CONVERGENCE OF FORMAL INVERTIBLE CR MAPPINGS BETWEEN MINIMAL HOLOMORPHICALLY NONDEGENERATE REAL ANALYTIC HYPERSURFACES

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Abstract. Recent advances in CR geometry have raised interesting fine questions about the regularity of CR mappings between real analytic hypersurfaces. In analogy with the known optimal results about the algebraicity of holomorphic mappings between real algebraic sets, some statements about the optimal regularity of formal CR mappings between real analytic CR manifolds can be naturally conjectured. Concentrating on the hypersurface case, we show in this paper that a formal invertible CR mapping between two minimal holomorphically nondegenerate real analytic hypersurfaces in $\mathbb{C}^n$ is convergent. The necessity of holomorphic nondegeneracy was known previously. Our technique is an adaptation the inductional study of the jets of formal CR maps which was discovered by Baouendi-Ebenfelt-Rothschild. However, as the manifolds we consider are far from being finitely nondegenerate, we must consider some new *conjugate reflection identities* which appear to be crucial in the proof. The higher codimensional case will be studied in a forthcoming paper.

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§1. Introduction and statement of the results

1.1. Main theorem. Let (M,p) and (M',p') be two small pieces of real analytic hypersurfaces of $\mathbb{C}^n$, with $n \geq 2$. Here, the two points $p \in M$ and $p' \in M'$ are considered to be "central points". Let $t=(t_1,\dots,t_n)$ be some holomorphic coordinates vanishing at p and let $\rho(t,\bar{t})=0$ be a real analytic power series defining equation for (M,p). Similarly, we choose a defining equation $\rho'(t',\bar{t}')=0$ for (M',p'). Let $h(t)=(h_1(t),\dots,h_n(t))$ be a collection of formal power series $h_j(t) \in \mathbb{C}[\![t]\!]$ with $h_j(0)=0$. We shall say that h induces a *formal CR mapping between (M,p) and (M',p')* if there exists a formal power series $b(t,\bar{t})$ such that $\rho'(h(t),\bar{h}(\bar{t})) \equiv b(t,\bar{t})\,\rho(t,\bar{t})$. Further, h will be said to be a *formal equivalence between (M,p) and (M',p')* if in addition the formal Jacobian determinant of h is

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nonzero, namely if $\det\left(\frac{\partial h_j}{\partial t_i}(0)\right)_{1\leq i,j\leq n} \neq 0$. If the formal power series $h_j(t)$ are convergent, it follows from the identity $\rho'(h(t), \bar{h}(\bar{t})) \equiv b(t,\bar{t})\,\rho(t,\bar{t})$ that h maps a neighborhood of p in M biholomorphically onto a neighborhood of p' in M'. We are interested in optimal sufficient conditions on the triple $\{M, M', h\}$ which insure that the formal equivalence h is convergent, namely the series $h_j(t)$ converge for t small enough. To specify that the mapping h is formal, we shall write it "$h\colon (M,p) \to_{\mathcal{F}} (M',p')$", with the index "$\mathcal{F}$" referring to the word "formal". The hypersurface (M,p) will be called *minimal* (at p, in the sense of Trépreau-Tumanov) if there does not exist a small piece of a complex $(n-1)$-dimensional manifold passing through p which is contained in (M,p). Recall also that (M',p') is called *holomorphically nondegenerate* if there does not exist a nonzero (1,0) vector field with holomorphic coefficients whose flow stabilizes (M',p'). The present paper is essentially devoted to establish the following assertion.

Theorem 1.2. *Let $h : (M,p) \to_{\mathcal{F}} (M',p')$ be a formal invertible CR mapping between two real analytic hypersurfaces in $\mathbb{C}^n$ and assume that (M,p) is minimal. If (M',p') is holomorphically nondegenerate, then h is convergent.*

(The reader is referred to the monograph [3] and to the articles [2,4,10,12] for further background material). This theorem provides a necessary and sufficient condition for the convergence of an invertible formal CR map of hypersurfaces. The necessity appears in a natural way (*see* Proposition 1.5 below). Geometrically, holomorphic nondegeneracy has a clear signification : it means that there exist *no* holomorphic tangent vector field to (M',p'). This condition is equivalent to the *nonexistence* of a local complex analytic foliation of $(\mathbb{C}^n,p')$ tangent to (M',p'). As matters stand, such a kind of characterization for the regularity of CR maps happens to be known already in case where at least *one* of the two hypersurfaces is algebraic, see *e.g.* [5,6,13]. In fact, in the algebraic case, one can apply the classical "polynomial identities" in the spirit of Baouendi-Jacobowitz-Treves. It was known that the true real analytic case requires deeper investigations.

1.3. Brief history. Formal invertible CR mappings $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ between two local pieces of real analytic hypersurfaces in $\mathbb{C}^n$ have been proved to be convergent in various circumstances. Firstly, in 1974 by Chern-Moser, assuming that (M',p') is Levi-nondegenerate. Secondly, in 1997 by Baouendi-Ebenfelt-Rothschild in [2], assuming that h is invertible (*i.e.* with nonzero Jacobian at p) and that (M',p') is finitely nondegenerate at p'. And more recently in 1999, by Baouendi-Ebenfelt-Rothschild [4], assuming for instance (but this work also contains other results) that (M',p') is essentially finite, that (M,p) is minimal and that h is not totally degenerate, a result which is valid in arbitrary codimension. (Again, the reader may consult [3] for essential background on the subject, for definitions, concepts and tools and also [10] for related topics.) In summary, the above-mentioned results have all exhibited some sufficient conditions.

1.4. Necessity. On the other hand, it is known (essentially since 1995, *cf.* [5]) that holomorphic nondegeneracy of the hypersurface (M',p') constitutes a *natural necessary condition* for h to be convergent, according to an important observation due to Baouendi-Rothschild [2,3,5] (this observation followed naturally from the characterization by Stanton of the finite-dimensionality of the space of infinitesimal CR automorphisms of (M,p) [16] ; Stanton's discovery is fundamental in the subject). We may restate this observation as follows (*see* its proof at the end of §4).

Proposition 1.5. *If (M', p') is holomorphically degenerate, then there exists a nonconvergent formal invertible CR self map of (M', p'), which is simply of the form $\mathbb{C}^n \ni t' \mapsto_{\mathcal{F}} \exp(\varpi'(t')L')(t') \in \mathbb{C}^n$, where L' is a nonzero holomorphic tangent vector to (M', p') and where the formal series $\varpi'(t') \in \mathbb{C}[\![t']\!]$, $\varpi'(0) = 0$, is nonconvergent.*

A geometric way to interpret this nonconvergent map would be to say that it is a map which "slides in nonconvergent complex time" along the complex analytic foliation induced by L', which is tangent to M' by assumption. By this, we mean that each point q' of an arbitrary complex curve γ' of the flow foliation induced by L' is "pushed" *inside* γ' by means of a nonconvergent series corresponding to the time parameter of the flow. This intuitive language can be illustrated adequately in the generic case where the vector field L' is nonzero at p'. Indeed, we can suppose that $L' = \partial/\partial t'_1$ after a straightening and the above nonconvergent formal mapping is simply $t' \mapsto_{\mathcal{F}} (t'_1 + \varpi(t'), t'_2, \dots, t'_n)$. Here, the t'_1-lines are the leaves of the flow foliation of L' and we indeed "push" or "translate" the point $(t'_1, t'_2, \dots, t'_n)$ by means of $\varpi'(t')$ inside a leaf.

Similar obstructions for the algebraic mapping problem stem from the existence of complex analytic (or algebraic) foliations tangent to (M', p'), see *e.g.* [5,6]. Again, this shows that the geometric notion of holomorphic nondegeneracy discovered by Stanton is crucial in the field.

1.6. Jets of Segre varieties. The holomorphically nondegenerate hypersurfaces are considerably more general and more difficult to handle than Levi-nondegenerate ones [14,15,17], finitely nondegenerate ones [2], essentially finite ones [3,4] or even Segre nondegenerate ones [10]. The explanation becomes clear after a reinterpretation of these conditions in the spirit of the important geometric definition of jets of Segre varieties due to Diederich-Webster [7]. In fact, these five distinct nondegeneracy conditions manifest themselves directly as nondegeneracy conditions of the morphism of k-th jets of Segre varieties attached to M', which is an invariant *holomorphic* map defined on its extrinsic complexification $\mathcal{M}' = (M')^c$ (we follow the notations of §2). Here, the letter "c" stands for the "complexification operator". In local holomorphic normal coordinates $t' = (w', z') \in \mathbb{C}^{n-1} \times \mathbb{C}$, vanishing at p' with $\tau' := (\zeta', \xi') \in \mathbb{C}^{n-1} \times \mathbb{C}$ denoting the complexied coordinates $(w', z')^c$, such that the holomorphic equation of the extrinsic complexification $\mathcal{M}'$ is written $\xi' = z' - i\Theta'(\zeta', t') = z' - i\sum_{\gamma \in \mathbb{N}^{n-1}_*} \zeta'^{\gamma}\, \Theta'_\gamma(t')$ (*cf.* (2.4)), the conjugate complexified Segre variety is defined by $\underline{\mathcal{S}}'_{t'} := \{\tau' : \xi' = z' - i\Theta'(\zeta', t')\}$ (here, t' is fixed; *see* [9] for a complete exposition of the geometry of complexified Segre varieties) and the jet of order k of the complex $(n-1)$-dimensional manifold $\underline{\mathcal{S}}'_{t'}$ at the point $\tau' \in \underline{\mathcal{S}}'_{t'}$ defines a holomorphic map

$$\varphi'_k : \mathcal{M}' \ni (t', \tau') \mapsto j^k_{\tau'} \underline{\mathcal{S}}'_{t'} \in \mathbb{C}^{n+N_{n-1,k}}, \qquad N_{n-1,k} = \frac{(n-1+k)!}{(n-1)!\,k!}, \tag{1.7}$$

given explicitely in terms of such a defining equation by a collection of power series:

$$\varphi'_k(t', \tau') := j^k_{\tau'} \underline{\mathcal{S}}'_{t'} = (\tau', \{\partial^\beta_{\zeta'}[\xi' - z' + i\Theta'(\zeta', t')]\}_{\beta \in \mathbb{N}^{n-1}, |\beta| \le k}). \tag{1.8}$$

