Harmonic Functions on the Real Hyperbolic Ball II : Hardy and Lipschitz Spaces

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Harmonic functions on the real hyperbolic ball II Hardy-Sobolev and Lipschitz spaces

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Abstract. In this paper, we pursue the study of harmonic functions on the real hyperbolic ball started in [13]. Our focus here is on the theory of Hardy-Sobolev and Lipschitz spaces of these functions. We prove here that these spaces admit Fefferman-Stein like characterizations in terms of maximal and square functionals. We further prove that the hyperbolic harmonic extension of Lipschitz functions on the boundary extend into Lipschitz functions on the whole ball. In doing so, we exhibit differences of behaviour of derivatives of harmonic functions depending on the parity of the dimension of the ball and on the parity of the order of derivation.

1. Introduction

In this article, the sequel of [13], we study Hardy-Sobolev and Lipschitz spaces of harmonic functions on the real hyperbolic ball.

The main motivation of this paper lies in the recent developments of the theory of Hardy-Sobolev spaces of $\mathcal{M}$-harmonic functions related to the complex hyperbolic metric on the unit ball, as exposed in [1] and [2]. Our aim here is to develop a similar theory in the case of the real hyperbolic ball.

Our starting point is a result of [13] stating that the Hardy spaces $\mathcal{H}^p$ of hyperbolic harmonic functions ($\mathcal{H}$-harmonic functions in the terminology of [13]) admit an atomic decomposition similar to the Euclidean harmonic functions. Then, for $0 < p < +\infty$, define the space $H^p(\mathbb{S}^{n-1})$ as $L^p(\mathbb{S}^{n-1})$ if $p > 1$ and as the equivalent of Garnett-Latter's atomic H^p-space if $0 < p \leq 1$ (*see* [13] for the exact definition). This space has been characterized in terms of square functionals of the Euclidean harmonic extensions of its elements by Colzani [5].

Further, Cifuentes [3] has extended the area integral characterization of H^p spaces for $\mathbb{R}^{n+1}_+$ to all symmetric spaces of rank one of the non-compact type, including the real hyperbolic spaces. For sake of completeness, we will give here the g-functional

characterization of $\mathcal{H}^p$ spaces, even though the proof being routine once the area characterization is obtained. We will then focus on developing a theory of Hardy-Sobolev spaces of $\mathcal{H}$-harmonic functions, similar to the one developed in [2]. The first step is to prove mean-value inequalities for $\mathcal{H}$-harmonic functions and their derivatives. This is done by adapting the proof in [2] using the theory of hypo-elliptic operators. We think that our mean-value inequalities have an interest of their own and that the proof should adapt to all rank one spaces of the non-compact type. The remaining of the proofs are direct adaptations of [2]. However, as in [13] where it is proved that the boundary behavior of derivatives of $\mathcal{H}$-harmonic functions depends on the parity of the dimension of $\mathbb{B}_n$, it is proved here that the characterizations of Hardy-Sobolev spaces depend on the parity of the order of derivation. Note that Graham [10] has already noticed a dependence of the behavior of harmonic functions on the parity of the dimension of the balls.

Finally, we will characterize some Lipschitz spaces on the sphere via their $\mathcal{H}$-harmonic extension on the real hyperbolic ball. In even dimension, we will take advantage of links between $\mathcal{H}$-harmonic functions and Euclidean harmonic functions exhibited in [18] (see (2.1) in section 2.3 below). This will allow us to characterize Lipschitz functions of any order. In odd dimension, the situation is different as the characterization only holds for Lipschitz functions of order $< n-1$ but may fail for Lipschitz functions of order $n-1$.

This article is organized as follows. In the next section we present the setting of our problem and state our main results. Section 3 is devoted to the proofs of the technical lemmas we will need, including the mean-value inequalities. We conclude this section by completing the Fefferman-Stein characterization of our $\mathcal{H}^p$ spaces. The following section is devoted to the proofs of similar characterizations for Hardy-Sobolev spaces while in the last section we give the results on Lipschitz spaces.

2. Statement of the problem and results

2.1. $SO(n,1)$ and its action on $\mathbb{B}_n$

We consider $G = SO_0(n,1) \subset GL_{n+1}(\mathbb{R})$, $(n \geq 3)$ the identity component of the group of matrices $g = (g_{ij})_{0 \leq i,j \leq n}$ such that $g_{00} \geq 1$, $\det g = 1$ and that leave invariant the quadratic form $-x_0^2 + x_1^2 + \ldots + x_n^2$. The group G admits both a Cartan decomposition $G = K\overline{A_+}K$ and an Iwasawa decomposition $G = NAK$ with

$$K = \left\{ k = \begin{pmatrix} 1 & 0 \\ 0 & \hat{k} \end{pmatrix} \ : \ \hat{k} \in SO(n) \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & Id_{n-1} \end{pmatrix} \ : \ t \in \mathbb{R} \right\} \quad \text{and} \quad A_+ = \{a_t \ : \ t \in \mathbb{R}^+\}$$

and

$$
N = \left\{ n_\xi = \begin{pmatrix} 1 + \frac{\delta^2}{2} & -\frac{\delta^2}{2} & \xi_2 & \cdots & \xi_n \\ \frac{\delta^2}{2} & 1 - \frac{\delta^2}{2} & \xi_2 & \cdots & \xi_n \\ \xi_2 & -\xi_2 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ \xi_n & -\xi_n & 0 & & 1 \end{pmatrix}, \ \xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1} \right\}
$$

In this decomposition, every $g \in G$ can either be written as $g = k_g a_{t(g)} k_g'$ with $k_g, \ k_g' \in K$ and $a_{t(g)} \in A_+$ or $g = n_g a_{s(g)} \tilde{k}_g$ with $n_g \in N$, $a_{s(g)} \in A$ and $\tilde{k}_g \in K$.
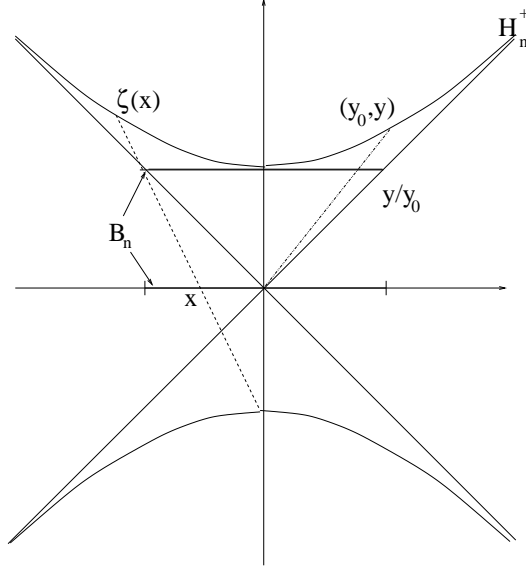
Figure 1: Two possible identifications of $\mathbb{H}_n^+$ with $\mathbb{B}_n$: the conformal model used in this paper where $\zeta(x) \in \mathbb{H}_n^+$ is identified with $x \in \mathbb{B}_n$ and the perhaps more usual model where $(y_0, y) \in \mathbb{H}_n^+$ is identified with $\frac{y}{y_0} \in \mathbb{B}_n$.

Let $|.|$ be the Euclidean norm on $\mathbb{R}^n$ and $\langle ., . \rangle$ the associated scalar product. Let $\mathbb{B}_n = \{ x \in \mathbb{R}^n \ : \ |x| < 1 \}$ and $\mathbb{S}^{n-1} = \partial \mathbb{B}_n = \{ x \in \mathbb{R}^n \ : \ |x| = 1 \}$. The homogeneous space G/K can be identified with the upper sheet $\mathbb{H}_+^n$ of the two-sheeted hyperboloïd $-x_0^2 + x_1^2 + \dots + x_n^2 = -1$. This in turn may be identified with $\mathbb{B}_n$. There are several ways to do so (see figure 2.1). We will here use the "conformal ball model" (see [17] page 127), where a point $x \in \mathbb{B}_n$ is identified with

$$
\zeta(x) = \left(\frac{1 + |x|^2}{1 - |x|^2}, \frac{2x_1}{1 - |x|^2}, \dots, \frac{2x_n}{1 - |x|^2} \right) \in \mathbb{H}_+^n.
$$

It is then easy (see [18]) to see that the linear action of $SO(n, 1)$ on $\mathbb{H}_+^n \simeq G/K$ is

identified with the conformal action on $\mathbb{B}_n$ given by $y = g.x$ with

$$y_p = \frac{\frac{1+|x|^2}{2}g_{p0} + \sum_{l=1}^n g_{pl}x_l}{\frac{1-|x|^2}{2} + \frac{1+|x|^2}{2}g_{00} + \sum_{l=1}^n g_{0l}x_l} \qquad \text{for } p = 1, \ldots, n.$$

An invariant measure on $\mathbb{B}_n$ is given by

$$d\mu = \frac{dx}{(1-|x|^2)^n} = \frac{r^{n-1}drd\sigma}{(1-r^2)^n}$$

where dx is the Lebesgue measure on $\mathbb{B}_n$ and $d\sigma$ is the normalized surface measure on $\mathbb{S}^{n-1}$.

We will need the following elementary facts about this action (*see* [12]):

Fact 2.1. *Let $g \in SO(n,1)$ and let $x_0 = g.0$. If $0 < \varepsilon < \frac{1}{6}$, then*

$$B\left(x_0, \frac{\sqrt{2}}{8}(1-|x_0|^2)\varepsilon\right) \subset g.B(0,\varepsilon) \subset B\left(x_0, 6(1-|x_0|^2)\varepsilon\right).$$

Fact 2.2. *Let $g \in SO(n,1)$ and let $x_0 = g.0$. Let v be a smooth function on $\mathbb{B}_n$ and define f on $\mathbb{B}_n$ by $f(x) = v(g.x)$. Then, for every k,*

$$(1-|x_0|^2)^k|\nabla^k v(x_0)| \leq C|\nabla^k f(0)|,$$

where $|\nabla^k f|$ means $\sup\left\{\left|\frac{\partial^{|\alpha|}f}{\partial x^\alpha}\right| \ : \ |\alpha| \leq k\right\}$.

2.2. The invariant Laplacian on $\mathbb{B}_n$ and the associated Poisson kernel

From [18] (*see* also [8],[7]), we know that the invariant Laplacian on $\mathbb{B}_n$ for the considered action can be written as

$$D = (1-r^2)^2\Delta + 2(n-2)(1-r^2)\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and Δ is the Euclidean Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Write Δ in radial-tangential coordinates:

$$\Delta = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\sigma = \frac{1}{r^2}N^2 + \frac{n-2}{r^2}N + \frac{1}{r^2}\Delta_\sigma$$

with $N = r\frac{d}{dr} = \sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$ and Δ_σ the tangential part of the Euclidean Laplacian. We then obtain that D is given in radial-tangential coordinates by

$$D = \frac{1-r^2}{r^2}\left[(1-r^2)N^2 + (n-2)(1+r^2)N + (1-r^2)\Delta_\sigma\right].$$

Note that, from $\Delta_\sigma = r^2\Delta - N^2 - (n-2)N$, Δ_σ is given in Cartesian coordinates by
$$\Delta_\sigma = \sum_{i<j} \mathcal{L}_{i,j}^2 \text{ with } \mathcal{L}_{i,j} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}.$$

Definition 2.3. A function u on $\mathbb{B}_n$ is $\mathcal{H}$-harmonic if $Du = 0$ on $\mathbb{B}_n$.

Computations will often be simpler when replacing D by
$$L = \frac{1}{r^2}\big[(1-r^2)N^2 + (n-2)(1+r^2)N + (1-r^2)\Delta_\sigma\big].$$

Note that $Du = 0$ if and only if $Lu = 0$.

The Poisson kernel that solves the Dirichlet problem associated to D is given by
$$\mathbb{P}_h(r\eta, \xi) = \left(\frac{1-r^2}{1+r^2-2r\langle\eta,\xi\rangle}\right)^{n-1}$$

for $0 \le r < 1$, $\eta, \xi \in \mathbb{S}^{n-1}$ *i.e.* for $r\eta \in \mathbb{B}_n$ and $\xi \in \mathbb{S}^{n-1}$.

Recall that the Euclidean Poisson kernel on the ball is given by
$$\mathbb{P}_e(r\eta, \xi) = \frac{1-r^2}{(1+r^2-2r\langle\eta,\xi\rangle)^{\frac{n}{2}}}.$$

Notation: For a distribution φ on $\mathbb{S}^{n-1}$, we define the *Poisson integral* of φ, $\mathbb{P}_e[\varphi]$: $\mathbb{B}_n \mapsto \mathbb{R}$ by
$$\mathbb{P}_e[\varphi](r\eta) = \langle\varphi, \mathbb{P}_e(r\eta, .)\rangle.$$

Similarly, we define the *$\mathcal{H}$-Poisson integral* of φ, $\mathbb{P}_h[\varphi] : \mathbb{B}_n \mapsto \mathbb{R}$ by
$$\mathbb{P}_h[\varphi](r\eta) = \langle\varphi, \mathbb{P}_h(r\eta, .)\rangle.$$

2.3. Expansion of $\mathcal{H}$-harmonic functions in spherical harmonics

Let $_2F_1$ denote Gauss' *hyper-geometric function* and let $F_l(x) = {_2F_1}(l, 1-\frac{n}{2}, l+\frac{n}{2}; x)$ (*see* [6] for properties of $_2F_1$ used here).

