$-harmonic functions.** We will now prove a “converse” to lemma 9.

Lemma 10 *There exists a function $\eta : [0,1] \times [0,1] \mapsto \mathbb{R}^+$ such that*

- **i:** $\mathbb{P}_e(r\zeta, \xi) = \int_0^1 \eta(r,\rho)\mathbb{P}_h(\rho r\zeta, \xi)d\rho$,
- **ii:** *there exists a constant C such that for every $r \in [0,1]$, $\int_0^1 \eta(r,\rho)d\rho \leq C$.*

Proof. Note that $\frac{1}{(x+y)^{\frac{n}{2}}} = c_n \int_0^\infty \frac{z^{\frac{n}{2}-2}}{(x+y+z)^{n-1}}dz$. Writing $X = 2(1- <\zeta,\xi>)$, with an obvious abuse of language, we then get

$$\begin{aligned}\mathbb{P}_e(r,X) =& \frac{1-r^2}{\big((1-r)^2+rX\big)^{\frac{n}{2}}} = \frac{1-r^2}{r^{\frac{n}{2}}}\frac{1}{\Big[\frac{(1-r)^2}{r}+X\Big]^{\frac{n}{2}}}\\ =& \frac{1-r^2}{r^{\frac{n}{2}}}c_n\int_0^\infty \frac{z^{\frac{n}{2}-2}}{\Big[X+\frac{(1-r)^2}{r}+z\Big]^{n-1}}dz\end{aligned}$$

The following change of variable $z=\frac{(1-\rho)^2}{\rho}-\frac{(1-r)^2}{r}=\frac{(r-\rho)(1-\rho r)}{\rho r}$, leads to

$$\begin{aligned}
\mathbb{P}_e(r,X)=&\frac{1-r^2}{r^{\frac{n}{2}}}c_n\int_0^r\frac{[(r-\rho)(1-\rho r)]^{\frac{n}{2}-2}}{\left[X+\frac{(1-\rho)^2}{\rho}\right]^{n-1}(\rho r)^{\frac{n}{2}-2}}\frac{1-\rho^2}{\rho^2}d\rho\\
=&\frac{1-r^2}{r^{n-2}}c_n\int_0^r\frac{[(r-\rho)(1-\rho r)]^{\frac{n}{2}-2}(1-\rho^2)}{[\rho X+(1-\rho^2)]^{n-1}\rho^{1-\frac{n}{2}}}d\rho\\
=&\frac{1-r^2}{r^{n-2}}c_n\int_0^r\mathbb{P}_h(\rho,X)(1-\rho^2)^{2-n}[(r-\rho)(1-\rho r)]^{\frac{n}{2}-2}\rho^{\frac{n}{2}-1}d\rho\\
=&c_n(1-r^2)\int_0^1\mathbb{P}_h(rs,X)(1-r^2s^2)^{2-n}\left[(1-s)(1-sr^2)\right]^{\frac{n}{2}-2}s^{\frac{n}{2}-1}ds
\end{aligned}$$

We thus obtain $i/$ with

$$\eta(r,s)=c_n(1-r^2)(1-r^2s^2)^{2-n}\left[(1-s)(1-sr^2)\right]^{\frac{n}{2}-2}s^{\frac{n}{2}-1}.$$

Of course $\eta\geq 0$ and one easily checks that $\int_0^1\eta(r,s)ds\leq C$, since $n\geq 3$. □

Corollary 11 *Let η be the function defined by lemma 10. Let f be a distribution on $\mathbb{S}^{n-1}$ and let $u=\mathbb{P}_h[f]$ and $v=\mathbb{P}_e[f]$. Then u and v are linked by*

$$v(r\zeta)=\int_0^1\eta(r,s)u(rs\zeta)ds.$$

In particular, if $u\in\mathcal{H}^p$, then $v\in H^p(\mathbb{B}_n)$ and $\|v\|_{H^p(\mathbb{B}_n)}\leq C\|u\|_{\mathcal{H}^p}$.

4.2. The inclusion $\mathcal{H}^p_{at}\subset\mathcal{H}^p$.

Definition *A function a on $\mathbb{S}^{n-1}$ is called a p-atom on $\mathbb{S}^{n-1}$ if either a is a constant or a is supported in a ball $\tilde{B}(\xi_0,r_0)$ and if*

1: $|a(\xi)|\leq\sigma\left[\tilde{B}(\xi_0,r_0)\right]^{-\frac{1}{p}}$, *for almost every $\xi\in\mathbb{S}^{n-1}$,*
2: *for every function $\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$,*

$$\left|\int_{\mathbb{S}^{n-1}}a(\xi)\Phi(\xi)d\sigma(\xi)\right|\leq\left\|\nabla^{k(p)}\Phi\right\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}r_0^{k(p)}\sigma\left[\tilde{B}(\xi_0,r_0)\right]^{1-\frac{1}{p}}$$

with $k(p)$ an integer strictly bigger than $(n-1)\left(\frac{1}{p}-1\right)$.