For k large enough, the various possible properties of this holomorphic map govern some different "*nondegeneracy conditions*" on M' which are appropriate for some generalizations of the Lewy-Pinchuk reflection principle. Let ${p'}^c := (p', \bar{p}') \in \mathcal{M}'$. We give here an account of five conditions, which can be understood as definitions:

$$\text{(I)} \qquad \begin{aligned} &(M', p') \text{ is } \textit{Levi-nondegenerate} \text{ at } p' \\ &\iff \varphi_1' \text{ is an immersion at } {p'}^c. \end{aligned}$$

$$\text{(II)} \qquad \begin{aligned} &(M', p') \text{ is } \textit{finitely nondegenerate} \text{ at } p' \\ &\iff \exists\, k_0 \in \mathbb{N}_*,\ \varphi_k' \text{ is an immersion at } {p'}^c,\ \forall\, k \geq k_0. \end{aligned}$$

$$\text{(III)} \qquad \begin{aligned} &(M', p') \text{ is } \textit{essentially finite} \text{ at } p' \\ &\iff \exists\, k_0 \in \mathbb{N}_*,\ \varphi_k' \text{ is a finite holomorphic map at } {p'}^c,\ \forall\, k \geq k_0. \end{aligned}$$

$$\text{(IV)} \qquad \begin{aligned} &(M', p') \text{ is } \textit{S-nondegenerate} \text{ at } p' \\ &\iff \exists\, k_0 \in \mathbb{N}_*,\ \varphi_k'|_{\mathcal{S}_{\bar{p}'}} \text{ is of generic rank } \dim_{\mathbb{C}} \mathcal{S}_{\bar{p}'} = n-1,\ \forall\, k \geq k_0. \end{aligned}$$

$$\text{(V)} \qquad \begin{aligned} &(M', p') \text{ is } \textit{holomorphically nondegenerate} \text{ at } p' \\ &\iff \exists\, k_0 \in \mathbb{N}_*,\ \varphi_k' \text{ is of generic rank } \dim_{\mathbb{C}} \mathcal{M}' = 2n-1,\ \forall\, k \geq k_0. \end{aligned}$$

Remarks. **1.** It follows from the biholomorphic invariance of Segre varieties that two Segre morphisms of k-jets associated to two different local coordinates for (M', p') are intertwined by a local biholomorphic map of $\mathbb{C}^{n+N_{n-1,k}}$. Consequently, the properties of φ_k' are invariant.

2. The condition (I) is classical. The condition (II) is studied by Baouendi-Ebenfelt-Rothschild [2,3] and appeared already in Pinchuk's thesis, in Diederich-Webster [7] and in some of Han's works. The condition (III) appears in Diederich-Webster [7] and was studied by Baouendi-Jacobowitz-Treves and by Diederich-Fornaess. The condition (IV) seems to be new and appears in [10]. The condition (V) was discovered by Stanton in her concrete study of infinitesimal CR automorphisms of real analytic hypersurfaces (*see* [16] and the references therein) and is equivalent to the *nonexistence* of a holomorphic vector field with *holomorphic* coefficients tangent to (M', p'). We claim that $(I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (IV) \Rightarrow (V)$ (only the implication $(IV) \Rightarrow (V)$ is not straightforward, *see* Lemma 5.15 below for a proof). Finally, this progressive list of nondegeneracy conditions is the same, word by word, in higher codimension.

1.9. A general commentary. To confirm evidence of the strong differences between these five levels of nondegeneracy, let us point out some facts which are clear at an intuitive and informal level. The immersive or finite local holomorphic maps $\varphi\colon (X, p) \to (Y, q)$ between local pieces of complex manifolds with $\dim_{\mathbb{C}} X \leq \dim_{\mathbb{C}} Y$ are very rare (from the point of view of complexity) in the set of maps of generic rank equal to $\dim_{\mathbb{C}} X$, or even in the set of maps having maximal generic rank m over a submanifold $(Z, p) \subset (X, p)$ of positive dimension $m \geq 1$. Thus condition (V) is by far the most general. Furthermore, an important difference between (V) and the other conditions is that (V) *is the only condition which is nonlocal*, in the sense that it happens to be satisfied at every point if it is satisfied at a single point only, provided, of course, that the local piece (M', p') is connected. On the contrary, it is obvious that the other four conditions are really local: even though they happen to be satisfied at one point, there exist in general many other points where they fail to be satisfied. In this concern, let us recall that any (M',p') satisfying (V) must satisfy (II) locally – hence also (III) and (IV) – over a Zariski dense open subset of points of (M',p') (this important fact is proved in [3]). Therefore, the points satisfying (III) but not (II), or (IV) but not (III), or (V) but not (IV), can appear to be more and more exceptional and rare from the point of view of a point moving at random in (M',p'), but however, from the point of view of local analytic geometry, which is the adequate viewpoint in this matter, they are more and more generic and general, in truth.

Remark. An important feature of the theory of CR manifolds is to propagate the properties of CR functions and CR maps along Segre chains, when (M,p) is minimal, like iteration of jets [3], support of CR functions, *etc.* Based on this heuristic idea, and believing that the generic rank of the Segre morphism over a Segre variety is a propagating property, I have claimed in February 1999 (and provided a too quick invalid proof) that any real analytic (M',p') which is minimal at p' happens to be holomorphically nondegenerate if and only if it is Segre nondegenerate at p'. This is not true for a general (M',p') as is shown for instance by an example from [4]: we take in $\mathbb{C}^3$ equipped with affine coordinates (z_1',z_2',z_3')

$$(1.10)\quad M': \quad y_3' = |z_1'|^2|1+z_1'\bar{z}_2'|^2(1+\mathrm{Re}(z_1'\bar{z}_2'))^{-1} - x_3'\,\mathrm{Im}(z_1'\bar{z}_2')(1+\mathrm{Re}(z_1'\bar{z}_2'))^{-1}.$$

This algebraic hypersurface is holomorphically nondegenerate but is not Segre nondegenerate at the origin (use Lemmas 3.3 and 5.15 for a checking).

1.11. Summary of the proof. To the mapping h, we will associate the so-called invariant *reflection function* $\mathcal{R}_h'(t,\bar{\nu}')$ as a $\mathbb{C}$-valued map of $(t,\bar{\nu}') \in (\mathbb{C}^n,p)\times(\overline{\mathbb{C}}^n,\bar{p})$ which is a series *a priori* only formal in t and holomorphic in $\bar{\nu}'$ (the interest of studying the reflection function *without any nondegeneracy condition on* (M',p') has been pointed out for the first time by the author and Meylan in [11]). We prove in a first step that $\mathcal{R}_h'$ and all its jets with respect to t converge on the first Segre chain. Then using Artin's approximation theorem [1] (the interest of this theorem of Artin for the subject has been pointed out by Derridj in 1986, *Séminaire sur les équations aux dérivées partielles*, Exposé no. XVI, *Sur le prolongement d'applications holomorphes*, 10pp., *see* p.5) and holomorphic nondegeneracy of (M',p'), we establish that the formal CR map h converges on the second Segre chain. Finally, the minimality of (M,p) together with a theorem of Gabrielov reproved elementarily by Eakin and Harris [8] will both imply that h is convergent in a neighborhood of p. An important novelty is the use of the conjugate reflection identities (5.9) below.

1.12. Closing remark. Two months after a first preliminary version of this paper was finished (November 1999), distributed (January 2000) and then circulated as a preprint, the author received in March 2000 a preprint (now published) [12] where Theorem 1.2 and Theorem 9.1 below were proved, using in the first steps an induction on the convergence of the mapping and its jets along Segre sets which was devised by Baouendi-Ebenfelt-Rothschild in [2]. But the proof that we provide here differs from the one in [12] in the last step essentially. For our part, we introduce here in equations (5.9) and (8.5) a crucial object which we call *conjugate reflection identities.* Essentially, this means that *both equivalent equations* $r'(t',\tau')=0$ *and* $\bar{r}'(\tau',t')=0$ *for* (M',p') (*see* §2.7) *must be considered and differentiated.* More precisely, we mean that the CR derivations $\underline{\mathcal{L}}^{\beta}$ of §5.1 below must be applied to equation (5.2), *and* to the conjugate of equation (5.2), which yields equations (5.9). The author knows no previous paper where such an observation is done and exploited. With this crucial remark at hand, the generalizations of Theorems 1.2 and 9.1 to higher codimension can be performed completely, *see* the preprint *Étude de la convergence de l'application de symétrie CR formelle* (in french), `arXiv.org/abs/math.CV/0005290` May 2000 (translated with the same proof in July 2000). The first version of that preprint (`0005290v1`) contained some explicit hints in §18 for a second proof using conjugation of reflection identities and the last step of the proof given in [12]. The author believes that without the use of the conjugation relation between $r'(t',\tau')$ and $\bar{r}'(\tau',t')$, no elementary proof of Theorems 1.2 and 9.1 can be provided in higher codimension.