In [15], [16] and [18], the spherical harmonic expansion of $\mathcal{H}$-harmonic functions has been obtained. Another proof, based on the method developed in [1] for $\mathcal{M}$-harmonic functions, can be found in [12]. One has the following:

Theorem 2.4. *Let u be an $\mathcal{H}$-harmonic function of class $\mathcal{C}^2$ on $\mathbb{B}_n$. Then the spherical harmonic expansion of u is given by*
$$u(r\zeta) = \sum_l \frac{F_l(r^2)}{F_l(1)} r^l \varphi_l(\zeta),$$

where φ_l is a spherical harmonic of degree l. Moreover, this series is absolutely convergent and uniformly convergent on every compact subset of $\mathbb{B}_n$.

It follows that, if we denote by $\mathbb{Z}_l^\zeta$ the zonal function of order l with pole ζ, then the hyperbolic Poisson kernel is given by

$$\mathbb{P}_h(r\zeta, \xi) = \sum_{l \geq 0} \frac{F_l(r^2)}{F_l(1)} r^l \mathbb{Z}_l^\zeta(\xi).$$

Note that, if the dimension n is even, then F_l is a polynomial of degree $\frac{n}{2} - 1$. This implies that, if u is $\mathcal{H}$-harmonic, then $\Delta^{\frac{n}{2}} u = 0$, a fact noticed in [14]. In particular, $\mathcal{H}$-harmonic functions should then behave like Euclidean harmonic functions, at least from the analysis point of view. However we will restrict our attention to common features of even and odd dimension.

The following lemma gives a further link between the hyperbolic Poisson kernel and the Euclidean one in even dimension:

Lemma 2.5. *Assume that n is even, and write $n = 2p$. There exist p polynomials $P_0, P_1, \ldots, P_{p-1}$ such that, for every $r\zeta \in \mathbb{B}_n, \xi \in \mathbb{S}^{n-1}$,*

$$(2.1) \qquad \mathbb{P}_h(r\zeta, \xi) = \sum_{k=0}^{p-1} P_k(r)(1 - r^2)^k \frac{\partial^k}{\partial r^k} \mathbb{P}_e(r\zeta, \xi).$$

Proof. For $a \in \mathbb{R}$, write $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. From [6] we get

$$F_l(x) = {}_2F_1(l, 1 - p, l + p, x) = \frac{1}{(l+p)_{p-1}} \frac{(1-x)^{2p-1}}{x^{l+p-1}} \frac{d^{p-1}}{dx^{p-1}} \left(x^{l+2(p-1)} (1-x)^{-p} \right).$$

Let $\alpha_{l,j}$ be defined by $\alpha_{l,0} = 1$ and $\alpha_{l,j+1} = \left(l + 2(p-1) - j \right) \alpha_{l,j}$, then by Leibniz' Formula

$${}_2F_1(l, 1 - p, l + p, x) = \frac{1}{(l+p)_{p-1}} \sum_{j=0}^{p-1} \binom{p-1}{j} (p)_j \alpha_{l,j} x^{p-1-j} (1-x)^j.$$

In particular ${}_2F_1(l, 1 - p, l + p, 1) = \frac{1}{(l+p)_{p-1}}$ thus

$$\frac{F_l(x)}{F_l(1)} = \sum_{j=0}^{p-1} \binom{p-1}{j} (p)_j \alpha_{l,j} x^{p-1-j} (1-x)^j.$$

Furthermore, it is easy to see that one can write

$$\alpha_{l,j} = \sum_{k=0}^{j} a_{k,j} l(l-1) \ldots (l-k+1)$$

where the coefficients $a_{k,j}$ are independent of l.

Now, recall that the spherical harmonic expansion of the Euclidean Poisson kernel is given by $\mathbb{P}_e(r\zeta, \xi) = \sum_{l \geq 0} r^l \mathbb{Z}_l^\zeta(\xi)$. Comparing this with the expansion of $\mathbb{P}_h$ and the

previous computations, we see that there exist polynomials $P_0, P_1, \ldots, P_{p-1}$ such that, for every $r\zeta \in \mathbb{B}_n, \xi \in \mathbb{S}^{n-1}$,

$$\mathbb{P}_h(r\zeta, \xi) = \sum_{k=0}^{p-1} P_k(r)(1 - r^2)^k \frac{\partial^k}{\partial r^k} \mathbb{P}_e(r\zeta, \xi),$$

which completes the proof. $\qquad \square$

In [13], another link between euclidean harmonic functions and $\mathcal{H}$-harmonic functions has been exhibited:

Lemma 2.6. *There exists a function* $\eta : [0, 1] \times [0, 1] \mapsto \mathbb{R}^+$ *such that*

i $\mathbb{P}_e(r\zeta, \xi) = \displaystyle\int_0^1 \eta(r, \rho)\mathbb{P}_h(\rho r\zeta, \xi)d\rho$, *in particular* $\int_0^1 \eta(r, \rho)d\rho = 1$;

ii *for every* $\alpha \geq 0$, *and every integer* $k > \alpha$, *there exists a constant* C *such that for every* $r \in [0, 1]$,

$$\int_0^1 \left|\left(r\frac{\partial}{\partial r}\right)^k \eta(r, \rho)\right|(1 - \rho)^\alpha d\rho \leq C(1 - r)^{\alpha - k}.$$

According to [13], the function η is given by

$$\eta(r, \rho) = c(1 - r^2)(1 - r^2\rho^2)^{2-n}\left[(1 - \rho)(1 - \rho r^2)\right]^{\frac{n}{2}-2}\rho^{\frac{n}{2}-1}.$$

The identity *(i)* is obtained by integration over $\mathbb{S}^{n-1}$ in the ξ variable. The estimate *(ii)* is obtained in a similar way as in [13] for $k = 0$ and $\alpha = 0$ after differentiation.

2.4. Hardy and Hardy-Sobolev spaces

The aim of this article is to extend Fefferman-Stein [9] theory to Hardy and Hardy-Sobolev spaces of $\mathcal{H}$-harmonic functions. We will therefore need to define analogues of nontangential maximal functions, area integrals and Littlewood-Paley g functions.

Definition 2.7. For $0 < \alpha < 1$ and $\zeta \in \mathbb{S}^{n-1}$, let $\mathcal{A}_\alpha(\zeta)$ be the interior of the convex hull of $B(0, \alpha)$ and ζ ; $\mathcal{A}_\alpha(\zeta)$ will be called *nontangential approach region*.

Note that these approach regions slightly differ from the region $\Gamma_\alpha(\zeta)$ used by Cifuentes [3]. There, the regions are defined in terms of the Iwasawa decomposition $SO_0(n, 1) = NAK$ by $\Gamma_\alpha(\zeta) = k\Gamma_\alpha^0$, where $k \in K$ is such that $k(1, 0, \ldots, 0) = \zeta$ and

$$\Gamma_\alpha^0 = \{n(\xi)a_t.0 : \|\xi\| < \alpha e^{-t}, \ t \in \mathbb{R}\}.$$

However, the Γ_α and $\mathcal{A}_\alpha$ regions are equivalent in the sense that, for every $\alpha < \beta < \gamma$, there exists $0 < r_0 < 1$ such that, for every $\zeta \in \mathbb{S}^{n-1}$,

(2.2) $\qquad \mathcal{A}_\alpha(\zeta) \setminus B(0, r_0) \subset \Gamma_\beta(\zeta) \setminus B(0, r_0) \subset \mathcal{A}_\gamma(\zeta) \setminus B(0, r_0)$.

We may now define the usual maximal and square functionals.

Definition 2.8. For a function u defined on $\mathbb{B}_n$, define the following functions on $\mathbb{S}^{n-1}$:
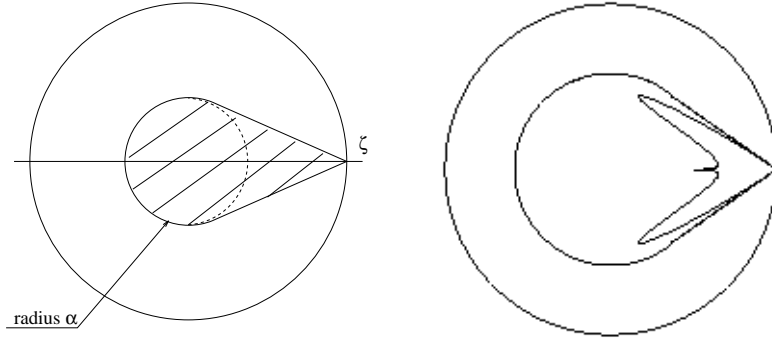
Figure 2: nontangential approach region $\mathcal{A}_\alpha(\zeta)$ and its comparison with $\Gamma_\alpha(\zeta)$

1. $\mathcal{M}[u](\xi) = \sup\limits_{0 < r < 1} |u(r\xi)|,$

2. $\mathcal{M}_\alpha[u](\xi) = \sup\limits_{x \in \mathcal{A}_\alpha(\xi)} |u(x)|.$

3. $S_\alpha[u](\xi) = \left[\int_{\mathcal{A}_\alpha(\xi)} |\nabla u(x)|^2 (1 - |x|^2)^{-n+2} dx \right]^{\frac{1}{2}}.$

4. $S_\alpha^N[u](\xi) = \left[\int_{\mathcal{A}_\alpha(\xi)} |Nu(x)|^2 (1 - |x|^2)^{-n+2} dx \right]^{\frac{1}{2}}.$

5. $g[u](\xi) = \left[\int_0^1 |\nabla u(t\xi)|^2 (1 - t^2) dt \right]^{\frac{1}{2}}.$

6. $g^N[u](\xi) = \left[\int_0^1 |Nu(t\xi)|^2 (1 - t^2) dt \right]^{\frac{1}{2}}.$

We can then define the Hardy spaces for $0 < p < +\infty$ as

$$\mathcal{H}^p = \{u \; \mathcal{H} - \text{harmonic} \; : \; \mathcal{M}[u] \in L^p(\mathbb{S}^{n-1})\}.$$

The atomic decomposition of these $\mathcal{H}^p$ for $0 < p \leq 1$ has been obtained in [13]. The following result holds:

Theorem A. *For $0 < p < +\infty$ and u $\mathcal{H}$-harmonic, the following are equivalent:*

1. $u \in \mathcal{H}^p.$

2. *u has a boundary distribution in $H^p(\mathbb{S}^{n-1})$.*

3. *$\mathcal{M}_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some (for all) $0 < \alpha < 1$.*

4. *$S_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some (for all) $0 < \alpha < 1$.*

5. $S_\alpha^N[u] \in L^p(\mathbb{S}^{n-1})$ *for some (for all)* $0 < \alpha < 1$.

6. $g[u] \in L^p(\mathbb{S}^{n-1})$,

7. $g^N[u] \in L^p(\mathbb{S}^{n-1})$.

Moreover, the L^p norms of these functionals are equivalent.

Some parts of this theorem have already been established by several authors. More precisely, the equivalence of 1 and 2 is proved in [13]. That 1 and 3 are equivalent is a direct consequence of the mean-value inequality (see Proposition 3.8). Cifuentes [3] proved the equivalence of 3 and 4 for the Γ_α domains. That these may be replaced by $\mathcal{A}_\alpha$ domains results from the equivalence property (2.2). Then, that 3 implies 4 is Theorem I and the reverse is Theorem II in [3]. Note that the constant L appearing there is removed by the fact that our $\mathcal{A}_\alpha$ domains include a ball around 0 and the mean-value property. The remaining parts, that is the equivalence of 4 to 7, will be proved in section 3.4.

Define now the Hardy-Sobolev spaces for $0 < p < +\infty$ and $k \in \mathbb{N}$ as

$$\mathcal{H}_k^p = \{u \; \mathcal{H} - \text{harmonic} \; : \; \text{for all } j \leq k, \; \mathcal{M}\big[\nabla^j u\big] \in L^p(\mathbb{S}^{n-1})\}.$$

and

$$H_k^p(\mathbb{S}^{n-1}) = \{f \in H^p(\mathbb{S}^{n-1}) \; ; \; \nabla^j f \in H^p(\mathbb{S}^{n-1}), \; 0 \leq j \leq k\}.$$

We will prove the following theorem:

Theorem B. *Let $0 < p < +\infty$ and k be an integer such that $0 \leq k \leq n - 2$. Then, for every $\mathcal{H}$-harmonic function u, the following are equivalent:*

1. $u \in \mathcal{H}_k^p$.