Proposition 12 *There exists a constant C_p such that, for every p-atom a on $\mathbb{S}^{n-1}$, $A=\mathbb{P}_h[a]$ satisfies*

$$\|A\|_{\mathcal{H}^p(\mathbb{B}_n)}\leq C_p.$$

Proof. Let a be a p-atom on $\mathbb{S}^{n-1}$, with support in $\tilde{B}(\xi_0,r_0)$. We want to estimate

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \sup_{t\in[0,1]} &\left|\int_{\tilde{B}(\xi_0,r_0)} \mathbb{P}_h(t\zeta,\xi)a(\xi)d\sigma(\xi)\right|^p d\sigma(\zeta)\\ =&\int_{\tilde{B}(\xi_0,cr_0)} \sup_{t\in[0,1]} \left|\int_{\tilde{B}(\xi_0,r_0)} \mathbb{P}_h(t\zeta,\xi)a(\xi)d\sigma(\xi)\right|^p d\sigma(\zeta)\\ &+\int_{\mathbb{S}^{n-1}\setminus\tilde{B}(\xi_0,cr_0)} \sup_{t\in[0,1]} \left|\int_{\tilde{B}(\xi_0,r_0)} \mathbb{P}_h(t\zeta,\xi)a(\xi)d\sigma(\xi)\right|^p d\sigma(\zeta)\\ =&I_1+I_2\end{aligned}$$

with $c>1$ a constant. But, by Hölder's inequality,

$$\begin{aligned}I_1=\int_{\tilde{B}(\xi_0,cr_0)} \sup_{t\in[0,1]} |\mathbb{P}_h[a](t\zeta)|^p d\sigma(\zeta) &\leq c\sigma\big(\tilde{B}(\xi_0,cr_0)\big)^{1-\frac{p}{2}}\left[\int_{\tilde{B}(\xi_0,cr_0)} \sup_{t\in[0,1]} |\mathbb{P}_h[a](t\zeta)|^2 d\sigma(\zeta)\right]^{\frac{p}{2}}\\ \leq& c\sigma\big(\tilde{B}(\xi_0,cr_0)\big)^{1-\frac{p}{2}}\|\mathbb{P}_h[a]\|^p_{\mathcal{H}^2(\mathbb{B}_n)} \leq c\sigma\big(\tilde{B}(\xi_0,cr_0)\big)^{1-\frac{p}{2}}\|a\|^p_{L^2(\mathbb{S}^{n-1})}\end{aligned}$$

since $\mathbb{P}_h$ is bounded $L^2(\mathbb{S}^{n-1})\mapsto\mathcal{H}^2(\mathbb{B}_n)$. Using property (1) of atoms, we see that

$$I_1\leq C\left(\frac{\sigma\big(\tilde{B}(\xi_0,cr_0)\big)}{\sigma\big(\tilde{B}(\xi_0,r_0)\big)}\right)^{1-\frac{p}{2}}\leq C_p.$$

Let us now estimate I_2. Using property (2) of atoms, we have, for $\zeta\in\mathbb{S}^{n-1}\setminus\tilde{B}(\xi_0,cr_0)$

$$\begin{aligned}\left|\int_{\tilde{B}(\xi_0,r_0)} \mathbb{P}_h(t\zeta,\xi)a(\xi)d\sigma(\xi)\right|^p \leq& r_0^{pk(p)}\left\|\nabla_\xi^{k(p)}\mathbb{P}_h(t\zeta,\xi)\right\|^p_{L^\infty}\sigma\big(\tilde{B}(\xi_0,r_0)\big)^{p-1}\\ \leq& C_p r_0^{pk(p)}(1-t^2)^{n-1}\times\sup_{\xi\in\tilde{B}(\xi_0,r_0)}\frac{1}{d(\zeta,\xi)^{p(n+k(p)-1)}}\sigma\big(\tilde{B}(\xi_0,r_0)\big)^{p-1}\end{aligned}$$