§2. Preliminaries and notations

2.1. Defining equations. *We shall never speak of a germ.* Thus, we shall assume constantly that we are given two small local real analytic manifold-pieces (M,p) and (M',p') of hypersurfaces in $\mathbb{C}^n$ with centered points $p\in M$ and $p'\in M'$. We first choose local holomorphic coordinates $t=(w,z)\in\mathbb{C}^{n-1}\times\mathbb{C}$, $z=x+iy$ and $t'=(w',z')\in\mathbb{C}^{n-1}\times\mathbb{C}$, $z'=x'+iy'$, vanishing at p and at p' such that the tangent spaces to M and to M' at 0 are given by $\{y=0\}$ and by $\{y'=0\}$ in these coordinates. By this choice, we carry out (*cf.* [3]) the equations of M and of M' in the form

$$(2.2)\qquad M\colon\quad z=\bar{z}+i\bar{\Theta}(w,\bar{w},\bar{z})\quad\text{and}\quad M'\colon\quad z'=\bar{z}'+i\bar{\Theta}'(w',\bar{w}',\bar{z}'),$$

where the power series $\bar{\Theta}$ and $\bar{\Theta}'$ converge normally in $(2r\Delta)^{2n-1}$ for some small $r>0$. We denote by $|t|:=\sup_{1\le i\le n}|t_i|$ the polydisc norm, so that $(2r\Delta)^{2n-1}=\{(w,\zeta,\xi)\colon |w|,|\zeta|,|\xi|<2r\}$. Here, if we denote by $\tau:=(\bar{t})^c:=(\zeta,\xi)$ the extrinsinc complexification of the variable $\bar{t}$, the equations of the complexified hypersurfaces $\mathcal{M}:=M^c$ and $\mathcal{M}':=(M')^c$ are simply obtained by complexifying the eqs. (2.2):

$$(2.3)\qquad \mathcal{M}\colon\quad z=\xi+i\bar{\Theta}(w,\zeta,\xi)\quad\text{and}\quad \mathcal{M}'\colon\quad z'=\xi'+i\bar{\Theta}'(w',\zeta',\xi').$$

As in [3], we shall assume for convenience that the coordinates (w,z) and (w',z') are *normal*, *i.e.* that they are already straightened in order that $\Theta(\zeta,0,z)\equiv 0$, $\Theta(0,w,z)\equiv 0$ and $\Theta'(\zeta',0,z')\equiv 0$, $\Theta'(0,w',z')\equiv 0$. This implies in particular that the Segre varieties $\mathcal{S}_0=\{(w,0)\colon |w|<2r\}$ and $\mathcal{S}_0'=\{(w',0)\colon |w'|<2r\}$ are straightened to the complex tangent plane to M at 0 and that, if we develop $\bar{\Theta}$ and $\bar{\Theta}'$ with respect to powers of w and w', then we can write

$$(2.4)\qquad z=\xi+i\sum_{\beta\in\mathbb{N}_*^{n-1}} w^\beta\,\bar{\Theta}_\beta(\zeta,\xi),\qquad z'=\xi'+i\sum_{\beta\in\mathbb{N}_*^{n-1}} {w'}^\beta\,\bar{\Theta}'_\beta(\zeta',\xi').$$

Here, we denote $\mathbb{N}_*^{n-1}:=\mathbb{N}^{n-1}\backslash\{0\}$. So we mean that the two above sums begin with a w and w' exponent of *positive length* $|\beta|=\beta_1+\cdots+\beta_{n-1}>0$. It is now natural to set for notational convenience $\bar{\Theta}_0(\zeta,\xi):=\xi$ and $\bar{\Theta}'_0(\zeta',\xi'):=\xi'$. Although normal coordinates are in principle unnecessary, the reduction to such normal coordinates will simplify a little the presentation of all our formal calculations below.

2.5. Complexification of the map. Now, the map h is by definition an n-vectorial formal power series $h(t)=(h_1(t),\dots,h_n(t))$ where $h_j(t)\in\mathbb{C}[\![t]\!]$, $h_j(0)=$

0 and det $(\partial h_j/\partial t_k(0))_{1\leq j,k\leq n} \neq 0$, which means that h is formally invertible. This map yields by extrinsic complexification a map $h^c = h^c(t,\tau) = (h(t), \bar{h}(\tau))$ between the two complexification $(\mathcal{M},0)$ and $(\mathcal{M}',0)$. In other words, if we denote $h = (g,f) \in \mathbb{C}^{n-1}\times\mathbb{C}$ in accordance with the splitting of coordinates in the target space, the assumption that $h^c(\mathcal{M}) \subset_{\mathcal{F}} \mathcal{M}'$ reads as two equivalent fundamental equations:

$$(2.6) \qquad \Updownarrow \begin{cases} f(w,z) = \left[\bar{f}(\zeta,\xi) + i\bar{\Theta}'(g(w,z),\bar{g}(\zeta,\xi),\bar{f}(\zeta,\xi))\right]_{\xi:=z-i\Theta(\zeta,w,z)}, \\ \bar{f}(\zeta,\xi) = \left[f(w,z) - i\Theta'(\bar{g}(\zeta,\xi),g(w,z),f(w,z))\right]_{z:=\xi+i\bar{\Theta}(w,\zeta,\xi)}, \end{cases}$$

after replacing ξ by $z - i\Theta(\zeta,w,z)$ in the first line and z by $\xi + i\bar{\Theta}(w,\zeta,\xi)$ in the second line. In fact, these (equivalent) identities must be interpreted as *formal identities* in the rings of *formal power series* $\mathbb{C}[\![\zeta,w,z]\!]$ and $\mathbb{C}[\![w,\zeta,\xi]\!]$ respectively. Of course, according to (2.3), we can equally choose the coordinates (ζ,w,z) or (w,ζ,ξ) over $\mathcal{M}$. In symbolic notation, we just write $h^c(\mathcal{M},0) \subset_{\mathcal{F}} (\mathcal{M}',0)$ to mean the identities (2.6).

2.7. Conjugate equations, vector fields and the reflection function. Let us also denote $r(t,\tau) := z - \xi - i\bar{\Theta}(w,\zeta,\xi)$, $\bar{r}(\tau,t) := \xi - z + i\Theta(\zeta,w,z)$ and similarly $r'(t',\tau') := z' - \xi' - i\bar{\Theta}'(w',\zeta',\xi')$, $\bar{r}'(\tau',t') := \xi' - z' + i\Theta'(\zeta',w',z')$, so that $\mathcal{M} = \{(t,\tau)\colon r(t,\tau) = 0\}$, $\mathcal{M}' = \{(t',\tau')\colon r'(t',\tau') = 0\}$ and the complexified Segre varieties are given by $\mathcal{S}_{\tau_p} = \{(t,\tau_p)\colon r(t,\tau_p) = 0\} \subset \mathcal{M}$ for fixed τ_p, and $\underline{\mathcal{S}}_{t_p} = \{(t_p,\tau)\colon r(t_p,\tau) = 0\} \subset \mathcal{M}$ for fixed t_p and similarly for $\mathcal{S}'_{\tau'_{p'}}$, $\underline{\mathcal{S}}'_{t'_{p'}}$ (again, the reader is referred to [9] for a complete exposition of the geometry of complexified Segre varieties). Finally, let us introduce the $(n-1)$ complexified (1,0) and (0,1) CR vector fields tangent to $\mathcal{M}$, that we will denote by $\mathcal{L} = (\mathcal{L}_1,\dots,\mathcal{L}_{n-1})$ and $\underline{\mathcal{L}} = (\underline{\mathcal{L}}_1,\dots,\underline{\mathcal{L}}_{n-1})$, and which can be given in symbolic vectorial notation by

$$(2.8) \qquad \mathcal{L} = \frac{\partial}{\partial w} + i\bar{\Theta}_w(w,\zeta,\xi)\frac{\partial}{\partial z} \qquad \text{and} \qquad \underline{\mathcal{L}} = \frac{\partial}{\partial \zeta} - i\Theta_\zeta(\zeta,w,z)\frac{\partial}{\partial \xi}.$$

The *reflection function* $\mathcal{R}'_h(t,\bar{\nu}')$, $t \in \mathbb{C}^n$, $\bar{\nu}' = (\bar{\lambda}',\bar{\mu}') \in \mathbb{C}^{n-1}\times\mathbb{C}$, will be by definition the formal power series

$$(2.9) \quad \mathcal{R}'_h(t,\bar{\nu}') = \mathcal{R}'_h(w,z,\bar{\lambda}',\bar{\mu}') = \bar{\mu}' - f(w,z) + i\sum_{\beta\in\mathbb{N}^{n-1}_*} \bar{\lambda}'^{\beta}\, \Theta'_\beta(g(w,z),f(w,z)).$$

We notice that this power series in fact belongs to the local "hybrid" ring $\mathbb{C}\{\bar{\nu}'\}[\![t]\!]$.

§3. Minimality and holomorphic nondegeneracy

3.1. Two characterizations. At first, we need to remind the two explicit characterizations of each one of the main two assumptions of Theorem 1.2. Let M be a real analytic CR hypersurface given in *normal coordinates* (w,z) as above in eq. (2.2).