2. u *has a boundary distribution in* $H_k^p(\mathbb{S}^{n-1})$.

3. u *has a boundary distribution f satisfying* $(-\Delta_\sigma)^{\frac{l}{2}} f \in H^p(\mathbb{S}^{n-1})$ *for* $0 \leq l \leq k$.

4. $u \in \mathcal{H}_{k-1}^p$ *and for some (for all) α such that* $0 < \alpha < 1$, $\mathcal{M}_\alpha\big[(-\Delta_\sigma)^{k/2}u\big] \in L^p(\mathbb{S}^{n-1})$.

5. $u \in \mathcal{H}_{k-1}^p$ *and for some (for all) α such that* $0 < \alpha < 1$, $S_\alpha\big[(-\Delta_\sigma)^{k/2}u\big] \in L^p(\mathbb{S}^{n-1})$.

6. $u \in \mathcal{H}_{k-1}^p$ *and for some (for all) α such that* $0 < \alpha < 1$, $S_\alpha^N\big[(-\Delta_\sigma)^{k/2}u\big] \in L^p(\mathbb{S}^{n-1})$.

7. $u \in \mathcal{H}_{k-1}^p$ *and for some (for all) α such that* $0 < \alpha < 1$, $S_\alpha\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$.

8. $u \in \mathcal{H}_{k-1}^p$ *and for some (for all) α such that* $0 < \alpha < 1$, $S_\alpha^N\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$.

9. $u \in \mathcal{H}_{k-1}^p$ *and for some (for all) α such that* $0 < \alpha < 1$, $S_\alpha\big[\nabla^k u\big] \in L^p(\mathbb{S}^{n-1})$.

Remark 2.9. As $(-\Delta_\sigma)^{k/2}$ preserves $\mathcal{H}$-harmonicity, the equivalence of 3, 4, 5 and 6 means that $(-\Delta_\sigma)^{k/2}u \in \mathcal{H}^p$ and follows from Theorem A.

For the equivalence between 2 and 3, note that for any differential operator $\mathbb{X}$ of order l, the operator $\mathbb{X}(-\Delta_\sigma)^{-l/2}$ is, in local coordinates, a pseudo-differential operator of order 0. Such operators map $H^p(\mathbb{R}^{n-1})$ to $H^p_{loc}(\mathbb{R}^{n-1})$ (*see* [20] page 264). Therefore, taking a system of local coordinates on $\mathbb{S}^{n-1}$ and using the atomic decomposition of $H^p(\mathbb{S}^{n-1})$, we get that $\mathbb{X}(-\Delta_\sigma)^{-l/2}$ maps $H^p(\mathbb{S}^{n-1})$ to itself.

Finally, this last fact also shows that one may replace $(-\Delta_\sigma)^{l/2}$ by the set of all products of l operators of the form $\mathcal{L}_{i,j}$.

The other equivalent properties will be established in section 4.

Some of the properties of Theorem B are equivalent with no restriction on k. This is obvious for Properties 2 to 6 and these always follow from Property 1. It is immediate from the proof that Properties 7, 8 and 9 are also equivalent for any k and these always imply Properties 2 to 6.

Moreover, in *even* dimension, the link between $\mathcal{H}$-harmonic and Euclidean harmonic functions exhibited in (2.1) allows to show that Property 2 implies Property 8. This follows from the area integral characterization for Hardy-Sobolev spaces of Euclidean harmonic functions. We will refrain from giving the details as the same technic will appear with full details in the proof of the similar fact about Lipschitz spaces. However, as we do not have a good converse link between $\mathcal{H}$-harmonic and Euclidean harmonic functions, we lack a tool that would allow us to see whether these properties imply Property 1 above the critical index.

On the other hand, the restriction $k \leq n-2$ is natural when n is *odd*: if $u \in \mathcal{H}^p_{n-1}$ and $\varphi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ then $\int_{\mathbb{S}^{n-1}} N^{n-1}u(r\zeta)\varphi(\zeta)d\sigma(\zeta)$ is bounded, thus, by Theorem 8 of [13], u is constant. The same is true for Theorem C below.

Define now

$$\mathcal{H}^p_{k,N} = \{u \in \mathcal{H}^p;\ \mathcal{M}\big[N^l u\big] \in L^p(\mathbb{S}^{n-1}),\ 0 \leq l \leq k\}.$$

In section 4, we study the relationship between $\mathcal{H}^p_k$ and $\mathcal{H}^p_{k,N}$. The situation is slightly different as parity of the order of derivation is involved.

Theorem C. *Let* $0 < \alpha < 1$, $0 < p < +\infty$, *and* k *be an integer such that* $0 \leq k \leq n-2$. *Then*

1. *If k is even, the following are equivalent:*

   (a) $u \in \mathcal{H}^p_{k,N}$.

   (b) *For some α such that $0 < \alpha < 1$, for every $0 \leq l \leq k$, $\mathcal{M}_\alpha\big[N^l u\big] \in L^p(\mathbb{S}^{n-1})$.*

   (c) $u \in \mathcal{H}^p_k$ *and hence all the equivalent properties stated in Theorem B are valid.*

2. *If k is odd, the following are equivalent:*

   (a) $u \in \mathcal{H}^p_{k,N}$

   (b) *For some α such that $0 < \alpha < 1$, for every $0 \leq l \leq k$, $\mathcal{M}_\alpha\big[N^l u\big] \in L^p(\mathbb{S}^{n-1})$.*

(c) $u \in \mathcal{H}_{k-1}^p$ and $\mathcal{M}\left[(1 - r^2)\Delta_\sigma^{\frac{k+1}{2}} u\right] \in L^p(\mathbb{S}^{n-1})$.

Corollary D. *If $k \leq n - 2$ is odd, $\mathcal{H}_k^p \subset \mathcal{H}_{k,N}^p$.*

The inclusion of $\mathcal{H}_k^p$ in $\mathcal{H}_{k,N}^p$ may be strict. For instance, let δ be the Dirac mass in $(1, 0, \ldots, 0)$ and let $u = \mathbb{P}_h[\delta]$. Then $u \in \mathcal{H}_{1,N}^p$ for $\frac{n}{2(n-1)} < p \leq 1$ but $\frac{\partial u}{\partial x_i}$ is not uniformly in L^p for $p < \frac{n}{n+1}$, thus $u \notin \mathcal{H}_1^p$ for this range of p's.

The fact that the space $\mathcal{H}_{k,N}^p$ is strictly bigger looks, at first sight, quite surprising since it is usually expected that the radial derivative dominates the gradient. In fact, this is naturally true in the interior of the domains and for instance, $S_\alpha(N^l u) \in L^p(\mathbb{S}^{n-1})$, $0 \leq l \leq k$ implies (and in fact is equivalent to) $u \in \mathcal{H}_k^p$ as stated in Theorem B.

It is no longer true for conditions involving the behavior of the radial derivatives near the boundary. For instance, when $k = 1$, we know from [13] that Nu has a boundary distribution that is identically zero. So, for u $\mathcal{H}$-harmonic, to be in $\mathcal{H}_{1,N}^p$ cannot be translated as a constraint on the boundary behaviour of u.

3. Mean-value inequalities and Hardy spaces

3.1. Mean-value inequalities

Recall that $\mathcal{H}$-harmonic functions satisfy the following mean-value equality:
Let $a \in \mathbb{B}_n$ and $g \in SO(n, 1)$ such that $g.0 = a$. Then, for every $\mathcal{H}$-harmonic function u,

$$u(a) = \frac{1}{\mu\big(B(0, r)\big)} \int_{g.B(0,r)} u(x) d\mu(x).$$

Thus, with fact 2.1 and $d\mu = \dfrac{dx}{(1 - |x|^2)^n}$, we get

$$(3.1) \qquad |u(a)| \leq \frac{C}{(1 - |a|^2)^n} \int_{B\big(a, (1-|a|^2)\varepsilon\big)} |u(x)| dx.$$

We will also need mean-value inequalities for normal derivatives of $\mathcal{H}$-harmonic functions, in particular when we study Hardy-Sobolev spaces. But, normal derivatives of $\mathcal{H}$-harmonic functions are no longer $\mathcal{H}$-harmonic, so that one may not directly apply Inequality (3.1) to $N^k u$.

To obtain these inequalities, we will follow the main lines of the proof in [2] for $\mathcal{M}$-harmonic functions.

Therefore, we will first study the commutator between N^k and L (which is easier to compute than the commutator between N^k and $D = (1 - r^2)L$). This leads us to the existence of an elliptic operator $\mathbb{N}_q$ such that for every $\mathcal{H}$-harmonic function u, $N^k u$ is annihilated by $\mathbb{N}_q$. We can then apply L^2-theory of elliptic operators and get estimates for $N^k u$ at 0 by its mean-value. To obtain the estimates in an arbitrary point a of $\mathbb{B}_n$, we transport the result from 0 to a with help of the action of $SO(n, 1)$ on $\mathbb{B}_n$ by computing the action of $g \in SO(n, 1)$ on $\mathbb{N}_q$.

Note that

$$(3.2) \qquad LN - NL = 2L + 2(N^2 + \Delta_\sigma) - 2(n-2)N.$$

Moreover an easy induction argument shows that there exist two sequences of polynomials $(P_k)_{k \geq 1}$ and $(Q_k)_{k \geq 1}$ of degree $k-1$ such that for $k \geq 1$,

$$LN^k = (N + 2I)^k L + P_k(N)N^2 + Q_k(N)\Delta_\sigma - 2(n-2)(N+2I)^{k-1}N.$$

From this, using the same induction as in [2], we get

Proposition 3.1. *For every k, there exist polynomials $S_k(x,y)$ of degree at most $q-1$ (with $q = 2^{k-1}$) and $R_k(x,y) = x^q + \ldots$ such that, if u is $\mathcal{H}$-harmonic, then*

$$L\big(R_k(L,\Delta_\sigma) - S_k(L,\Delta_\sigma)N\big)N^k u = 0.$$

Note that $\mathbb{N}_q := L\big(R_k(L,\Delta_\sigma) - S_k(L,\Delta_\sigma)N\big) = L^{q+1} +$ terms of order $\leq 2q$ in L, Δ_σ with $\mathcal{C}^\infty$ coefficients and $q = 2^{k-1}$. In particular, if u is $\mathcal{H}$-harmonic, then $v = N^k u$ is a solution of an equation $\mathbb{N}_q v = 0$.

We will use the following formalism: for $\mathbb{M}$ a differential operator and Φ a diffeomorphism of $\mathbb{B}_n$, if $f = v \circ \Phi$, then define $\Phi^* \mathbb{M}$ by

$$\Phi^* \mathbb{M}(f) = (\mathbb{M}v) \circ \Phi.$$

It is then obvious that

$$(3.3) \qquad \begin{aligned} \Phi^*(\mathbb{M}_1 \circ M_2) &= (\Phi^* \mathbb{M}_1) \circ (\Phi^* \mathbb{M}_2) \\ \Phi^*(h\mathbb{M}) &= h(\Phi).\Phi^* \mathbb{M} \end{aligned}$$

Let $g \in SO(n,1)$ be such that $g.0 = \rho\zeta = a \in \mathbb{B}_n$ and let $\Phi_a : \begin{array}{ccc} \mathbb{B}_n & \to & \mathbb{B}_n \\ x & \mapsto & g.x \end{array}$.

But, by definition, D is invariant by the action of $SO(n,1)$ on $\mathbb{B}_n$, that is $\Phi_a^* D = D$. On the other hand, $D = (1-|x|^2)L$ thus (3.3) tells us that $\Phi_a^* D = (1-|\Phi_a(x)|^2)\Phi_a^* L$, which implies that $\Phi_a^* L = \frac{1-|x|^2}{1-|g.x|^2}L$, and the formula of [18] page 39 gives

$$\Phi_a^* L = \frac{(1 + \rho^2|x|^2 - 2\rho\langle x,\zeta\rangle)^2}{1 - \rho^2}L.$$

Further $\Phi_a^* N$ is a differential operator of order 1 with $\mathcal{C}^\infty$ coefficients defined by

$$\begin{aligned} \Phi_a^* N f(x) &= \big\langle \Phi_a(x), dv_{\Phi_a(x)} \big\rangle = \big\langle \Phi_a(x), d(f \circ \Phi_a^{-1})_{\Phi_a(x)} \big\rangle \\ &= \big\langle \Phi_a(x), df_z.d(\Phi_a^{-1})_{\Phi_a(x)} \big\rangle \end{aligned}$$

thus, if $x \in B\big(0, \frac{\varepsilon}{6}\big)$ then, with Fact 2.1, $\Phi_a(x) \in B\big(a, (1-a^2)\varepsilon\big)$, and with Fact 2.2 (applied to $v(x) = x$), the coefficients of $(1-|a|^2)\Phi_a^* N$ as well as their derivatives are $\mathcal{C}^\infty$ and bounded independently of a.