thus

$$\begin{aligned}I_2\leq& C_p r_0^{pk(p)}\sigma\big(\tilde{B}(\xi_0,r_0)\big)^{p-1}\int_{\mathbb{S}^{n-1}\setminus\tilde{B}(\xi_0,cr_0)}\sup_{\xi\in\tilde{B}(\xi_0,r_0)}\frac{1}{d(\zeta,\xi)^{p(n+k(p)-1)}}d\sigma(\zeta)\\ \leq& C_p\frac{r_0^{pk(p)}r_0^{(n-1)(p-1)}}{r_0^{\left[p\left(1+\frac{k(p)}{n-1}\right)-1\right](n-1)}}\end{aligned}$$

since $p(n+k(p)-1)>n-1$ *i.e.* $k(p)>(n-1)\left(\frac{1}{p}-1\right)$. Thus $I_2\leq C_p$. □

Remark 1 : Condition (2) implies with $\Phi=1$ that

$$\int_{\mathbb{S}^{n-1}} a(\xi)d\sigma(\xi)=0.$$

Remark 2 : Condition (2) is equivalent to the *a priori* weaker condition :

2′: For every spherical harmonic P of degree $\leq k(p)$,

$$\left|\int_{\mathbb{S}^{n-1}} a(\xi)P(\xi)d\sigma(\xi)\right|\leq\left\|\nabla^{k(p)}P\right\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}r_0^{k(p)}\sigma\left[\tilde{B}(\xi_0,r_0)\right]^{1-\frac{1}{p}}.$$

Proof. Assume this condition is fulfilled and let $\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$. There exists P, a linear combination of spherical harmonics of degree $\leq k(p)$ and $R\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ such that

(1) $\Phi=P+R$,
(2) $\|R\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}\leq C_p r_0^{k(p)}\left\|\nabla^{k(p)}\Phi\right\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}$.

Then

$$\begin{aligned}\left|\int_{\mathbb{S}^{n-1}}a(\xi)\Phi(\xi)d\sigma(\xi)\right|\leq&\left|\int_{\mathbb{S}^{n-1}}a(\xi)P(\xi)d\sigma(\xi)\right|+\left|\int_{\mathbb{S}^{n-1}}a(\xi)R(\xi)d\sigma(\xi)\right|\\ \leq& C_p\left\|\nabla^{k(p)}P\right\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}r_0^{k(p)}\left[\sigma\left(\tilde{B}(\xi_0,r_0)\right)\right]^{1-\frac{1}{p}}\\ &+\|a\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}\|R\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}\sigma\left(\tilde{B}(\xi_0,r_0)\right)\\ \leq& C\left\|\nabla^{k(p)}\Phi\right\|_{L^\infty\left(\tilde{B}(\xi_0,r_0)\right)}r_0^{k(p)}\left[\sigma\left(\tilde{B}(\xi_0,r_0)\right)\right]^{1-\frac{1}{p}}.\end{aligned}$$

We could also impose the following weaker condition

3: For every spherical harmonic P of degree $\leq k(p)$,

$$\left|\int_{\mathbb{S}^{n-1}}a(\xi)P(\xi)d\sigma(\xi)\right|=0.$$

We would than obtain a stronger atomic decomposition theorem. However this version is sufficient for our needs. It is also more intrinsic, the estimates we impose are directly those that are needed in the proof and finally it allows us to stay near to the proof in [8].

Definition *A function A on $\mathbb{B}_n$ is called an $\mathcal{H}^p$-*atom *on $\mathbb{B}_n$ if there exists a p-atom a on $\mathbb{S}^{n-1}$ such that $A=\mathbb{P}_h[a]$.*

We define $\mathcal{H}^p_{at}(\mathbb{B}_n)$ as the space of distributions u on $\mathbb{B}_n$ such that there exists :

(1) *a sequence of $\mathcal{H}_p$-atoms $(A_j)_{j=1}^\infty$ on $\mathbb{B}_n$,*
(2) *a sequence $(\lambda_j)_{j=1}^\infty\in\ell^p$ such that*

$$u=\sum_{j=1}^\infty\lambda_jA_j,\tag{4.1}$$

with uniform convergence on compact subsets of $\mathbb{B}_n$.

We write

$$\|u\|_{\mathcal{H}^p_{at}}=\inf\left\{\left(\sum_{i=1}^\infty|\lambda_j|^p\right)^{\frac{1}{p}}\right\}$$

where the infimum is taken over all decompositions of u of the form (4.1).