Lemma 3.2. ([3]) *The following properties are equivalent:*

(1) $\bar{\Theta}(w,\zeta,0) \not\equiv 0$.
(2) $\frac{\partial\bar{\Theta}}{\partial\zeta}(w,\zeta,0) \not\equiv 0$.
(3) *M is minimal at* 0.
(4) *The Segre variety S_0 is not contained in M.*
(5) *The holomorphic map $\mathbb{C}^{2n-2} \ni (w,\zeta) \mapsto (w, i\bar{\Theta}(w,\zeta,0)) \in \mathbb{C}^n$ has generic rank n.*

Lemma 3.3. ([2,3,16]) *If the coordinates (w', z') are normal as above, then the real analytic hypersurface M' is holomorphically nondegenerate at 0 if and only if there exist $\beta^1, \dots, \beta^{n-1} \in \mathbb{N}_*^{n-1}$, $\beta^n := 0$, such that*

$$\text{(3.4)} \qquad det\left(\frac{\partial \Theta'_{\beta^i}}{\partial t'_j}(w', z')\right)_{1\leq i,j\leq n} \not\equiv 0 \qquad in\ \mathbb{C}\{w', z'\}.$$

Remark. Since for $\beta = 0$, we have $\Theta'_\beta(t') = \Theta'_0(t') = z'$, we see that (3.4) holds if and only if det $\left(\frac{\partial \Theta'_{\beta^i}}{\partial w'_j}(w', z')\right)_{1\leq i,j\leq n-1} \not\equiv 0$. Further, we can precise the other classical nondegeneracy conditions (I), (II) and (III) of §1 (for condition (IV), *see* Lemma 5.15):

Lemma 3.5. *The following concrete characterizations hold in normal coordinates:*

(1) *M' is Levi nondegenerate at 0 if and only if the map $w' \mapsto (\Theta_\beta(w', 0))_{|\beta|=1}$ is immersive at 0.*

(2) *M' is finitely nondegenerate at 0 if and only if there exists $k_0 \in \mathbb{N}_*$ such that the map $\mathbb{C}^{n-1} \ni w' \mapsto (\Theta'_\beta(w', 0))_{1\leq|\beta|\leq k_0}$ is immersive at 0 for all $k \geq k_0$.*

(3) *M' is essentially finite at 0 if and only if there exists $k_0 \in \mathbb{N}_*$ such that the map $\mathbb{C}^{n-1} \ni w' \mapsto (\Theta'_\beta(w', 0))_{1\leq|\beta|\leq k_0}$ is finite at 0 for all $k \geq k_0$.*

3.6. Switch of the assumptions. It is now easy to observe that the nondegeneracy conditions upon M transfer to M' through h and vice versa.

Lemma 3.7. *Let $h\colon (M, 0) \to_{\mathcal{F}} (M', 0)$ be a formal invertible CR map between two real analytic hypersurfaces. Then*

1) *$(M, 0)$ is minimal if and only if $(M', 0)$ is minimal.*

2) *$(M, 0)$ is holomorphically nondegenerate if and only if $(M', 0)$ is holomorphically nondegenerate.*

Proof. We admit and use in the proof that minimality and holomorphic nondegeneracy are biholomorphically invariant properties. Let $N \in \mathbb{N}_*$ be arbitrary. Since h is invertible, after composing h with a biholomorphic and polynomial mapping $\Phi\colon (M', 0) \to (M'', 0)$ which cancels low order terms in the Taylor series of h at the origin, we can achieve that $h(t) = t + O(|t|^N)$. Since the coordinates for $(M'', 0)$ may be nonnormal, we must compose $\Phi \circ h$ with a biholomorphism $\Psi\colon (M'', 0) \to (M''', 0)$ which straightens the real analytic Levi-flat union of Segre varieties $\bigcup_{|x|\leq r} S''_{\overline{h(0,x)}}$ into the real hyperplane $\{y''' = 0\}$ (this is how one constructs normal coordinates). One can verify that $\Psi(t) = t + O(|t|^N)$ also. Then all terms of degre $\leq N$ in the power series of Θ''' coincide with those of Θ. Each one of the two characterizing properties **(1)** of Lemma 3.2 and (3.4) of Lemma 3.3 is therefore satisfied by Θ if and only if it is satisfied by Θ'''. $\square$

§4. Formal versus analytic

4.1. Approximation theorem. We collect here some useful statements from local analytic geometry that we will repeatedly apply in the article. One of the essential arguments in the proof of the main Theorem 2.1 rests on the existence of analytic solutions arbitrarily close in the Krull topology to formal solutions of some analytic equations, a fact which is known as *Artin's approximation theorem*. Let $\mathfrak{m}(w)$ denote the maximal ideal of the local ring $\mathbb{C}[\![w]\!]$ of formal power series in $w \in \mathbb{C}^n$, $n \in \mathbb{N}_*$. Here is the first of our three fundamental tools, which will be used to get the Cauchy estimates which show that the reflection function converges on the first Segre chain (*see* Lemma 6.6).

Theorem 4.2. (Artin [1]) *Let $R(w, y) = 0$, $R = (R_1, \dots, R_J)$, where $w \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, $R_j \in \mathbb{C}\{w, y\}$, $R_j(0) = 0$, be a converging system of* holomorphic *equations. Suppose $\hat{g}(w) = (\hat{g}_1(w), \dots, \hat{g}_m(w))$, $\hat{g}_k(w) \in \mathbb{C}[\![w]\!]$, $\hat{g}_k(0) = 0$, are* formal *power series which solve $R(w, \hat{g}(w)) \equiv 0$ in $\mathbb{C}[\![w]\!]$. Then for every integer $N \in \mathbb{N}_*$, there exists a* convergent series solution *$g(w) = (g_1(w), \dots, g_m(w))$, i.e. satisfying $R(w, g(w)) \equiv 0$, such that $g(w) \equiv \hat{g}(w)$* (mod $\mathfrak{m}(w)^N$).

4.3. Formal implies convergent : first recipe. The second tool will be used to prove that h is convergent on the second Segre chain, *i.e.* that $h(w, i\bar{\Theta}(\zeta, w, 0)) \in \mathbb{C}\{w, \zeta\}$ (*see* §8).

Theorem 4.4. *Let $R(w, y) = 0$, where $R = (R_1, \dots, R_J)$, $w \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, $R_j \in \mathbb{C}\{w, y\}$, $R_j(0) = 0$, be a system of* holomorphic *equations. Suppose that $\hat{g}(w) = (\hat{g}_1(w), \dots, \hat{g}_m(w)) \in \mathbb{C}[\![w]\!]^m$, $\hat{g}_k(0) = 0$ are* formal *power series solving $R(w, \hat{g}(w)) \equiv 0$ in $\mathbb{C}[\![w]\!]$. If $J \geq m$ and if there exist $j_1, \dots, j_m$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq J$ such that*

$$\det \left(\frac{\partial R_{j_k}}{\partial y_l}(w, \hat{g}(w)) \right)_{1 \leq k, l \leq m} \not\equiv 0 \quad \text{in} \quad \mathbb{C}[\![w]\!], \tag{4.5}$$

then the formal power series $\hat{g}(w) \in \mathbb{C}\{w\}$ is in fact already convergent.

Remark. This theorem is a direct corollary of Artin's Theorem 4.2. The reader can find an elementary proof of it for instance in §12 of [10].

4.6. Formal implies convergent : second recipe. The third statement will be applied to the canonical map of the second Segre chain, namely to the map $(w, \zeta) \mapsto (w, i\bar{\Theta}(\zeta, w, 0))$, which is of generic rank n by Lemma 3.2 **(5)**.

Theorem 4.7. ([8]) *Let $a(y) \in \mathbb{C}[\![y]\!]$, $y \in \mathbb{C}^\mu$, $a(0) = 0$, be a formal power series and assume that there exists a local holomorphic map $\varphi\colon (\mathbb{C}_x^\nu, 0) \to (\mathbb{C}_y^\mu, 0)$, of maximal generic rank μ, i.e. satisfying*

$$\exists\ j_1, \dots, j_\mu,\ 1 \leq j_1 < \cdots < j_\mu \leq \nu, \quad s.t. \quad det \left(\frac{\partial \varphi_k}{\partial x_{j_l}}(x) \right)_{1 \leq k, l \leq \mu} \not\equiv 0, \tag{4.8}$$

and such that $a(\varphi(x)) \in \mathbb{C}\{x\}$ is convergent. Then $a(y) \in \mathbb{C}\{y\}$ is convergent.

4.9. Application. We can now give an important application of Theorem 4.2 : the Cauchy estimates for the convergence of the reflection function come for free after one knows that all the formal power series $\Theta'_\beta(h(w, z)) \in \mathbb{C}[\![w, z]\!]$ are convergent.

Lemma 4.10. *Assume that $h\colon (M, 0) \to_{\mathcal{F}} (M', 0)$ is a formal invertible CR mapping and that M' is holomorphically nondegenerate. Then the following properties are equivalent :*

(1) $h(w, z) \in \mathbb{C}\{w, z\}^n$.
(2) $\mathcal{R}'_h(w, z, \bar{\lambda}, \bar{\mu}) \in \mathbb{C}\{w, z, \bar{\lambda}, \bar{\mu}\}$.
(3) $\Theta'_\beta(h(w, z)) \in \mathbb{C}\{w, z\}$, $\forall\ \beta \in \mathbb{N}^{n-1}$ *and* $\exists\ \varepsilon > 0\ \exists\ C > 0$ *such that* $|\Theta'_\beta(h(w, z))| \leq C^{|\beta|+1}$, *for all (w, z) with $|(w, z)| < \varepsilon$ and all $\beta \in \mathbb{N}^{n-1}$.*
(4) $\Theta'_\beta(h(w, z)) \in \mathbb{C}\{w, z\}$, $\forall\ \beta \in \mathbb{N}^{n-1}$.