As $\Phi_a^* N^2 = (\Phi_a^* N) \circ (\Phi_a^* N)$, $(1-|a|^2)^2 \Phi_a^* N^2$ is a differential operator of order 2 with $\mathcal{C}^\infty$ coefficients bounded (as well as their derivatives) independently of a.

At last, $\Delta_\sigma = \frac{r^2}{1-r^2}L - N^2 - (n-2)\frac{1+r^2}{1-r^2}N$ thus $(1-|a|^2)\Phi_a^*\Delta_\sigma$ is also a differential operator of order 2 with $\mathcal{C}^\infty$ coefficients bounded (as well as their derivatives) independently of a.

Finally, as $\mathbb{N}_q = L^{q+1}+$ terms of order $\leq 2q$ in L, Δ_σ and N with $\mathcal{C}^\infty$ coefficients, thus

$$
\begin{aligned}
\Phi_a^*\mathbb{N}_q &= \Phi_a^*L^{q+1} + \text{terms of order } \leq 2q \text{ with } \mathcal{C}^\infty \text{ coefficients} \\
&= \frac{(1+\rho^2|x|^2 - 2\rho\langle x,\zeta\rangle)^{2(q+1)}}{(1-\rho^2)^{q+1}}L^{q+1} \\
&\quad +\text{terms of order } \leq 2q \text{ with } \mathcal{C}^\infty \text{ coefficients} \\
&= \frac{|\rho x - \zeta|^{2(q+1)}}{(1-\rho^2)^{q+1}}L^{q+1} + \mathbb{R}_{q,a}
\end{aligned}
$$

where $\mathbb{R}_{q,a}$ is a differential operator of order $\leq 2q$ with $\mathcal{C}^\infty$ coefficients.

Let u be an $\mathcal{H}$-harmonic function, $v = N^k u$ and $f = v \circ \Phi_a$. As v satisfies $\mathbb{N}_q v = 0$, f satisfies $(1-|a|^2)^{q+1}\Phi_a^*\mathbb{N}_q f = 0$ on $B\left(0,\frac{\varepsilon}{6}\right)$ (with $e.g.$ $\varepsilon < 1$). We have thus shown that $(1-\rho^2)^{q+1}\Phi_a^*\mathbb{N}_q$ satisfies on $B\left(0,\frac{\varepsilon}{6}\right)$, all the hypotheses (with constants independent on a) of the following theorem (see [2] page 678):

Theorem 3.2. *Suppose $P(D)$ is a differential operator in $\mathbb{R}^N$,*

$$
P(D) = \sum_{|\alpha|\leq 2q} h_\alpha(x)D^\alpha \quad \text{where} \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}},
$$

which is elliptic with constant c_0 in $B(0,\varepsilon)$, that is,

$$
\sum_{|\alpha|=2q} h_\alpha(x)\xi^\alpha \geq c_0|\xi|^{2q} \quad \text{for} \quad \xi \in \mathbb{R}^N,
$$

and with $h_\alpha \in \mathcal{C}^\infty\left(\overline{B(0,\varepsilon)}\right)$. Assume that $P(D)f = 0$ in $B(0,\varepsilon)$. Then, for all non-negative integers m and all p such that $0 < p < \infty$,

$$
|\nabla^m f(0)| \leq C\left(\int_{|x|\leq\varepsilon}|f(x)|^p dx\right)^{1/p},
$$

where C depends only on c_0,ε,m,p and a bound of the norms of the functions h_α in some $\mathcal{C}^l\left(\overline{B(0,\varepsilon)}\right)$-space with $l = l(m)$.

From this, we get

$$
\left|\nabla^d f(0)\right| \leq c\left(\int_{|x|\leq\frac{\varepsilon}{6}}|f(x)|^p dx\right)^{\frac{1}{p}} \leq c\left(\int_{|x|\leq\frac{\varepsilon}{6}}|f(x)|^p\frac{dx}{(1-|x|^2)^n}\right)^{\frac{1}{p}}
$$

or, with *Fact 2.2*,

$$
\left|\nabla^d v(a)\right| \leq c\left(\int_{B\left(0,\frac{\varepsilon}{6}\right)}|v \circ \Phi_a(x)|^p d\mu(x)\right)^{\frac{1}{p}} \times (1-|a|^2)^{-d}
$$

where μ is the G-invariant measure on $\mathbb{B}_n$. Thus

$$\left|\nabla^d v(a)\right| \le c \left(\int_{g.B\left(0,\frac{\varepsilon}{6}\right)} |v(x)|^p d\mu(x)\right)^{\frac{1}{p}} \times (1-|a|^2)^{-d}$$

and, with *Fact 2.1*,

$$\begin{aligned}
\left|\nabla^d v(a)\right| &\le & c \left(\int_{B\left(a,(1-|a|^2)\varepsilon\right)} |v(x)|^p \frac{dx}{(1-|x|^2)^n}\right)^{\frac{1}{p}} \times (1-|a|^2)^{-d} \\
&\le & c(1-|a|^2)^{-d-\frac{n}{p}} \left(\int_{B\left(a,(1-|a|^2)\varepsilon\right)} |v(x)|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

In conclusion, we have just proved the following lemma :

Lemma 3.3. *(Mean-value inequality) For every $0 < \varepsilon < 1$, $k,d \in \mathbb{N}$, $0 < p < +\infty$, there exists a constant c such that, for every $\mathcal{H}$-harmonic function u, and every $a \in \mathbb{B}_n$,*

$$\left|\nabla^d N^k u(a)\right| \le c(1-|a|)^{-d-\frac{n}{p}} \left(\int_{B\left(a,(1-|a|^2)\varepsilon\right)} \left|N^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

We now show that this mean-value inequality applies to any derivative of an $\mathcal{H}$-harmonic function:

Proposition 3.4. *For every $0 < \varepsilon < 1$ and every $0 < p < +\infty$, there exists a constant C such that for every $\mathcal{H}$-harmonic function u, every $k \in \mathbb{N}$, $d \ge 0$, and for every $a \in \mathbb{B}_n$,*

$$\left|\nabla^{k+d} u(a)\right| \le C(1-|a|)^{-d-\frac{n}{p}} \left(\int_{B\left(a,(1-|a|^2)\varepsilon\right)} \left|\nabla^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Proof. Recall that $\mathcal{L}_{i,j} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ $(1 \le i < j \le n)$, in particular, these operators preserve $\mathcal{H}$-harmonicity and $N\mathcal{L}_{i,j} = \mathcal{L}_{i,j}N$.

Applying Lemma 3.3 to $\mathcal{L}_{i,j}^k u$, for every $0 < \varepsilon < 1$ and every $0 < p < +\infty$, there exists a constant C such that for every $\mathcal{H}$-harmonic function u, for every $1 \le i < j \le n$ and every $k \in \mathbb{N}$, for every d, and every $a \in \mathbb{B}_n$,

$$(3.4) \qquad \left|\nabla^d \mathcal{L}_{i,j}^k u(a)\right| \le C(1-|a|)^{-d-\frac{n}{p}} \left(\int_{B\left(a,(1-|a|^2)\varepsilon\right)} \left|\mathcal{L}_{i,j}^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

The same applies to $(-\Delta_\sigma)^\gamma$ $(\gamma \in \mathbb{R}^+)$.

Let $\hat{\nabla}^k u$ be defined by

$$\{\mathbb{X} N^q u \ : \ \mathbb{X} = \prod_{l=1}^p \mathcal{L}_{i_l,j_l}, \ p+q \le k\},$$

then (3.4) implies that Lemma 3.3 stays true if we replace N^k by $\tilde{\nabla}^k u$. But, outside a fixed neighborhood V of 0, $\left|\tilde{\nabla}^k u\right| \simeq \left|\nabla^k u\right|$, thus for every $a \in \mathbb{B}_n \setminus V$

$$\left|\nabla^d \nabla^k u(a)\right| \leq C(1-|a|)^{-d-\frac{n}{p}} \left(\int_{B\left(a,(1-|a|^2)\varepsilon\right)} \left|\nabla^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

As for $a \in V$ one can apply Theorem 3.2 on $B\left(a, (1-|a|^2)\varepsilon\right)$ with constants independent of a, we get the previous inequality on V (recall that ∇^k means the set of all derivatives of order less than k). $\qquad\square$

3.2. First consequences of the mean-value inequalities

As an immediate consequence of Lemma 3.3, we obtain the following:

Corollary 3.5. *Let* $0 < \alpha < \beta < 1$, $k, d \in \mathbb{N}$. *Then there exists a constant c such that for every $\mathcal{H}$-harmonic function u,*

$$\mathcal{M}_\alpha\left((1-|z|)^d |\nabla^d N^k u|\right) \leq c\mathcal{M}_\beta(N^k u).$$

Proof. This follows from Lemma 3.3 and the fact that if $\alpha < \beta$ and if ε is small enough then, for every $\xi \in \mathbb{S}^{n-1}$ and every $a \in \mathcal{A}_\alpha(\xi)$, $B(a, (1-|a|^2)\varepsilon) \subset \mathcal{A}_\beta(\xi)$. $\quad\square$

Let $l \in \mathbb{R}$ and f a function defined on $\mathbb{B}_n$. Define $I_l f$ by

$$I_l f(r\zeta) = \int_0^r f(t\zeta)(1-t)^{l-1} dt, \qquad 0 < r < 1, \ \zeta \in \mathbb{S}^{n-1}.$$

If $l \in \mathbb{R}^+$, I_l is a "fractional integration operator" of order l in the normal direction. In particular, if l is a positive integer and $N^l h = g$, then

$$(3.5) \qquad |h| \leq C\left[I_l|g| + \sup_{j \leq l-1, |z| \leq \varepsilon} \left|\nabla^j h(z)\right|\right].$$

The following lemma is a direct consequence of the mean-value inequalities and its proof follows the main lines of the upper half-line case (*see* [19] pages 214–216) or the $\mathcal{M}$-harmonic function case in [2].

Lemma 3.6. *For* $0 < \alpha < \beta < 1$, $\gamma > -\frac{n}{2}$, $l \in \mathbb{R}$ *and* $d \in \mathbb{N}$, *there exists a constant C such that, for every $\zeta \in \mathbb{S}^{n-1}$, and for every $\mathcal{H}$-harmonic function u*

$$\int_{\mathcal{A}_\alpha(\zeta)} \left[I_l\left|\nabla^d N^k u\right|(z)\right]^2 (1-|z|)^{2\gamma} dz \leq C \int_{\mathcal{A}_\beta(\zeta)} \left|N^k u(z)\right|^2 (1-|z|)^{2(l+\gamma-d)} dz.$$

From this and Formula (3.5), one deduces the following (*see* [2] for the proof in case of $\mathcal{M}$-harmonic functions):

Lemma 3.7. *For $0 < \alpha < \beta < 1$, $\gamma > -\frac{n}{2}$ and $d \in \mathbb{N}$, there exists a constant C such that, for every $\zeta \in \mathbb{S}^{n-1}$, and every $\mathcal{H}$-harmonic function u,*

$$\int_{\mathcal{A}_\alpha(\zeta)} \left|\nabla^d u(z)\right|^2 (1-|z|)^{2\gamma} dz \quad \leq \quad C \int_{\mathcal{A}_\beta(\zeta)} \left|N^k u(z)\right|^2 (1-|z|)^{2(k+\gamma-d)} dz$$
$$+ C \sup_{|z|<\varepsilon} \left|\nabla^{k-1} u(z)\right|^2.$$

3.3. Maximal Characterization of $\mathcal{H}^p$

From the definition of $\mathcal{M}[u]$ and $\mathcal{M}_\alpha[u]$, it is obvious that $\mathcal{M}[u] \leq \mathcal{M}_\alpha[u]$, in particular, if $\mathcal{M}_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ then $\mathcal{M}[u] \in L^p(\mathbb{S}^{n-1})$. The next proposition claims that the converse is true for $\mathcal{H}$-harmonic functions as well as for their normal derivatives. In particular, this proves the equivalence of statements 1 and 3 of Theorem A.

Proposition 3.8. *For $0 < \alpha < 1$, $0 < p < +\infty$, for every integer $k \geq 0$ and for every $\mathcal{H}$-harmonic function u, the following are equivalent :*

1. $\mathcal{M}\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$,

2. $\mathcal{M}_\alpha\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$.