Proposition 13 *For $0<p\leq1$, $\mathcal{H}^p_{at}(\mathbb{B}_n)\subset\mathcal{H}^p(\mathbb{B}_n)$ there exists a constant C_p such that for every $u\in\mathcal{H}^p_{at}(\mathbb{B}_n)$,*

$$\|u\|_{\mathcal{H}^p}\leq C_p\|u\|_{\mathcal{H}^p_{at}}.$$

Proof. It is *mutatis mutandis* the proof of theorem 2.2 in [8].

Let $\varepsilon > 0$ and let $u = \sum_{j=1}^{\infty} \lambda_j A_j$ be a function in $\mathcal{H}^p_{at}$ and take an atomic decomposition such that $\sum_{i=1}^{\infty} |\lambda_j|^p \leq (1+\varepsilon)\|u\|^p_{\mathcal{H}^p_{at}}$.

Property 2 of atoms implies that

$$\begin{aligned}\left|\nabla^k A_j(x)\right| =&\left|\nabla^k \mathbb{P}_h[a_j]\right| \leq \left\|\nabla_\xi^{k(p)} \nabla_x^k \mathbb{P}_h(x,.)\right\|_{L^\infty\left(B(\xi_0,r_0)\right)} r_0^{k(p)} \sigma\big(B(\xi_0,r_0)\big)^{1-\frac{1}{p}}\\ \leq&\frac{C_{p,k}}{(1-|x|)^{k_{p,l}}}\end{aligned}$$

the series $\sum_{j=1}^{\infty} \lambda_j \nabla^k A_j(x)$ converge uniformly on every compact subset of $\mathbb{B}_n$, thus $\sum_{j=1}^{\infty} \lambda_j A_j(x)$ defines an $\mathcal{H}$-harmonic function on $\mathbb{B}_n$.

Moreover

$$\left|\sum_{j=1}^{\infty} \lambda_j A_j(x)\right|^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p |A_j(x)|^p.$$

Therefore

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \sup_{0<r<1} \left|\sum_{j=1}^{\infty} \lambda_j A_j(r\zeta)\right|^p d\sigma(\zeta) \leq& \int_{\mathbb{S}^{n-1}} \sup_{0<r<1} \sum_{j=1}^{\infty} |\lambda_j|^p |A_j(r\zeta)|^p d\sigma(\zeta)\\ \leq& C_p^p \sum_{j=1}^{\infty} |\lambda_j|^p\\ \leq&(1+\varepsilon)^p C_p^p \|u\|^p_{\mathcal{H}^p_{at}}\end{aligned}$$

which means that $\|u\|_{\mathcal{H}^p} \leq C\|u\|^p_{\mathcal{H}^p_{at}}$. □

4.3. **The inclusion $\mathcal{H}^p \subset \mathcal{H}^p_{at}$.** We will here use the fact that the space $H^p(\mathbb{B}_n)$ of Euclidean harmonic functions v such that $\mathcal{M}[v] \in L^p(\mathbb{S}^{n-1})$ admits an atomic decomposition *i.e.* that for every function $v \in H^p$, there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^p$ and a sequence $(a_k)_{k\in\mathbb{N}}$ of p-atoms on $\mathbb{S}^{n-1}$ such that

$$v(r\zeta) = \sum_{k\in\mathbb{N}} \lambda_k \mathbb{P}_e[a_k](r\zeta) \tag{4.2}$$

and moreover

$$\|v\|_{H^p} \simeq \left(\sum_{k\in\mathbb{N}} |\lambda_k|^p\right)^{\frac{1}{p}}.$$

This result is well known, however it seems difficult to find an adequate reference. One may for instance adapt the proof of Garnett and Latter [6] as outlined in [3].

Let $u \in \mathcal{H}^p$, then u admits a boundary distribution f and $u = \mathbb{P}_h[f]$. Then let $v = \mathbb{P}_e[f]$. By lemma 10, $v \in H^p(\mathbb{B}_n)$ thus v admits an atomic decomposition *i.e.* there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^p$ and a sequence $(a_k)_{k\in\mathbb{N}}$ of p-atoms on $\mathbb{S}^{n-1}$ such that v is given by (4.2), thus

$$f = \sum_{k=0}^{\infty} \lambda_k a_k$$

in the sense of distribution. Therefore $u = \mathbb{P}_h[\sum \lambda_k a_k] = \sum \lambda_k \mathbb{P}_h[a_k]$, the series being convergent in $\mathcal{H}^p$ by proposition 13. We have thus proved the following theorem :

Theorem 14 *For every $0 < p \leq 1$, $\mathcal{H}^p = \mathcal{H}^p_{at}$ and the norms are equivalent.*

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