Proof. The implications **(1)** $\Rightarrow$ **(2)** $\Rightarrow$ **(3)** $\Rightarrow$ **(4)** are straightforward. On the other hand, consider the implication **(4)** $\Rightarrow$ **(1)**. By assumption, there exist convergent power series $\varphi'_\beta(w,z) \in \mathbb{C}\{w,z\}$ such that $\Theta'_\beta(h(w,z)) \equiv \varphi'_\beta(w,z)$ in $\mathbb{C}[\![w,z]\!]$. It then follows that $h(t)$ is convergent by an application of Theorem 4.4 with $R_n(t,t') := z' - \varphi_0(t)$ and $R_i(t,t') := \Theta'_{\beta^i}(t') - \varphi'_{\beta^i}(t)$, $1 \le i \le n-1$ and where the multiindices $\beta^1, \dots, \beta^{n-1}$ are chosen as in Lemma 3.3 (use the property $\det(\frac{\partial h_j}{\partial t_k}(0))_{1\le j,k\le n} \neq 0$ and the composition formula for Jacobian matrices to check that (4.5) holds). $\square$

Proof of Proposition 1.5. Let $\varphi' \colon (t',u') \mapsto \exp(u'L')(t') = \varphi'(t',u')$ be the local flow of the holomorphic vector field $L' = \sum_{k=1}^n a'_k(t')\partial/\partial t'_k$ tangent to M'. Of course, this flow is holomorphic with respect to $t' \in \mathbb{C}^n$ and $u' \in \mathbb{C}$, for $|t'|$, $|u'| \le \varepsilon$, $\varepsilon > 0$. This flow satisfies $\varphi'(t',0) \equiv t'$ and $\partial_{u'}\varphi'_k(t',u') \equiv a'_k(\varphi'(t',u'))$. As $L' \neq 0$, we have $\partial_{u'}\varphi'(t',u') \not\equiv 0$. We can assume that $\partial_{u'}\varphi'_1(t',u') \not\equiv 0$. Let $\varpi'(t') \in \mathbb{C}[\![t']\!]\backslash\mathbb{C}\{t'\}$, $\varpi'(0) = 0$, be a *nonconvergent* formal power series which satisfies further $\partial_{u'}\varphi'_1(t',\varpi'(t')) \not\equiv 0$ in $\mathbb{C}[\![t']\!]$ (there exist many of such). If the formal power series $h^\sharp \colon t' \mapsto_{\mathcal{F}} \varphi'(t',\varpi'(t'))$ would be convergent, then $t' \mapsto_{\mathcal{F}} \varpi'(t')$ would also be convergent, because of Theorem 4.4, contrarily to the choice of ϖ'. Finally, L' being tangent to $(M',0)$, it is clear that $h^\sharp(M',0) \subset_{\mathcal{F}} (M',0)$. $\square$

§5. Classical reflection identities

5.1. The fundamental identities. In this paragraph, we start the proof of our main Theorem 1.2 by deriving the classical reflection identities. Thus let $\beta \in \mathbb{N}_*^{n-1}$. By $\gamma \le \beta$, we shall mean $\gamma_1 \le \beta_1, \dots, \gamma_{n-1} \le \beta_{n-1}$. Denote $|\beta| := \beta_1 + \cdots + \beta_{n-1}$ and $\underline{\mathcal{L}}^\beta := \underline{\mathcal{L}}_1^{\beta_1} \cdots \underline{\mathcal{L}}_{n-1}^{\beta_{n-1}}$. Then applying all these derivations of any order (*i.e.* for each $\beta \in \mathbb{N}^{n-1}$) to the identity $\bar{r}'(\bar{h}(\tau), h(t))$, *i.e.* to

$$\bar{f}(\zeta,\xi) \equiv f(w,z) - i \sum_{\gamma\in\mathbb{N}_*^{n-1}} \bar{g}(\zeta,\xi)^\gamma\ \Theta'_\gamma(g(w,z),f(w,z)), \tag{5.2}$$

as $(w,z,\zeta,\xi) \in \mathcal{M}$, it is well-known that we obtain an infinite family of formal identities that we recollect here in an independent technical statement (for the proof, *see* [3,10]).

Lemma 5.3. *Let $h : (M,0) \to_{\mathcal{F}} (M',0)$ be a formal invertible CR mapping between $\mathcal{C}^\omega$ hypersurfaces in $\mathbb{C}^n$. Then for every $\beta \in \mathbb{N}_*^{n-1}$, there exists a collection of universal polynomial $\underline{u}_{\beta,\gamma}$, $|\gamma| \le |\beta|$ in $(n-1)N_{n-1,|\beta|}$ variables, where $N_{k,l} := \frac{(k+l)!}{k!\,l!}$ and there exist holomorphic $\mathbb{C}$-valued functions $\underline{\Omega}_\beta$ in $(2n-1+nN_{n,|\beta|})$ variables near $0\times 0\times 0\times (\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(0))_{|\alpha^1|+|\gamma^1|\le|\beta|}$ in $\mathbb{C}^{n-1}\times\mathbb{C}^{n-1}\times\mathbb{C}\times\mathbb{C}^{nN_{n,|\beta|}}$ such that the following identities*

$$\tag{5.4} \left\{ \begin{aligned} &\frac{1}{\beta!}\partial_{\zeta'}^{\beta}\Theta'(\bar{g}(\zeta,\xi), g(w,z), f(w,z)) = \\ &= \Theta'_\beta(g(w,z),f(w,z)) + \sum_{\gamma\in\mathbb{N}_*^{n-1}} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\bar{g}(\zeta,\xi)^\gamma\ \Theta'_{\beta+\gamma}(g(w,z),f(w,z)) \\ &\equiv \sum_{|\gamma|\le|\beta|} \frac{\underline{\mathcal{L}}^\gamma \bar{f}(\zeta,\xi)\ \underline{u}_{\beta,\gamma}((\underline{\mathcal{L}}^\delta \bar{g}(\zeta,\xi))_{|\delta|\le|\beta|})}{\underline{\Delta}(w,\zeta,\xi)^{2|\beta|-1}} \\ &=: \underline{\Omega}_\beta(w,\zeta,\xi,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(\zeta,\xi))_{|\alpha^1|+|\gamma^1|\le|\beta|}) \\ &=: \underline{\omega}_\beta(w,\zeta,\xi), \end{aligned} \right.$$

hold as formal power series in $\mathbb{C}[\![w,\zeta,\xi]\!]$, *where*

$$\left\{\begin{aligned}&\underline{\Delta}(w,\zeta,\xi)=\underline{\Delta}(w,z,\zeta,\xi)|_{z=\xi+i\bar{\Theta}(w,\zeta,\xi)}:=\det(\underline{\mathcal{L}}\bar{g})=\\&=\det\left(\frac{\partial\bar{g}}{\partial\zeta}(\zeta,\xi)-i\Theta_\zeta(\zeta,w,z)\frac{\partial\bar{g}}{\partial\xi}(\zeta,\xi)\right)|_{z=\xi+i\bar{\Theta}(w,\zeta,\xi)}.\end{aligned}\right. \tag{5.5}$$

Remark. The terms $\underline{\Omega}_\beta$, holomorphic in their variables, arise after writing $\underline{\mathcal{L}}^\delta\bar{h}(\zeta,\xi)$ as $\chi_\delta(w,z,\zeta,\xi,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(\zeta,\xi))_{|\alpha^1|+|\gamma^1|\leq|\delta|})$ (by noticing that the coefficients of $\underline{\mathcal{L}}$ are analytic in (w,z,ζ,ξ)) and by replacing again z by $\xi+i\bar{\Theta}(w,\zeta,\xi)$).

5.6. Convergence over a uniform domain. From this lemma which we have written down in the most explicit way, we deduce the following useful observations. First, as we have by the formal stabilization of Segre varieties $h(\{w=0\})\subset_{\mathcal{F}}\{w'=0\}$ and as h is invertible, then it holds $\det(\underline{\mathcal{L}}\bar{g}(0))=\det(\partial g_j/\partial w_k(0))_{1\leq j,k\leq n-1}\neq 0$ also, whence the rational term $1/\underline{\Delta}^{2|\beta|-1}\in\mathbb{C}[\![w,\zeta,\xi]\!]$ defines a true formal power series at the origin. Putting now $(\zeta,\xi)=(0,0)$ in eqs. (5.5) and shrinking r if necessary, we then readily observe that $\underline{\Delta}^{1-2|\beta|}(w,0,0)\in\mathcal{O}((r\Delta)^{n-1},\mathbb{C})$, since $\Theta_\zeta(0,w,0)\in\mathcal{O}((r\Delta)^{n-1},\mathbb{C})$ and since the terms $\partial_\zeta^{\gamma^1}\bar{g}(0,0)$ for $|\gamma^1|=1$ and $\partial_\xi^1\bar{g}(0,0)$ are *constants*. Clearly, the numerator in the middle identity (5.4) is also convergent in $(r\Delta)^{n-1}$ after putting $(\zeta,\xi)=(0,0)$, and we deduce finally the following important property:

$$\underline{\Omega}_\beta(w,0,0,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(0,0))_{|\alpha^1|+|\gamma^1|\leq|\beta|})\in\mathcal{O}((r\Delta)^{n-1},\mathbb{C}), \tag{5.7}$$

for all $\beta\in\mathbb{N}_*^{n-1}$. In other words, *the domains of convergence of the* $\underline{\omega}_\beta(w,0,0)$ *are independent of* β.