Moreover, there exists $C = C_{\alpha,p}$ such that for every $\mathcal{H}$-harmonic function u,

$$\big\|\mathcal{M}\big[N^k u\big]\big\|_p \leq \big\|\mathcal{M}_\alpha\big[N^k u\big]\big\|_p \leq C\big\|\mathcal{M}\big[N^k u\big]\big\|_p.$$

P r o o f. According to Lemma 3.3, for $a \in \mathcal{A}_\alpha(\zeta)$

$$\left|N^k u(a)\right|^{\frac{p}{2}} \leq C(1-|a|)^{-n} \int_{B\left(a, 2(1-|a|)\varepsilon\right)} \left|N^k u(\omega)\right|^{\frac{p}{2}} d\omega.$$

Integrating in polar coordinates $\omega = r\eta$, we see that $\eta \in B\big(\zeta, c(1-|a|)\big)$ and bounding $\left|N^k u(\omega)\right|$ by $\mathcal{M}\big[N^k u\big](\eta)$ we get

$$\left|N^k u(a)\right|^{\frac{p}{2}} \leq C(1-|a|)^{-n} \int_{B\left(\zeta, c(1-|a|)\right) \cap \mathbb{S}^{n-1}} \big[\mathcal{M}\big[N^k u\big](\eta)\big]^{\frac{p}{2}} d\sigma(\eta)$$
$$\times \int_{|a|-2(1-|a|)\varepsilon}^{|a|+2(1-|a|)\varepsilon} r^{n-1} dr$$
$$\leq C(1-|a|)^{-n+1} \int_{B\left(\zeta, c(1-|a|)\right) \cap \mathbb{S}^{n-1}} \big[\mathcal{M}\big[N^k u\big](\eta)\big]^{\frac{p}{2}} d\sigma(\eta).$$

But $\sigma\big[B\big(\zeta, c(1-|a|)\big) \cap \mathbb{S}^{n-1}\big] \sim (1-|a|)^{n-1}$ therefore

$$\mathcal{M}_\alpha\big[N^k u(\zeta)\big]^{\frac{p}{2}} \leq C\mathcal{M}_{HL}\Big[\mathcal{M}\big[N^k u\big]^{\frac{p}{2}}\Big](\zeta)$$

where $\mathcal{M}_{HL}$ is Hardy-Littlewood's maximal function on $\mathbb{S}^{n-1}$. We just have to use the fact that $\mathcal{M}_{HL}$ is bounded $L^2(\mathbb{S}^{n-1}) \mapsto L^2(\mathbb{S}^{n-1})$ to complete the proof. $\qquad\square$

This proposition, whose proof is directly inspired from the $\mathbb{R}^{n+1}_+$ case in [9] depends only on the mean-value inequalities (Lemma 3.3). Thus, it remains true if we replace

N^k by ∇^k or by $\mathcal{L}_{i,j}^k$ (thus also by $(-\Delta_\sigma)^{k/2}$) as long as we replace Lemma 3.3 by Proposition 3.4 or by Inequality (3.4).

Further, this proposition implies that the opening α of the non-tangential approach region plays no role. In the sequel, we will appeal to this classical fact with no further reference.

3.4. Characterization by Littlewood-Paley's g-function

Due to the mean-value inequality for $\mathcal{H}$-harmonic functions, one immediately gets :

Lemma 3.9. *For every α with $0 < \alpha < 1$ there exists a constant C such that for every $\mathcal{H}$-harmonic function u and every $\xi \in \mathbb{S}^{n-1}$,*

$$g^N[u](\xi) \leq CS_\alpha^N[u](\xi) \quad \text{and} \quad g[u](\xi) \leq CS_\alpha[u](\xi).$$

Proof. Simply adapt the $\mathbb{R}_+^{n+1}$ case from [19]. □

Theorem 3.10. *Let $0 < p < +\infty$. For every $\mathcal{H}$-harmonic function u, the following are equivalent :*

1. $g[u] \in L^p(\mathbb{S}^{n-1})$,

2. $g^N[u] \in L^p(\mathbb{S}^{n-1})$,

3. $S_\alpha^N[u] \in L^p(\mathbb{S}^{n-1})$ *for some α, $0 < \alpha < 1$ (thus for every α).*

4. $S_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ *for some α, $0 < \alpha < 1$ (thus for every α).*

Proof. The implications $1\Rightarrow2$ and $4\Rightarrow3$ are trivial. The implication $4\Rightarrow1$ (and $3\Rightarrow2$) follow from Lemma 3.9. Further, the implication $3\Rightarrow4$ follows directly from Lemma 3.7. The proof will be complete once we have established $2\Rightarrow3$.

Let $\mathbb{H}$ be the Hilbert space defined by

$$\mathbb{H} = \left\{ \varphi : [0,1] \mapsto \mathbb{C} \ : \ \|\varphi\|_{\mathbb{H}}^2 = \int_0^1 |\varphi(s)|^2(1-s^2)ds < +\infty \right\}.$$

Let u be an $\mathcal{H}$-harmonic function such that $g^N[u] \in L^p(\mathbb{S}^{n-1})$. For $0 < s < 1$ define $U(r\zeta) = Nu(rs\zeta)$, then

$$
\begin{aligned}
\|U(r\zeta)\|_{\mathbb{H}} = \|s \mapsto Nu(rs\zeta)\|_{\mathbb{H}} &= \int_0^1 |Nu(rs\zeta)|^2(1-s^2)ds \\
&\leq \int_0^r |Nu(s\zeta)|^2 \left(1 - \left(\frac{s}{r}\right)^2\right) \frac{ds}{r} \\
&= \frac{1}{r^3} \int_0^r |Nu(s\zeta)|^2(r^2-s^2)ds \\
&\leq Cg^N[u](\zeta)^2
\end{aligned}
$$

so

$$\mathcal{M}[U](\xi) = \sup_{0<r<1} \|U(r\zeta)\|_{\mathbb{H}} \leq Cg^N[u](\zeta) \in L^p(\$^{n-1}).$$

Next, it is easy to see that the equivalence of the L^p norms of the area integral and the non-tangential maximal functions extends to Hilbert space valued $\mathcal{H}$-harmonic functions. The key fact is that equality $D|u|^2 = 2(1-|x|^2)^2|\nabla u|^2$ is valid in Hilbert spaces. It follows that, for every $0 < p < +\infty$ there exists C such that

$$(3.6) \qquad \|S_\alpha[U]\|_p \leq C\|\mathcal{M}[U]\|_p \leq C'\|g^N[u]\|_p.$$

Write $S_\alpha[U](\zeta)$ with the parameterization $r(\xi)\xi$ of $\partial\mathcal{A}_\alpha(\zeta)$:

$$S_\alpha[U](\zeta)^2 = \int_{\$^{n-1}} \int_0^{r(\xi)} \int_0^1 |\nabla Nu(rs\xi)|^2(1-s^2)sds(1-r^2)^{2-n}r^{n-1}drd\sigma(\xi)$$

and, with the change of variables $t = rs$, we get, changing order of integration

$$
\begin{aligned}
S_\alpha[U](\zeta)^2 &= \int_{\$^{n-1}} \int_0^{r(\xi)} \int_t^{r(\xi)} |\nabla Nu(t\xi)|^2 \left(1 - \left(\frac{t}{r}\right)^2\right) \frac{t}{r}(1-r^2)^{2-n}r^{n-2}drdtd\sigma \\
&\geq \int_{\$^{n-1}} \int_0^{r(\xi)} |\nabla Nu(t\xi)|^2 \int_t^{r(\xi)} (r-t)(1-r)^{2-n}drt^{n-3}dtd\sigma(\xi)
\end{aligned}
$$

But, if $1 - t > 2(1 - r(\xi))$

$$\int_t^{r(\xi)} (r-t)(1-r)^{2-n}dr = \int_{1-r(\xi)}^{1-t} s^{2-n}(1-t-s)ds \geq C(1-t)^{4-n}$$

thus there exists $\beta < \alpha$ such that

$$
\begin{aligned}
S_\alpha[U](\zeta)^2 &\geq \int_{\mathcal{A}_\beta(\zeta)} |N^2u(t\xi)|^2(1-t)^{4-n}t^{n-1}dtd\sigma(\xi) \\
&\geq C\int_{\mathcal{A}_{\beta'}(\zeta)} \left[I_1|N^2u|(t\xi)\right]^2(1-t)^{2-n}t^{n-1}dtd\sigma(\xi)
\end{aligned}
$$

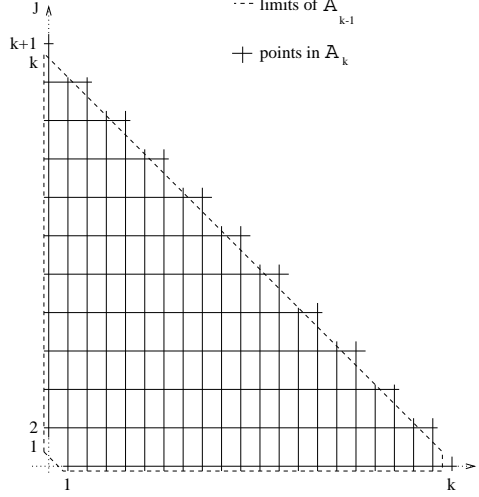with $\beta' < \beta$ according to Lemma 3.6. Using (3.5), we get

$$S_\alpha[U](\zeta)^2 + C\sup_{|z|\leq\varepsilon} |Nu(z)| \geq CS_{\beta'}^N[u](\zeta)^2.$$

Equation (3.6), allows to conclude that $2\Rightarrow3$.   □

4. Characterization of Hardy-Sobolev spaces

In this section, we prove Theorems B and C.

In these theorems, that $\mathcal{M}$ can be replaced by $\mathcal{M}_\alpha$ is a direct consequence of the mean-value inequality (*see* Proposition 3.8 and the remarks following it). We will need the following.

Figure 3: The set A_k (here with k even)

Notation : For an integer $k \geq 1$, write $\mathbb{A}_k$ for the set of indices

$$\mathbb{A}_k = \{(i,j) \in \mathbb{N} \times \mathbb{N} \ : \ 0 \leq i \leq k, \ 0 \leq j \leq k+1, \ j \text{ even}, \ 1 \leq i+j \leq k+1\}.$$

Lemma 4.1. *For every $k \geq 1$, there exist two families of polynomials $\left(P_j^{(k)}\right)_{j=1\ldots\left[\frac{k}{2}\right]}$ and $\left(Q_{i,j}^{(k)}\right)_{(i,j)\in\mathbb{A}_k}$ such that for every $\mathcal{H}$-harmonic function u and every k,*

$$(1-r^2)N^{k+1}u + 2(n-1-k)N^k u = \sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u + (1-r^2)\sum_{(i,j)\in\mathbb{A}_k} Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}}u.$$

Moreover, the polynomials $P_{[k/2]}^{(k)}$ and $Q_{0,\left[\frac{k+1}{2}\right]}^{(k)}$ are not zero on the boundary.

Proof. Using the radial-tangential expression of D, we can see that if $Du = 0$ then

(4.1) $$(1-r^2)N^2 u + 2(n-2)Nu = (1-r^2)[(n-2)Nu - \Delta_\sigma u].$$

The lemma is thus verified for $k=1$ with $Q_{1,0}^{(1)}(r) = n-2$, $Q_{0,2}^{(1)}(r) = -1$.

Applying N^{k-1} to equation (4.1) leads to

$$\begin{aligned}
(1-r^2)N^{k+1}u + (n-1-k)N^k u \ = \ & r^2\sum_{l=1}^{k-1} a_l^{(k)}N^l u + r^2\sum_{l=0}^{k-2} b_l^{(k)}N^l\Delta_\sigma u \\
& + (1-r^2)\big[(n-2k)N^k u - N^{k-1}\Delta_\sigma u\big]
\end{aligned}$$

and we conclude with the induction hypothesis.   □

The equivalence of 1a and 1b as well as the equivalence of 2a and 2b in Theorem C have already been shown. We will now prove the remaining of this theorem.

Theorem 4.2. *Let $0 < \alpha < 1$, $0 < p < +\infty$, and $k \le n - 2$ be an integer. The following are equivalent:*

1. *If k is even*

   (a) $\mathcal{M}_\alpha\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$, *for every integer j with $0 \le j \le k$,*

   (b) $\mathcal{M}_\alpha\big[(-\Delta_\sigma)^j u\big] \in L^p(\mathbb{S}^{n-1})$, *for every integer j with $0 \le j \le \frac{k}{2}$,*

   (c) $\mathcal{M}_\alpha\big[\nabla^j u\big] \in L^p(\mathbb{S}^{n-1})$ *for every integer j with $0 \le j \le k$.*

2. *If k is odd*

   (a) $\mathcal{M}_\alpha\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$, *for every integer j with $0 \le j \le k$,*

   (b) $\mathcal{M}_\alpha\big[(-\Delta_\sigma)^j u\big] \in L^p(\mathbb{S}^{n-1})$, *for every integer j with $0 \le j \le \frac{k-1}{2}$, and*

   $$\mathcal{M}_\alpha\Big[(1 - r^2)(-\Delta_\sigma)^{\frac{k+1}{2}} u\Big] \in L^p(\mathbb{S}^{n-1}).$$

Proof of Theorem 4.2. The theorem is of course true for $k = 0$. Assume the result holds up to rank $k - 1$.

Assume first that k is *even*. The implication (1c)$\Rightarrow$(1b) is obvious, so let us now show that (1b)$\Rightarrow$(1a).

Let u be an $\mathcal{H}$-harmonic function that satisfies (1b), and let

$$(4.2) \qquad\qquad v = 2(n - 1 - k)N^k u + (1 - r^2)N^{k+1} u.$$

According to Lemma 4.1,

$$v(r\zeta) = \sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u + (1 - r^2)\sum_{(i,j)\in\mathbb{A}_k} Q_{i,j}^{(k)}(r)N^i \Delta_\sigma^{\frac{j}{2}} u.$$

Then, with hypothesis (1b), for $0 \le j \le \frac{k}{2}$, $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p$ thus

$$\mathcal{M}_\alpha\left[\sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u\right] \in L^p(\mathbb{S}^{n-1}).$$

On the other hand, Corollary 3.5 implies that

$$\mathcal{M}_\alpha\left[(1 - r^2)\sum_{(i,j)\in\mathbb{A}_k} Q_{i,j}^{(k)}(r)N^i \Delta_\sigma^{\frac{j}{2}} u\right] \le C \sum_{(i,j)\in\mathbb{A}_{k-1}\cup(0,0)} \mathcal{M}_\beta\left[N^i \Delta_\sigma^{\frac{j}{2}} u\right] \in L^p(\mathbb{S}^{n-1}),$$

by induction hypothesis. We deduce from it that $\mathcal{M}_\alpha[v] \in L^p(\mathbb{S}^{n-1})$. But, solving the differential equation (4.2), we get

$$(4.3) \qquad\qquad N^k u(r\zeta) = \left(\frac{1 - r^2}{r^2}\right)^{n-1-k} \int_0^r v(t\zeta)\frac{t^{2n-3-2k}}{(1 - t^2)^{n-k}}dt$$

thus $\mathcal{M}_\alpha\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$ since $k \leq n-2$.

Assume now that u satisfies (1a) *i.e.* that $\mathcal{M}_\alpha\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$ for $j \leq k$ and let us show that $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p(\mathbb{S}^{n-1})$ for $j \leq \frac{k}{2}$. Let $1 > \beta > \alpha$.

According to the induction hypothesis, for $j \leq \frac{k}{2} - 1$, $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p(\mathbb{S}^{n-1})$ then

$$\mathcal{M}_\alpha\left[\sum_{j=1}^{\frac{k}{2}-1} P_j^{(k)}(r)\Delta_\sigma^j u\right] \in L^p(\mathbb{S}^{n-1}).$$

Further, note that for a regular function φ,

(4.4) $$\mathcal{M}_\alpha\big[(1-r^2)\varphi\big] \leq C\big(\mathcal{M}_\alpha\big[(1-r^2)^2 N\varphi\big] + |\varphi(0)|\big).$$

Assume now that $(i,j) \in \mathbb{A}_k$ so that $k-i+j \geq 1$, iterating inequality (4.4), we get

$$\begin{aligned}
\mathcal{M}_\alpha\left[(1-r^2)Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}} u\right] &\leq C\mathcal{M}_\alpha\left[(1-r^2)N^i\Delta_\sigma^{\frac{j}{2}} u\right] \\
&\leq C\mathcal{M}_\alpha\left[(1-r^2)^{k-i+1}N^k\Delta_\sigma^{\frac{j}{2}} u\right] + C\sum_{j=0}^{k-1}\big|\nabla^j u(0)\big| \\
&\leq C\sum_{j=0}^{k}\mathcal{M}_\beta\big[N^j u\big],
\end{aligned}$$

by Corollary 3.5. So, $C\displaystyle\sum_{j=0}^{k}\mathcal{M}_\beta\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$, thus

$$\mathcal{M}_\alpha\left[\sum_{(i,j)\in\mathbb{A}_k}(1-r^2)Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}} u\right] \in L^p(\mathbb{S}^{n-1}).$$

We then get from Lemma 4.1 that $\mathcal{M}_\alpha\left[P_{k/2}^{(k)}(r)(-\Delta_\sigma)^{k/2}u\right] \in L^p(\mathbb{S}^{n-1})$, and as $P_{k/2}^{(k)}$ is not zero on the boundary, $\mathcal{M}_\alpha\big[(-\Delta_\sigma)^{k/2}u\big] \in L^p(\mathbb{S}^{n-1})$. So (1a) and (1b) are equivalent.

Let us now show that (1a)+(1b) implies (1c). It is enough to show this implication for $\mathbb{X}$ a differential operator of the form $\mathbb{X} = N^j\mathbb{Y}$ with $\mathbb{Y}$ a product of $k-j$ operators of the form $\mathcal{L}_{p,q}$. Since (1a) holds, we can assume that $j < k$. Let

$$v = (1-r^2)N^{j+1}u + 2(n-1-j)N^j u$$

and compose with $\mathbb{Y}$, it results that

$$\mathbb{Y}v = (1-r^2)N^{j+1}\mathbb{Y}u + 2(n-1-j)N^j\mathbb{Y}u.$$

Using as previously formula (4.3), we see that

$$\mathcal{M}_\alpha[\mathbb{X}u] \in L^p(\mathbb{S}^{n-1})$$

which completes the proof in the case k is even. $\diamond$

Assume now that k is *odd*. The proof of (2b)$\Rightarrow$(2a) is similar to the case k even. The converse is again based on Lemma 4.1. According to the induction hypothesis, $\mathcal{M}_\alpha\big[\Delta_\sigma^l u\big] \in L^p(\mathbb{S}^{n-1})$ for $0 \leq l \leq \frac{k-1}{2} = \big[\frac{k}{2}\big]$ so that

$$\mathcal{M}_\alpha\left[\sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u\right] \in L^p(\mathbb{S}^{n-1}).$$

One has, as before,

$$\mathcal{M}_\alpha\left[(1-r^2)\sum_{(i,j)\in\mathbb{A}_k\backslash(0,k+1)} Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}} u\right] \in L^p(\mathbb{S}^{n-1})$$

and that $\mathcal{M}_\alpha\big[(1-r^2)N^{k+1}\big] \in L^p(\mathbb{S}^{n-1})$.

Combining all this, we get that

$$\mathcal{M}_\alpha\left[(1-r^2)Q_{0,k+1}^{(k)}(r)\Delta_\sigma^{\frac{k+1}{2}} u\right] \in L^p(\mathbb{S}^{n-1})$$

and as $Q_{0,k+1}^{(k)}$ is non-zero on the boundary, we finally get

$$\mathcal{M}_\alpha\left[(1-r^2)\Delta_\sigma^{\frac{k+1}{2}} u\right] \in L^p(\mathbb{S}^{n-1})$$

and (*a*) and (*b*) are equivalent.   □

Proof of Theorem B. First, (4) is an immediate consequence of (1). For the converse, assume Property 4, that is $\mathcal{M}_\alpha[(-\Delta_\sigma)^{k/2}u] \in L^p(\mathbb{S}^{n-1})$. We want to prove that

(4.5) $$\mathcal{M}_\alpha[N^{k-l}Xu] \in L^p(\mathbb{S}^{n-1})$$

for any $0 \leq l \leq k$ and any operator X of the form X that is the product of l of the form $\mathcal{L}_{i,j}$.

But, according to Remark 2.9, we also have $\mathcal{M}_\alpha[Yu] \in L^p(\mathbb{S}^{n-1})$ for any operator Y of the form $Y = \prod_{p=1}^q \mathcal{L}_{i_p,j_p}$ for $0 \leq q \leq k$.

It follows that Xu satisfies Conditions (1b) or (2b) of Theorem 4.2 depending on the parity of $k-l$ and the estimate (4.5) follows from that theorem.

We will now complete the proof of Theorem B by establishing the area integral characterizations.

The equivalence of Assertions 5 and 6 as well as the equivalence of Assertions 7, 8 and 9 are direct consequences of Lemma 3.7 (with $\gamma = -\frac{n}{2} + 1$), thus hold without restriction on k.

Further, as

$$\left|N(-\Delta_\sigma)^{k/2}u\right| \leq CI_k\left|N^{k+1}(-\Delta_\sigma)^{k/2}u\right| + \sup_{0\leq j\leq 2k,|z|\leq\varepsilon}\left|\nabla^j u\right|,$$

so, Lemma 3.7 and the mean-value inequality imply that

$$S_\alpha^N\left[(-\Delta_\sigma)^{k/2}u\right] \leq S_\beta^N\big[N^k u\big] + \|u\|_{\mathcal{H}^p},$$

so that Assertion 8 implies Assertion 6.

Let us now prove that if, for $0 \leq j \leq \frac{k}{2}$, $S_\alpha[\Delta_\sigma^j u] \in L^p(\$^{n-1})$, then $S_\alpha^N[N^k u] \in L^p(\$^{n-1})$. The proof goes according to the method developped for the equivalence of maximal functions.

For simplicity, we will restrict our attention to the case $k = 1$. In order to estimate $S_\alpha^N[Nu]$, we have to estimate Nu. While trying to use the previous method, Lemma 4.1 for $k = 2$ does not give a satisfying estimate. However, we can obtain the desired estimate as follows. Denote by v the function

$$v = 2(n-2)Nu + (1-r^2)N^2u,$$

then $Nv = 2(n-3)N^2u + (1-r^2)N^3u + 2(1-r^2)N^2u$ and write this in the form $Nv = w + 2(1-r^2)N^2u$. As before, solving the differential equation $(1-r^2)N^3u + 2(n-3)N^2u = w$, we have

$$N^2u(r\zeta) = \left(\frac{1-r^2}{r^2}\right)^{n-3} \int_0^r w(tz)t^{2(n-3)+1}(1-t^2)^{2-n}dt$$

so that

(4.6) $$\left|N^2u(r\zeta)\right| \leq C(1-r^2)^{n-3}I_{3-n}(|w|).$$

On the other hand, by Lemma 4.1 for $k = 1$, $v = (n-2)(1-r^2)Nu - (1-r^2)\Delta_\sigma u$, so $Nv = (n-2)(1-r^2)N^2u - (1-r^2)N\Delta_\sigma u + 2r^2\Delta_\sigma u - 2r^2(n-2)Nu$.

Recall that $|f| \leq CI_1(|N^1f|) + C\sup_{|z|<\varepsilon, j\leq 1}|\nabla^j f|$. One then gets

$$\begin{aligned}
|w| &\leq |Nv| + 2(1-r^2)|N^2u| \\
&\leq C\left(I_1(|N\Delta_\sigma u|) + I_1(|N^2u|) + (1-r^2)|N^2u| + (1-r^2)|N\Delta_\sigma u|\right) \\
&\quad + \sup_{0\leq j\leq 3, |z|\leq\varepsilon}|\nabla^j u|.
\end{aligned}$$

Inserting this in (4.6), and invoking the facts that $I_l((1-r^2)|f|) = I_{l+1}(|f|)$ and that $I_l(I_s|f|) \leq CI_{l+s}(|f|)$, one gets

$$\left|N^2u(r\zeta)\right| \leq C(1-r^2)^{n-3}\left(I_{4-n}(|N\Delta_\sigma u|) + I_{4-n}(|N^2u|) + \sup_{|z|<\varepsilon, 0\leq j\leq 3}|\nabla^j u|\right).$$

We are now in position to estimate $S_\alpha^N[Nu]$:

$$\begin{aligned}
S_\alpha^N[Nu](\zeta)^2 &= \int_{\mathcal{A}_\alpha(\zeta)} \left|N^2u(x)\right|^2(1-|x|)^{2-n}dx \\
&\leq C\left(\int_{\mathcal{A}_\alpha(\zeta)} [I_{4-n}(|N\Delta_\sigma u|)]^2(1-|x|^2)^{n-4}dx \right.\\
&\quad \left. + \int_{\mathcal{A}_\alpha(\zeta)} [I_{4-n}(|N^2u|)]^2(1-|x|^2)^{n-4}dx + \sup_{|z|<\varepsilon, 0\leq j\leq 3}|\nabla^j u|^2\right)
\end{aligned}$$

A further appeal to Lemma 3.6, with $l = 4-n$, $d = 0$, $k = 2$ and $\gamma = \frac{n-4}{2}$ leads to

$$\int_{\mathcal{A}_\alpha(\zeta)} [I_{4-n}(|N^2u|)]^2(1-|x|^2)^{n-4}dx \leq C\int_{\mathcal{A}_\beta(\zeta)} |N^2u|^2(1-|x|^2)^{4-n}dx.$$