5.8. Conjugate reflection identities. On the other hand, applying the same derivations $\underline{\mathcal{L}}^\beta$'s to the conjugate identity $r'(h(t),\bar{h}(\tau))=0$, we would get another family of what we shall call *conjugate reflection identities*:

$$0\equiv\underline{\mathcal{L}}^\beta\bar{f}(\zeta,\xi)+i\sum_{\gamma\in\mathbb{N}_*^{n-1}}g(w,z)^\gamma\;\underline{\mathcal{L}}^\beta(\bar{\Theta}'_\gamma(\bar{g}(\zeta,\xi),\bar{f}(\zeta,\xi))). \tag{5.9}$$

But these equations furnish essentially no more information for the reflection principle, because:

Lemma 5.10. *If* $(t,\tau)\in\mathcal{M}$, *then*

$$\left\langle\underline{\mathcal{L}}^\beta(r'(h(t),\bar{h}(\tau)))=0,\ \forall\beta\in\mathbb{N}^{n-1}\right\rangle\iff\left\langle\underline{\mathcal{L}}^\beta(\bar{r}'(\bar{h}(\tau),h(t)))=0,\ \forall\beta\in\mathbb{N}^{n-1}\right\rangle. \tag{5.11}$$

Proof. As the two equations for $\mathcal{M}'$ are equivalent, there exists an invertible formal series $\alpha(t,\tau)$ such that $r'(h(t),\bar{h}(\tau))\equiv\alpha(t,\tau)\,\bar{r}'(\bar{h}(\tau),h(t))$. Thus

$$\left\{\begin{aligned}\underline{\mathcal{L}}^\beta(r'(h(t),\bar{h}(\tau)))\;&\equiv\alpha(t,\tau)\,\underline{\mathcal{L}}^\beta(\bar{r}'(\bar{h}(\tau),h(t)))+\\&\quad+\sum_{\gamma\leq\beta,\gamma\neq\beta}\alpha_\gamma^\beta(t,\tau)\,\underline{\mathcal{L}}^\gamma(\bar{r}'(\bar{h}(\tau),h(t))),\end{aligned}\right. \tag{5.12}$$

for some formal series $\alpha_\gamma^\beta(t,\tau)$ depending on the derivatives of $\alpha(t,\tau)$. The implication "$\Leftarrow$" follows at once and the reverse implication is totally similar. □

5.13. Heuristics. Nevertheless, in the last step of the proof of our main Theorem 1.2, the equations (5.9) above will be of crucial use, in place of the equations (5.4) which will happen to be unusable. The explanation is the following. Whereas the jets $(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(\zeta,\xi))_{|\alpha^1|+|\gamma^1|\leq|\beta|}$ of the mapping $\bar{h}$ cannot be seen directly to be convergent on the first Segre chain $\mathcal{S}_0^1 := \{(w,0)\}$, a convergence which would be a necessary fact to be able to use formula (5.4) again in order to pass from the first to the second Segre chain $\mathcal{S}_2^0 := \{(w, i\bar{\Theta}(w,\zeta,0))\}$ it will be possible – fortunately! – to show in §7 below that *the jets of the reflection function* $\mathcal{R}'_h$ *itself converge on the first Segre chain*, namely that all the derivatives $\underline{\mathcal{L}}^\beta(\bar{\Theta}'_\gamma(\bar{g}(\zeta,\xi),\bar{f}(\zeta,\xi)))$, restricted to the conjugate first Segre chain $\underline{\mathcal{S}}_0^1 = \{(\zeta,0)\}$, converge. In summary, we will only be able *a priori* to show that the jets of $\mathcal{R}'_h$ converge on the first Segre chain, and thus only the equations (5.9) will be usable in the next step, but not the classical reflection identities (5.4). *This shows immediately why the conjugate reflection identities* (5.9) *should be undertaken naturally in this context.*

5.14. The Segre-nondegenerate case. Nonetheless, in the Segre-nondegenerate case, which is less general than the holomorphically nondegenerate case, we have been able to show directly that the jets of h converge on the first Segre chain (*see* [10]), and so on by induction, without using conjugate reflection identities. The explanation is simple: in the Segre nondegenerate case, we have first the following characterization, which shows that we can *separate* the w' variables from the z' variable:

Lemma 5.15. *The $\mathcal{C}^\omega$ hypersurface M', given in normal coordinates (w',z'), is Segre-nondegenerate at 0 if and only if there exist $\beta^1,\ldots,\beta^{n-1}\in\mathbb{N}_*^{n-1}$ such that*

$$\text{(5.16)}\qquad det\left(\frac{\partial\Theta'_{\beta^i}}{\partial w'_j}(w',0)\right)_{1\leq i,j\leq n-1}\not\equiv 0 \qquad in\ \mathbb{C}\{w'\}.$$

Also, M' is holomorphically nondegenerate at 0 if it is Segre nondegenerate at 0.

Proof. In our normal coordinates, it follows that $\mathcal{S}_{p'} = \mathcal{S}'_0 = \{(w',0,0,0)\}$ and $\varphi'_k|_{\mathcal{S}'_0} \cong w' \mapsto (\{\Theta'_\beta(w',0)\}_{|\beta|\leq k})$, whence the rephrasing (5.16) of definition (IV). As we can take $\beta^n = 0$ in (3.4) above, we see that the determinant of (3.4) does not vanish if (5.16) holds. This proves the promised implication $(IV)\Rightarrow(V)$. □

Thanks to this characterization, we can delineate an analog to Lemma 4.10, whose proof goes exactly the same way:

Lemma 5.17. *Assume that h is invertible, that M is given in normal coordinates 2.1 and that M' is Segre nondegenerate. Then the following properties are equivalent*

- **(1)** $h(w,0)\in\mathbb{C}\{w\}$.
- **(2)** $\mathcal{R}'_h(w,0,\bar{\lambda},\bar{\mu})\in\mathbb{C}\{w,\bar{\lambda},\bar{\mu}\}$.
- **(3)** $\Theta'_\beta(h(w,0))\in\mathbb{C}\{w\}$, $\forall\ \beta\in\mathbb{N}^{n-1}$ *and* $\exists\ \varepsilon>0\ \exists\ C>0$ *such that* $|\Theta'_\beta(h(w,0))|\leq C^{|\beta|+1}$, $\forall\ |w|<\varepsilon\ \ \forall\ \beta\in\mathbb{N}^{n-1}$.
- **(4)** $\Theta'_\beta(h(w,0))\in\mathbb{C}\{w\}$, $\forall\ \beta\in\mathbb{N}^{n-1}$.

5.18. Comment. In conclusion, in the Segre nondegenerate case (only) the convergence of all the components $\Theta'_\beta(h)$ of the reflection mapping *after restriction to the Segre variety* $\mathcal{S}_0=\{(w,0)\}$ is *equivalent to the convergence of all the components* *of* h. The same property holds for jets. Thus, in the Segre nondegenerate case, one can use the classical reflection identities (5.4) (in which appear the jets of $\bar{h}$, see $\underline{\Omega}_\beta$) by induction on the Segre chains [10]. This is not so in the general holomorphically nondegenerate case, because it can happen that (3.4) holds whereas (5.16) does not hold, as shows the example (1.10). In substance, one has therefore to use the *conjugate reflection identities*. Now, the proof of our main Theorem 1.2 will be subdivided in three steps, which will be achieved in §6, §7 and §8 below.

§6. Convergence of the reflection function on $\mathcal{S}_0^1$

6.1. Examination of the reflection identities. The purpose of this paragraph is to prove as a first step that the reflection function $\mathcal{R}'_h$ converges on the first Segre chain $\mathcal{S}_0 = \{(w, 0)\}$ or more precisely :

Lemma 6.2. *After perhaps shrinking the radius* $r > 0$ *of* (5.7)*, the formal power series* $\mathcal{R}'_h(w, 0, \bar{\lambda}', \bar{\mu}')$ *is holomorphic in* $(r\Delta)^{n-1} \times \{0\} \times (r\Delta)^n$.

Proof. We specify the infinite family of identities (5.2) (for $\beta = 0$) and (5.4) (for $\beta \in \mathbb{N}_*^{n-1}$) on $\mathcal{S}_0$, to obtain first that $f(w, 0) \equiv 0 \in \mathbb{C}\{w\}$ and that for all $\beta \in \mathbb{N}_*^{n-1}$

$$(6.3)\quad \Theta'_\beta(g(w,0), f(w,0)) \equiv \underline{\Omega}_\beta(w, 0, 0, (\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(0,0))_{|\alpha^1|+|\gamma^1|\leq|\beta|}) \quad \in \ \mathbb{C}\{w\}.$$

Furthermore, since by (5.7) the $\underline{\Omega}_\beta$'s converge for $|w| < r$ and $\zeta = \xi = 0$, we have got $\Theta'_\beta(g(w,0), f(w,0)) \in \mathcal{O}((r\Delta)^{n-1}, \mathbb{C})$, $\forall\ \beta \in \mathbb{N}^{n-1}$. It remains to establish a Cauchy estimate like in **(3)** of Lemma 5.17. To this aim, we introduce some notation. We set $\varphi'_0(w, z) := f(w, z)$ and $\varphi'_\beta(w, z) := \Theta'_\beta(g(w, z), f(w, z))$ for all $\beta \in \mathbb{N}_*^{n-1}$. By (6.3), we already know that all the series $\varphi'_\beta(w, 0)$ are holomorphic in $\{|w| < r\}$. Thus, in order to prove that the reflection function restricted to the first Segre chain, namely that the series $= \mathcal{R}'_h|_{\mathcal{S}_0} = \bar{\mu}' + i \sum_{\beta\in\mathbb{N}_*^{n-1}} \bar{\lambda}'^\beta\ \varphi'_\beta(w, 0)$ is convergent with respect to all its variables, we must establish a crucial assertion.