A last appeal to Lemma 3.6, with $l = 4 - n$, $d = 0$, $k = 1$ and $\gamma = \frac{n-4}{2}$ leads to

$$
\begin{aligned}
\int_{\mathcal{A}_\alpha(\zeta)} \left[I_{4-n}(|N\Delta_\sigma u|) \right]^2 (1 - |x|^2)^{n-4} dx &\leq C \int_{\mathcal{A}_\beta(\zeta)} |N\Delta_\sigma u|^2 (1 - |x|^2)^{4-n} dx \\
&\leq C S_\gamma^N \left[(-\Delta_\sigma)^{\frac{1}{2}} u \right],
\end{aligned}
$$

by the mean-value properties. As the only part that matters in this last integral is the part near to the boundary, we will cut it into two parts. Let κ be a constant that we will fix later. Then

$$
\begin{aligned}
\int_{\mathcal{A}_\beta(\zeta)} \left| N^2 u \right|^2 (1 - |x|^2)^{4-n} dx &\leq \int_{\mathcal{A}_\beta(\zeta) \cap B(0,\kappa)} \left| N^2 u \right|^2 (1 - |x|^2)^{4-n} dx \\
&\quad + \int_{\mathcal{A}_\beta(\zeta) \cap \left(\mathbb{B}_n \setminus B(0,\kappa) \right)} \left| N^2 u \right|^2 (1 - |x|^2)^{4-n} dx \\
&\leq C \sup_{|z| \leq \kappa, 0 \leq j \leq 3} \left| \nabla^j u \right| \\
&\quad + (1 - \kappa^2)^2 \int_{\mathcal{A}_\beta(\zeta)} \left| N^2 u \right|^2 (1 - |x|^2)^{2-n} dx.
\end{aligned}
$$

Gathering the above estimates, we finally get

$$
\begin{aligned}
\text{(4.7)} \qquad S_\alpha^N[Nu](\zeta)^2 &\leq C S_\gamma^N \left[(-\Delta_\sigma)^{\frac{1}{2}} u \right](\zeta)^2 + C(1 - \kappa^2)^2 S_\beta[Nu](\zeta)^2 \\
&\quad + C \sup_{|z| \leq \kappa, 0 \leq j \leq 3} \left| \nabla^j u \right|.
\end{aligned}
$$

Note that this inequality depends only on the mean-value inequality. Now, consider the function on $\mathbb{B}_n$ defined by $u_\delta(x) = u(\delta x)$ ($0 < \delta < 1$) and note that u_δ satisfies

$$
\left[(1 - \delta^2 r^2)^2 \Delta + 2(n-2)\delta^2 (1 - \delta^2 r^2) N \right] u_\delta = 0
$$

so that, repeating the arguments of section 3.1, we see that u_δ satisfies the same mean-value inequalities as u, with constants independent on δ. In particular, one can replace u in (4.7) by u_δ and get

$$
S_\alpha^N[Nu_\delta](\zeta)^2 \leq C S_\gamma^N \left[(-\Delta_\sigma)^{\frac{1}{2}} u_\delta \right](\zeta)^2 + C(1 - \kappa^2)^2 S_\beta^N[Nu_\delta](\zeta)^2 + C \sup_{|z| \leq \kappa, 0 \leq j \leq 3} \left| \nabla^j u_\delta \right|
$$

with constants independent on $\frac{1}{2} < \delta < 1$. Then taking $L^p(\mathbb{S}^{n-1})$ norms, one gets

$$
\begin{aligned}
\left\| S_\alpha^N[Nu_\delta] \right\|_p &\leq C \left\| S_\gamma^N \left[(-\Delta_\sigma)^{\frac{1}{2}} u_\delta \right] \right\|_p + C(1 - \kappa^2) \left\| S_\beta^N[Nu_\delta] \right\|_p + C\|u\|_{\mathcal{H}^p} \\
&\leq C \left\| S_\gamma^N \left[(-\Delta_\sigma)^{\frac{1}{2}} u \right] \right\|_p + C'(1 - \kappa^2) \left\| S_\alpha^N[Nu_\delta] \right\|_p + C\|u\|_{\mathcal{H}^p},
\end{aligned}
$$

since, as is well known (*see* Proposition 4 of [4]), that the L^p norms of S_α^N and S_β^N are equivalent. It is now enough to choose κ such that $C'(1 - \kappa^2) = \frac{1}{2}$, then

$$
\left\| S_\alpha^N[Nu_\delta] \right\|_p \leq 2C \left\| S_\gamma^N \left[(-\Delta_\sigma)^{\frac{1}{2}} u \right] \right\|_p + 2C\|u\|_{\mathcal{H}^p}.
$$

We then conclude by monotone convergence when $\delta \to 1$. $\qquad\qquad\square$

5. Lipschitz spaces

In this section, we transpose to hyperbolic-harmonic extensions the following well known characterization of Lipschitz spaces $\Lambda_\alpha(\mathbb{S}^{n-1})$ through harmonic extensions (see [11]): a bounded function f is in $\Lambda_\alpha(\mathbb{S}^{n-1})$ if and only if its Euclidean harmonic extension $v = \mathbb{P}_e[f]$ satisfies the following property for some integer $k > \alpha$

$$(5.1) \qquad \sup_{z \in B_n} (1 - |z|)^{k-\alpha}|N^k v(z)| < \infty.$$

Moreover, this quantity is equivalent to the Lipschitz norm of f and, if (5.1) holds for some $k > \alpha$, it holds for any $k > \alpha$.

To be more precise, let us recall first the definitions of the Lipschitz spaces.

Definition 5.1. For $0 < \alpha < 1$, $\Lambda_\alpha(\mathbb{S}^{n-1})$ is defined as the space of bounded functions f on $\mathbb{S}^{n-1}$ for which there exists a constant $C > 0$ such that $|f(\zeta) - f(\eta)| \leq C|\zeta - \eta|^\alpha$ for all $\eta, \zeta \in \mathbb{S}^{n-1}$.

For $k < \alpha < k + 1$, $k \in \mathbb{N}$, $\Lambda_\alpha(\mathbb{S}^{n-1})$ is defined as the space of bounded functions f on $\mathbb{S}^{n-1}$ such that every derivative of f of order k belongs to $\Lambda_{\alpha-k}(\mathbb{S}^{n-1})$.

When α is an integer, $\Lambda_\alpha(\mathbb{S}^{n-1})$ is defined by real interpolation and the corresponding Lipschitz spaces are usually designed as *Zygmund classes*.

Remark 5.2. It is well known that the Zygmund classes of order $k \in \mathbb{N}$ are larger than the Hölder classes H_k. These are the classes of $\mathcal{C}^{k-1}$ functions f such that, for every derivative X of order $k - 1$, $|Xf(\zeta) - Xf(\eta)| \leq C|\zeta - \eta|$, for some constant C that does not depend on ζ, η.

We prove the following result for hyperbolic extensions.

Theorem 5.3. *Let $\alpha > 0$ and assume further that $\alpha < n - 1$ if n is odd. Let f be a bounded function on $\mathbb{S}^{n-1}$. The following are equivalent*

i. *$f \in \Lambda_\alpha(\mathbb{S}^{n-1})$.*

ii. *There exists an integer $k > \alpha$, and a constant $C > 0$ such that, for every $z \in \mathbb{B}_n$,*

$$(5.2) \qquad (1 - |z|)^k |N^k \mathbb{P}_h[f](z)| \leq C(1 - |z|)^\alpha.$$

Remark 5.4. It is easy to prove that condition (ii) implies that for any integer $j > \alpha$, one has

$$(1 - |z|)^j |N^j \mathbb{P}_h[f](z)| \leq C(1 - |z|)^\alpha.$$

For $j > k$, this follows from the mean-value property (3.3) and for $j < k$ from radial integration. One can show as well that the analogous estimate holds for any derivative of order $j > \alpha$: for any differential operator X of order j,

$$(1 - |z|)^j |X\mathbb{P}_h[f](z)| \leq C(1 - |z|)^\alpha.$$

As a consequence, condition (ii) implies that, if $\mathbb{P}_h[f]$ satisfies (5.2), then $\mathbb{P}_h[f] \in \Lambda_\alpha(\mathbb{B}_n)$. This in turn implies that $f \in \Lambda_\alpha(\mathbb{S}^{n-1})$.

Proof. According to the remark, it is enough to prove that (i) implies (ii) with $k = [\alpha] + 1$.

Assume first that n is even and let $f \in \Lambda_\alpha(\mathbb{S}^{n-1})$. Then, for any $j > \alpha$

$$(1 - |z|)^j \big| N^j \mathbb{P}_e[f](z) \big| \leq C(1 - |z|)^\alpha.$$

Next, Formula (2.1) implies

$$(1 - |z|)^k \big| N^k \mathbb{P}_h[f](z) \big| = \sum_{j=0}^{n/2-1} \sum_{l=0}^{k} O\big((1 - |z|)^{k+(j+l-k)_+} N^{j+l} \mathbb{P}_e[f](z)\big)$$

and each of these terms has the desired behaviour, thus we have established (ii).

The case n odd will be obtained in a different way (which also gives the result for n even). However, the argument below will only be valid for $\alpha < n - 1$.

Let $f \in \Lambda_\alpha(\mathbb{S}^{n-1})$, with $\alpha < n - 1$. Let $u = \mathbb{P}_h[f]$. We will prove in two steps that u satisfies the estimate (5.2). The first step deals with the case $0 < \alpha < 2$, the general case will follow.

First Step. If $0 < \alpha < 2$ then

$$N^2 u(z) = \mathcal{O}\big((1 - |z|)^{\alpha-2}\big).$$

For $\zeta, \xi \in \mathbb{S}^{n-1}$, denote by $S_\zeta \xi$ the symmetric of ξ with respect to ζ on $\mathbb{S}^{n-1}$. In other words, we have $S_\zeta \xi = 2\langle \xi, \zeta \rangle \zeta - \xi$. It follows from the definition of $\Lambda_\alpha(\mathbb{S}^{n-1})$ that $|2f(\zeta) - f(\xi) - f(S_\zeta \xi)| \leq ||f||_\alpha |\zeta - \xi|^\alpha$.

Note that, since $\mathbb{P}_h(r\xi, .)$ has integral 1 on $\mathbb{S}^{n-1}$, we have

$$\int_{\mathbb{S}^{n-1}} \left(r \frac{\partial}{\partial r} \right)^2 \mathbb{P}_h(r\xi, \zeta) d\sigma(\zeta) = 0.$$

Using this fact and the symmetry under rotations of $\mathbb{P}_h(r\xi, .)$, we get that

$$\left| \left(r \frac{\partial}{\partial r} \right)^2 u(r\xi) \right| = \left| \int_{S^{n-1}} \left(r \frac{\partial}{\partial r} \right)^2 \mathbb{P}_h(r\xi, \zeta) f(\zeta) d\sigma(\zeta) \right|$$

$$= \left| \int_{\mathbb{S}^{n-1}} \left(r \frac{\partial}{\partial r} \right)^2 \mathbb{P}_h(r\xi, \zeta)[2f(\zeta) - f(\xi) - f(S_\zeta \xi)] d\sigma(\zeta) \right|.$$

Further, a direct computation shows that

$$\left| \left(r \frac{\partial}{\partial r} \right)^2 \mathbb{P}_h(r\xi, \zeta) \right| \leq \frac{C(1 - r)^{n-3}}{\big((1 - r) + |\xi - \zeta|\big)^{2(n-1)}}.$$

Combining these two facts, we get

$$\left| \left(r \frac{\partial}{\partial r} \right)^2 u(r\xi) \right| \leq C ||f||_\alpha (1 - r)^{n-3} \int_{\mathbb{S}^{n-1}} \frac{|\xi - \zeta|^\alpha}{\big((1 - r) + |\xi - \zeta|\big)^{2(n-1)}} d\sigma(\zeta).$$

Now, cutting the integral into two parts $|\xi - \zeta| > c(1 - r)$ and $|\xi - \zeta| < c(1 - r)$, we get $|N^2u(r\xi)| \leq C||f||_\alpha(1 - r)^{\alpha-2}$. This completes the proof of the case $0 < \alpha < 2$.

Second Step. If $\alpha \geq 2$, then $\left|N^{[\alpha]+1}u\right| \leq C(1 - |z|)^{\alpha-[\alpha]-1}$.
First, to simplify the notations, we denote by $\mathcal{L}$ the set of vector fields $\mathcal{L}_{ij}$, $1 \leq i < j \leq n$, and by $\mathcal{L}^l$ the set of vector fields consisting of the products of l elements of $\mathcal{L}$.

For any $0 < \alpha' < \alpha$ and any integer $l < \alpha'$, $\mathcal{L}^lf \subset \Lambda_{\alpha'-l}(\mathbb{S}^{n-1})$. For a fixed integer $l \leq [\alpha] - 1$, let $\alpha' = l + \alpha - [\alpha] + 1$ so that $\alpha' - l \in [1, 2[$. As $\mathcal{L}$ and D commute, $\mathcal{L}^lu = \mathbb{P}_h(\mathcal{L}^lf)$ (with the obvious abuse of notation) so that, from the first part of the proof, we conclude that
$$\left|N^2\mathcal{L}^lu(z)\right| \leq C(1 - |z|)^{\alpha-[\alpha]-1}.$$
But,
$$(1 - |z|)^2\left|N^2\mathcal{L}^{l+2}u(z)\right| = (1 - |z|)^2\left|\mathcal{L}^2N^2\mathcal{L}^lu(z)\right| \leq C(1 - |z|)^2\left|\nabla^2N^2\mathcal{L}^lu(z)\right|$$
so that, by the mean-value inequality (3.3), we get
$$(1 - |z|)^2\left|N^2\mathcal{L}^{l+2}u(z)\right| \leq C(1 - |z|)^{\alpha-[\alpha]-1}.$$
Integrating twice in the radial direction gives $\left|\mathcal{L}^{l+2}u\right| \leq C(1 - |z|)^{\alpha-[\alpha]-1}$ (since $\alpha < [\alpha] + 1$).

The proof of the theorem is therefore completed with the next lemma.

Lemma 5.5. *Let $A > 0$ and k be an integer such that $k + 2 < A + (n - 1)$. Let u be a $\mathcal{H}$-harmonic function in $\mathbb{B}^n$ and assume that $u(z) = \mathcal{O}\big((1 - |z|)^{-A}\big)$. The following are equivalent.*

1. $\left|N^{k+2}u(z)\right| = \mathcal{O}\big((1 - |z|)^{-A}\big)$,

2. *for $l = 2, \ldots, k + 2$, $\left|\mathcal{L}^lu(z)\right| = \mathcal{O}\big((1 - |z|)^{-A}\big)$.*

Proof of Lemma 5.5. The proof that (1) implies (2) is classical as pointed out in the remark after the statement of the theorem. We will now prove the converse by induction on k.

Assume first that $k = 0$. By Lemma 4.1, $(1 - |z|^2)N^3u + 2(n - 3)N^2u$ is equal to
$$\mathcal{O}(1)\Delta_\sigma u + \mathcal{O}(1)(1 - |z|^2)[N\Delta_\sigma u + N^2u + \Delta_\sigma u + Nu].$$
As $\Delta_\sigma u = \sum \mathcal{L}_{ij}^2u$, Assumption 2 gives $\Delta_\sigma u(z) = \mathcal{O}\big((1 - |z|)^{-A}\big)$. The mean-value property then implies that $(1 - |z|)|N\Delta_\sigma u(z)| = \mathcal{O}\big((1 - |z|)^{-A}\big)$. From the hypothesis $u(z) = \mathcal{O}\big((1 - |z|)^{-A}\big)$ and the mean-value property, we also get that $(1 - |z|)|Nu(z)|$ and $(1 - |z|)^2\left|N^2u(z)\right|$ are $\mathcal{O}\big((1 - |z|)^{-A}\big)$. Summarizing these estimates, we get
$$(1 - |z|^2)N^3u(z) + 2(n - 3)N^2u(z) = \mathcal{O}\big((1 - |z|)^{-A-1}\big).$$

We thus get a differential equation of the form $(1 - r^2)N^3u + 2(n - 3)N^2u = w$ with $w(z) = \mathcal{O}\big((1 - |z|)^{-A-1}\big)$. Solving it as in (4.2)-(4.3), we get the estimate $N^2u(z) = \mathcal{O}\big((1 - |z|)^{-A-1}\big)$, since $n - 3 + A > 0$.

Bootstraping the argument and re-estimating w, we then get $w = \mathcal{O}\big((1-|z|)^{-A}\big)$ from which we deduce that $N^2 u(z)$ is in fact $\mathcal{O}\big((1-|z|)^{-A}\big)$. This completes the proof in the case $k = 0$.

Consider now the case $k \geq 1$. Assume that $k + 2 < A + (n-1)$ and that the lemma is proved for any integer $k' \leq k-1$.

As before, it suffices to show that

$$(1-|z|^2)N^{k+3}u + 2(n-3-k)N^{k+2}u = \mathcal{O}\big((1-|z|^2)^{-A}\big)$$

and to solve the differential equation as in (4.2-4.3) to get the result (since here $n - 3 - k + A > 0$). By Lemma 4.1, this quantity equals

$$\mathcal{O}(1)\sum_{j=1}^{[(k+2)/2]}\Delta_\sigma^j u + \mathcal{O}(1)(1-|z|^2)\sum_{\substack{0\leq i\leq k+2 \\ 0\leq 2j\leq k+3,\, 1\leq i+2j\leq k+3}} N^i\Delta_\sigma^j u.$$

The first sum is by assumption equal to $\mathcal{O}\big((1-|z|^2)^{-A}\big)$. By the induction hypothesis, one has $\big|N^{k+1}u(z)\big| = \mathcal{O}\big((1-|z|)^{-A}\big)$. As usual, it allows to say that any derivative of u of order less than $k+1$ has the same bound. By the mean-value inequality, we further get that any derivative of order $k+2$ is an $\mathcal{O}\big((1-|z|)^{-A-1}\big)$. It thus suffices to consider only the term of order $k+3$ in the second sum.

We split the corresponding remaining terms into two parts:

$$\mathcal{O}(1)(1-|z|^2)N^2\Delta_\sigma^{(k+1)/2}u + \mathcal{O}(1)(1-|z|^2)\sum_{\substack{0\leq i\leq k+2,\, i\neq 2 \\ 0\leq 2j\leq k+3,\, i+2j=k+3}} N^i\Delta_\sigma^j u.$$

Note that the first term only appears when k is odd. For this term, we write $(1-|z|^2)\big|N^2\Delta_\sigma^{(k+1)/2}u\big| \simeq (1-|z|^2)\big|\mathcal{L}N^2\mathcal{L}^k u\big|$ and we bound it by the mean-value of $N^2\mathcal{L}^k u$. Now, we apply the induction hypothesis to the $\mathcal{H}$-harmonic function $\mathcal{L}^k u$ to get that it is $\mathcal{O}\big((1-|z|^2)^{-A}\big)$.

Using the mean-value property, we get that the second sum is bounded (up to a constant) by the mean-value of

$$\sum_{\substack{1\leq i\leq k+2,\, i\neq 2 \\ 0\leq 2j\leq k+3,\, i+2j=k+3}} \big|N^{i-1}\Delta_\sigma^j u\big| + \big|\mathcal{L}^{k+2}u\big|.$$

Each of these terms is again $\mathcal{O}\big((1-|z|^2)^{-A}\big)$, either by assumption (2) or by the induction hypothesis. This completes the proof of the lemma. $\diamond$

The proof of Theorem 5.3 is thus complete. $\square$

As a conclusion, let us emphasize that the restriction on α can not be removed in odd dimension:

Proposition 5.6. *Assume that n is odd. Then,*

1. *if $f \in H_{n-1}(\mathbb{S}^{n-1})$, then $(1-|z|)|N^n\mathbb{P}_h[f]|$ is bounded;*

2. *there is an $f \in \Lambda_{n-1}(\mathbb{S}^{n-1})$, such that $(1-|z|)|N^n\mathbb{P}_h[f]|$ is unbounded.*

Remark 5.7. We know from [13, 14] that if $(1 - |z|^2)N^n\mathbb{P}_h[f] = o(1)$ then f is constant. Therefore this theorem gives the optimal Lipschitz-regularity of $\mathcal{H}$-harmonic functions in odd dimensional hyperbolic balls.

Proof. Assume that $f \in H_{n-1}$. Then there is a constant $C > 0$ such that f satisfies $|Xf(\zeta) - Xf(\eta)| \leq C|\zeta - \eta|$ for any differential operator $X \in \mathcal{L}^{n-2}$ and any $\eta, \zeta \in \mathbb{S}^{n-1}$. This implies that $\Delta_\sigma^{\frac{n-1}{2}} P_h[f]$ is bounded.

Set $u = \mathbb{P}_h[f]$ and note that, when n is odd, Lemma 4.1 leads to,

$$(1 - |z|^2)N^n u = P_{(n-1)/2}^{(n-1)}(r)\Delta_\sigma^{(n-1)/2}u + \mathcal{O}(1)\sum_{j=1}^{(n-3)/2}\Delta_\sigma^j u$$
$$+ \mathcal{O}(1)(1 - |z|^2)\sum_{\substack{0 \leq 2j \leq n,\ 1 \leq i+2j \leq n \\ 0 \leq i \leq n-1}} N^i\Delta_\sigma^j u.$$

From $f \in \mathcal{C}^{n-2}(\mathbb{S}^{n-1})$, we get that all terms on the right hand side are bounded, excepted eventually $\mathcal{O}(1)(1 - |z|^2)N^i\Delta_\sigma^j u$ with $i + 2j = n$. These may all be written as $\mathcal{O}(1)(1 - |z|^2)N^i\mathbb{P}_h[\Delta_\sigma^{\frac{n-i}{2}}f]$ with $i \leq n - 1$. As $\Delta_\sigma^{\frac{n-i}{2}}f \in \Lambda_{i-1}(\mathbb{S}^{n-1})$, by theorem 5.3, these terms are therefore also bounded.

Let us now prove point 2. Using again Lemma 4.1, if $(1 - |z|)|N^n\mathbb{P}_h[f]|$ is bounded and $f \in \Lambda_{n-1}(\mathbb{S}^{n-1})$, then $\Delta_\sigma^{\frac{n-1}{2}}\mathbb{P}_h[f]$ is also bounded. Then, as $\Delta_\sigma^{\frac{n-1}{2}}\mathbb{P}_h[f] = \mathbb{P}_h[\Delta_\sigma^{\frac{n-1}{2}}f]$, it follows that $\Delta_\sigma^{\frac{n-1}{2}}f$ is also bounded, by Fatou's Theorem. But it is well known that there exists $f \in \Lambda_{n-1}(\mathbb{S}^{n-1})$ such that $\Delta_\sigma^{\frac{n-1}{2}}f$ is unbounded. $\square$

Acknowledgements

References

[1] P. AHERN, J. BRUNA & C. CASCANTE H^p-theory for generalized $\mathcal{M}$-harmonic functions in the unit ball. *Indiana Univ. Math. J.*, 45 (1996) 103–145.

[2] A. BONAMI, J. BRUNA & S. GRELLIER On Hardy, *BMO* and Lipschitz spaces of invariant harmonic functions in the unit ball. *Proc. London Math. Soc.*, 77 (1998) 665–696.

[3] P. CIFUENTES H^p-classes on rank one symmetric spaces of noncompact type. II. Nontangential maximal function and area integral.. *Proc. Amer. Math. Soc.*, 106 (1989), 519–525.

[4] R. R. COIFMAN, Y. MEYER & E. M. STEIN Some new function spaces and their applications to harmonic analysis. *Jour. Func. Anal.*, 62 (1985) 304–335.

[5] L. COLZANI Hardy spaces on unit spheres. *Boll. U.M.I. Analisi Funzionale e Applicazioni VI*, IV - C (1985) 219–244.

[6] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER & F. TRICOMI *Higher transcendental functions* **1** McGraw-Hill, New-York, 1953.

[7] P. EYMARD Le noyau de Poisson et l'analyse harmonique non usuelle. In *Topics in Harmonic Analysis*, pages 353–404. Torino/Milano, 1982.

[8] P. EYMARD Transfert des fonctions harmoniques et fonctions propres de l'opérateur de Laplace-Beltrami. In *Conference on Harmonic Analysis (Lyon 1982)*, page Exp 11. Publ. Dep. Math. Lyon, 1982.

[9] C. FEFFERMAN & E .M. STEIN H^p spaces of several variables. *Acta Math.*, 129 (1972) 137–193.

[10] C. R. GRAHAM The Dirichlet problem for the Bergman Laplacian II. *Comm. Partial Differential Equations*, 8 (1983) 563–641.

[11] H. C. GREENWALD Lipschitz spaces on the surface of the unit sphere in Euclidean n-space. *Pacific J. Math.*, 50 (1974) 63–80.

[12] PH. JAMING *Trois problèmes d'analyse harmonique.* PhD thesis, Université d'Orléans, 1998.

[13] PH. JAMING Harmonic functions on the real hyperbolic ball I : Boundary values and atomic decomposition of Hardy spaces. *Colloq. Math.*, 80 (1999) 63–82.

[14] PH. JAMING Harmonic functions on classical rank one balls. *Boll. U.M.I.*, 4-B Vol 8 (2001) 685–702

[15] K. MINEMURA Harmonic functions on real hyperbolic spaces. *Hiroshima Math. J.*, 3 (1973) 121–151.

[16] K. MINEMURA Eigenfunctions of the Laplacian on a real hyperbolic spaces. *J. Math. Soc. Japan*, 27 (1975) 82–105.

[17] J. G. RATCLIFFE *Foundations of hyperbolic manifolds.* Graduate Texts in Mathematics, 149 Springer 1994.

[18] H. SAMII *Les Transformations de Poisson dans la Boule Hyperbolique.* PhD thesis, Université Nancy 1, 1982.

[19] E. M. STEIN *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.

[20] E. M. STEIN *Harmonic Analysis.* Princeton University Press, 1993.

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