Lemma 6.4. *After perhaps shrinking* $r > 0$*, there exists a constant* $C > 0$ *with*

$$(6.5)\qquad |\varphi'_\beta(w, 0)| \leq C^{|\beta|+1}, \quad \forall\ |w| < r,\ \forall\ \beta \in \mathbb{N}^{n-1}.$$

Proof. Actually, this Cauchy estimate will follow, by construction, from eq. (6.3) and from the property $|\Theta'_\beta(w', z')| \leq C'^{|\beta|+1}$ when (w', z') satisfy $|(w', z')| < r'$ (the natural Cauchy estimate for Θ'), once we have proved the following independent and important proposition, which is a rather direct application Artin's approximation Theorem 4.2 : □

Lemma 6.6. *Let* $w \in \mathbb{C}^\mu$*,* $\mu \in \mathbb{N}_*$*,* $\lambda(w) \in \mathbb{C}[\![w]\!]^\nu$*,* $\lambda(0) = 0$*,* $\nu \in \mathbb{N}_*$*, and let* $\Xi_\beta(w, \lambda) \in \mathbb{C}\{w, \lambda\}$*,* $\Xi_\beta(0, 0) = 0$*,* $\beta \in \mathbb{N}^m$*,* $m \in \mathbb{N}_*$*, be a collection of holomorphic functions satisfying*

$$(6.7)\quad \exists\ r > 0\ \ \exists\ C > 0\ \ s.t. \quad |\Xi_\beta(w, \lambda)| \leq C^{|\beta|+1}, \quad \forall\ \beta \in \mathbb{N}^m,\ \forall\ |(w, \lambda)| < r.$$

Assume that $\Xi_\beta(w, \lambda(w)) \in \mathcal{O}((r\Delta)^\mu, \mathbb{C})$, $\forall\ \beta \in \mathbb{N}^m$ *and put* $\Phi_\beta(w) := \Xi_\beta(w, \lambda(w))$. *Then the following Cauchy inequalities are satisfied by the* Φ_β*'s :*

$$(6.8)\quad \exists\ 0 < r_1 \leq r,\ \exists\ C_1 > 0\ \ s.t. \quad |\Phi_\beta(w)| \leq C_1^{|\beta|+1},\ \forall\ \beta \in \mathbb{N}^m,\ \forall\ |w| < r_1.$$

Proof. We set $R_\beta(w,\lambda) := \Xi_\beta(w,\lambda) - \Phi_\beta(w)$. Then $R_\beta \in \mathcal{O}(\{|(w,\lambda)| < r\}, \mathbb{C})$. By noetherianity, we can assume that a finite subfamily $(R_\beta)_{|\beta|\leq\kappa_0}$ generates the ideal $(R_\beta)_{\beta\in\mathbb{N}^m}$, for some $\kappa_0 \in \mathbb{N}_*$ large enough. Applying now Theorem 4.2 to the collection of equations $R_\beta(w,\lambda) = 0$, $|\beta| \leq \kappa_0$, of which a formal solution $\lambda(w)$ exists by assumption, we get that there exists a convergent solution $\lambda_1(w) \in \mathbb{C}\{w\}^\nu$ vanishing at the origin, *i.e.* some $\lambda_1(w) \in \mathcal{O}((r_1\Delta)^\mu, \mathbb{C}^\nu)$, for some $0 < r_1 \leq r$, with $\lambda_1(0) = 0$, which satisfies $R_\beta(w,\lambda_1(w)) \equiv 0$, $\forall\ |\beta| \leq \kappa_0$. This implies that $R_\beta(w,\lambda_1(w)) \equiv 0$, $\forall\ \beta \in \mathbb{N}^m$. Now, we have obtained

$$\Xi_\beta(w,\lambda(w)) \equiv \Phi_\beta(w) \equiv \Xi_\beta(w,\lambda_1(w)), \qquad \forall\ \beta \in \mathbb{N}^m. \tag{6.9}$$

The composition formula for analytic function then yields at once $|\Xi_\beta(w,\lambda_1(w))| \leq C_1^{|\beta|+1}$ for $|w| < r_1$, after perhaps shrinking once more this positive number r_1 in order that $|\lambda_1(w)| < r/2$ if $|w| < r_1$. Thanks to eq. (6.9), this gives the desired inequality for $\Phi_\beta(w)$. The Proof of Lemmas 6.7 and 6.2 are thus complete now. □

§7. Convergence of the jets of the reflection function on $\mathcal{S}_0^1$

7.1. Transversal differentiation of the reflection identities. The next step in our proof consists in showing that all the jets of the reflection function converge on the first Segre chain $\mathcal{S}_0$, or more precisely:

Lemma 7.2. *For all $\alpha \in \mathbb{N}$ and all $\gamma \in \mathbb{N}^{n-1}$, we have*

$$\left[\partial_z^\alpha \partial_w^\gamma \mathcal{R}'_h(w,z,\bar\lambda,\bar\mu)\right]|_{z=0} \quad \in \quad \mathbb{C}\{w,\bar\lambda,\bar\mu\}. \tag{7.3}$$

Equivalently, $\forall\ \alpha \in \mathbb{N}$, $\forall\ \gamma \in \mathbb{N}^{n-1}$, $\exists\ r(\alpha,\gamma) > 0$, $\exists\ C(\alpha,\gamma) > 0$ *such that*

$$|[\partial_z^\alpha \partial_w^\gamma \varphi'_\beta(w,z)]|_{z=0}| \leq C(\alpha,\gamma)^{|\beta|+1} \quad if \quad |w| < r(\alpha,\gamma), \quad \forall\ \beta \in \mathbb{N}_*^{n-1}. \tag{7.4}$$

Remark. Fortunately, the fact that $r(\alpha,\gamma)$ depends on α and γ will cause no particular obstruction for the achievement of the last third step in §8 below. We believe however that this dependence should be avoided, but we get no immediate control of $r(\alpha,\gamma)$ as $\alpha + |\gamma| \to \infty$, in our proof – although it can be seen by induction that $[\partial_z^\alpha \partial_w^\gamma \varphi'_\beta(w,z)]|_{z=0} \in \mathcal{O}((r\Delta)^{n-1}, \mathbb{C})$ (*cf.* the proof of Lemma 7.2 below).

Proof. If we denote by $\mathcal{E}_{\alpha,\gamma}$ the statement of the lemma, then it is clear that

$$\mathcal{E}_{\alpha,0} \ \Rightarrow\ (\mathcal{E}_{\alpha,\gamma} \ \ \forall\ \gamma \in \mathbb{N}^{n-1}). \tag{7.5}$$

It suffices therefore to establish the truth of $\mathcal{E}_{\alpha,0}$ for all $\alpha \in \mathbb{N}$. Let us first establish that $\partial_z^\alpha|_{z=0}[\varphi'_\beta(w,z)] \in \mathcal{O}((r\Delta)^{n-1}, \mathbb{C})$, $\forall\ \alpha \in \mathbb{N}$, $\forall\ \beta \in \mathbb{N}^{n-1}$. To this aim, we specify the variables $(w,z,\zeta,\xi) := (w,z,0,z) \in \mathcal{M}$ (because $\Theta(0,w,z) \equiv 0$) in the equations (5.2) and (5.4) to obtain firstly

$$\bar f(0,z) \equiv f(w,z) - i \sum_{\gamma\in\mathbb{N}_*^{n-1}} \bar g(0,z)^\gamma\ \Theta'_\gamma(g(w,z), f(w,z)) \tag{7.6}$$

and secondly the following infinite number of relations:

$$\begin{cases} \underline{\Omega}_\beta(w,0,z,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar h(0,z))_{|\alpha^1|+|\gamma^1|\leq|\beta|}) \equiv \\ \equiv \Theta'_\beta(g(w,z), f(w,z)) + \displaystyle\sum_{\gamma\in\mathbb{N}_*^{n-1}} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\ \bar g(0,z)^\gamma\ \Theta'_{\beta+\gamma}(g(w,z), f(w,z)). \end{cases} \tag{7.7}$$

Essentially, the game will consist in differentiating the equalities (7.6) and (7.7) with respect to z at 0 up to arbitrary order α, in the aim to obtain new identities which will yield $\partial_z^\alpha|_{z=0}[\Theta'_\beta(g(w,z),f(w,z))] \in \mathcal{O}((r\Delta)^{n-1},\mathbb{C})$, $\forall\ \beta\in\mathbb{N}^{n-1}$, $\forall\ \alpha\in\mathbb{N}$, by an induction process of "trigonal type". Let us complete this informal description. To begin with, for $\alpha=1$, after applying the derivation operator $\partial_z^1|_{z=0}$ to eqs. (7.6) and (7.7), we get immediately

$$\partial_z^1 f(w,0) \equiv \partial_z^1 \bar f(0,0) + i\sum_{j=1}^{n-1}[\partial\bar g_j/\partial z](0,0)\ \Theta'_\gamma(g(w,0),f(w,0)) \in \mathbb{C}\{w\}, \tag{7.8}$$

since $\bar g(0,0)=0$ (so $\partial_z^1[\bar g(0,z)^\gamma]_{z=0}=0$ for $|\gamma|\geq 2$), and

$$\begin{cases} \partial_z^1\Theta'_\beta(g(w,0),f(w,0)) = [\partial_z^1[\underline{\Omega}_\beta(w,0,z,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar h(0,z))_{|\alpha^1|+|\gamma^1|\leq|\beta|})]]|_{z=0} - \\ \qquad - \sum_{|\gamma|=1}\dfrac{(\beta+\gamma)!}{\beta!\ \gamma!}\partial_z^1\bar g(0,0)^\gamma\ \Theta'_{\beta+\gamma}(g(w,0),f(w,0)), \end{cases} \tag{7.9}$$

making the slight abuse of notation $\partial_z^1\chi(w,0)$ instead of writing $\partial_z^1|_{z=0}[\chi(w,z)]$ for any formal power series $\chi(w,z)\in\mathbb{C}[\![w,z]\!]$. For instance, $\partial_z^1\bar g(0,0)^\gamma$ significates $[\partial_z^1(\bar g(0,z)^\gamma)]|_{z=0} = \sum_{k=1}^{n-1}\gamma_k\partial_z^1\bar g_k(0,0)[\bar g(0,0)^{\gamma_1}\cdots\bar g_k(0,0)^{\gamma_k-1}\cdots\bar g_{n-1}(0,0)^{\gamma_{n-1}}]$. All these expressions are convergent, because we know already (thanks to the first step) that $\Theta'_\beta(g(w,0),f(w,0))\in\mathbb{C}\{w\}$ $\forall\,\beta\in\mathbb{N}_*^{n-1}$ (and even $\Theta'_\beta(g(w,0),f(w,0))\in\mathcal{O}((r\Delta)^{n-1},\mathbb{C})$) and because the derivative $\partial_z^1|_{z=0}(\underline{\Omega}_\beta)$ can be expressed (thanks to the chain rule) in terms of the derivatives $\partial\underline{\Omega}_\beta/\partial z$, in terms of the derivatives $\partial\underline{\Omega}_\beta/(\partial^{\alpha^1}\partial^{\gamma^1})$ (considering $\partial^{\alpha^1}\partial^{\gamma^1}$ as independent variables), and in terms of the derivatives $\partial_z^1|_{z=0}(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar h(0,z))$, all taken at $z=0$, which are terms obviously converging and even which belong to the space $\mathcal{O}((r\Delta)^{n-1},\mathbb{C})$. Thus, we have got that $\partial_z^1\varphi'_\beta(w,0)\in\mathcal{O}((r\Delta)^{n-1},\mathbb{C})$, $\forall\ \beta\in\mathbb{N}^{n-1}$ (including $\beta=0$). More generally, for arbitrary $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}_*^{n-1}$, we observe readily that $\partial_z^\alpha|_{z=0}[\bar f(0,z)]$ is constant and, for the same reasons as explained above, that

$$\partial_z^\alpha|_{z=0}\left[\underline{\Omega}_\beta(w,0,z,\ (\partial_\xi^{\alpha^1}\partial\zeta^{\gamma^1}\bar h(0,z))_{|\alpha^1|+|\gamma^1|\leq|\beta|})\right] \in \mathcal{O}((r\Delta)^{n-1},\mathbb{C}). \tag{7.10}$$

We can use this observation to perform a "trigonal" induction as follows. Let $\alpha_0\in\mathbb{N}_*$ and suppose by induction that $\partial_z^\alpha\varphi'_\beta(w,0)\in\mathcal{O}((r\Delta)^{n-1},\mathbb{C})$ $\forall\ \alpha\leq\alpha_0$, $\forall\ \beta\in\mathbb{N}_*^{n-1}$. Then applying the derivation $\partial_z^{\alpha_0+1}|_{z=0}$ to (7.6), developing the expression according to Leibniz' formula and using the fact that $\partial_z^{\alpha_0+1}|_{z=0}[\bar g(0,z)^\gamma]=0$ for all $|\gamma|\geq\alpha_0+2$, we get the expression:

$$\begin{cases} \partial_z^{\alpha_0+1}f(w,0)\equiv\partial_z^{\alpha_0+1}\bar f(0,0)+i\sum_{0<|\gamma|\leq\alpha_0+1} \\ \sum_{\kappa=1}^{\alpha_0+1}\dfrac{(\alpha_0+1)!}{\kappa!(\alpha_0+1-\kappa)!}\ \partial_z^\kappa\bar g(0,0)^\gamma\ \ \partial_z^{\alpha_0+1-\kappa}\varphi'_\gamma(w,0). \end{cases} \tag{7.11}$$

Now, thanks to the induction assumption and *because the order of derivation in the expression* $\partial_z^{\alpha_0+1-\kappa}\varphi'_\gamma(w,0)$ *for* $1\leq\kappa\leq\kappa_0+1$ *is less or equal to* α_0, we obtain that this expression (7.11) belongs to $\mathcal{O}((r\Delta)^{n-1}, \mathbb{C})$. Concerning the differentiation of (7.9) with respect to z, we also get that the term

$$(7.12)\qquad \begin{cases} \partial_z^{\alpha_0+1}\varphi'_\beta(w,0) \equiv \partial_z^{\alpha_0+1}\underline{\omega}_\beta(w,0,0) - \sum_{0<|\gamma|\leq\alpha_0+1} \frac{(\beta+\gamma)!}{\beta!\ \gamma!} \\ \left(\sum_{\kappa=1}^{\alpha_0+1} \frac{(\alpha_0+1)!}{\kappa!(\alpha_0+1-\kappa)!}\ \partial_z^\kappa \bar{g}(0,0)^\gamma\ \ \partial_z^{\alpha_0+1-\kappa}\varphi'_{\beta+\gamma}(w,0)\right) \end{cases}$$

belongs to $\mathcal{O}((r\Delta)^{n-1}, \mathbb{C})$. Again, the important fact is that in the sum $\sum_{\kappa=1}^{\alpha_0+1}$, only the derivations $\partial_z^\alpha\varphi'_\beta(w,0)$ for $0 \leq \alpha \leq \alpha_0$ occur. In summary, we have shown that $\partial_z\varphi_\beta(w,0)$ is convergent for all $\alpha \in \mathbb{N}$.

7.13. Intermezzo. The induction process can be said to be of "trigonal type" because we are dealing with the infinite collection of identities (5.4) which can be interpreted as a linear system $Y = AX$, where X denotes the unknown $(\Theta'_\beta)_{\beta\in\mathbb{N}^{n-1}}$ and A is an infinite trigonal matrix, as shows an examination of (5.4). Further, when we consider the jets, we still get a trigonal system. *The main point is that after restriction to the first Segre chain $\{\zeta = \xi = 0\}$, this trigonal system becomes diagonal (or with only finitely many nonzero elements after applying ∂_z^α), but this crucial simplifying property fails to be satisfied after passing to the next Segre chains.* To be honest, we should recognize that the proof we are conducting here unfortunately fails (for this reason) to be generalizable to higher codimension... However, an important natural idea will appear during the course of the proof, namely the appearance of the natural (and new) *conjugate reflection identities* (5.9) which we will heavily use in §8 below. For reasons of symmetry, we have naturally wondered whether they can be exploited more deeply. A complete investigation is contained in our subsequent work on the subject (quoted in §1.12).

7.14. End of proof of Lemma 7.2. It remains to show that there exist constants $r(\alpha) > 0$, $C(\alpha) > 0$ such that the estimate (7.4) holds for $(\alpha, \gamma) = (\alpha, 0)$: $|\partial_z^\alpha\varphi'_\beta(w,0)| \leq C(\alpha)^{|\beta|+1}$ if $|w| < r(\alpha)$, $\forall\ \beta \in \mathbb{N}_*^{n-1}$. To this aim, we shall apply Lemma 6.6 with the suitable functions and variables. First, it is clear that there exist universal polynomials[1] such that the following composite derivatives can be written

$$(7.15)\qquad \partial_z^\alpha[\Theta'_\beta(h(w,z))] = P_\alpha(\nabla_z^{*\alpha}h(w,z)), (\nabla_{t'}^{*\alpha}\Theta'_\beta)(h(w,z))),$$

where the $n\alpha$-tuple $\nabla^{*\alpha}h(w,z) := (((\partial_z^k h_1(w,z), \ldots, \partial_z^k h_n(w,z))_{1\leq k\leq\alpha})$ and the $(\frac{(\alpha+n)!}{\alpha!\ n!} - 1)$-tuple $\nabla_{t'}^{*\alpha}\Theta'_\beta(t') := ((\partial_{t'}^\beta\Theta'_\beta(t'))_{1\leq|\beta|\leq\alpha}$. We now consider these polynomials as holomorphic functions $G_\beta^\alpha = G_\beta^\alpha(\nabla_z^\alpha h)$ of the $n(\alpha+1)$ variables $\nabla_z^\alpha h = ((\partial_z^k h_1, \ldots, \partial_z^k h_n)_{0\leq k\leq\alpha})$ which satisfy, by eq. (7.15):

$$(7.16)\qquad \partial_z^\alpha\Theta'_\beta(h(w,z)) = G_\beta^\alpha(\nabla_z^\alpha h(w,z)) = G_\beta^\alpha(\nabla_z^\alpha h)|_{\nabla_z^\alpha h := \nabla_z^\alpha h(w,z)},$$

[1]The explicit formula in dimension one for the derivative of a composition $\frac{d^n}{dx^n}(\psi \circ \phi(x)) = (\psi \circ \phi)^{(n)}(x)$ is known as *Faa di Bruno's formula*, (one of the favorite students of Cauchy):

$$\frac{1}{n!}(\psi\circ\phi)^{(n)}(x) = \sum_{\alpha_1+2\alpha_2+\cdots+n\alpha_n=n} \frac{1}{\alpha_1!\,\alpha_2!\cdots\alpha_n!\,(1!)^{\alpha_1}\,(2!)^{\alpha_2}\cdots(n!)^{\alpha_n}}$$
$$\times(\phi'(x))^{\alpha_1}\,(\phi''(x))^{\alpha_2}\cdots(\phi^{(n)}(x))^{\alpha_n}\ \psi^{(\alpha_1+\alpha_2+\cdots+\alpha_n)}(\phi(x)).$$

where the $n\alpha$-tuple $\nabla_z^\alpha h(w,z) = (\partial_z^k h_j(w,z))_{0\leq k\leq \alpha}^{1\leq j\leq n}$ and $\nabla_z^\alpha h := (\partial_z^k h_j)_{0\leq k\leq\alpha}^{1\leq j\leq n}$ are $n(\alpha{+}1)$ *independent variables* as we have just said above. Obviously, these functions $G_\beta^\alpha(\nabla_z^\alpha h)$ satisfy an estimate of the form $|G_\beta^\alpha(\nabla_z^\alpha h)| \leq C(\alpha)^{|\beta|+1}$ if $|\nabla_z^\alpha h| < r$, because the functions $\nabla_{t'}^{*\alpha}\Theta'_\beta(t')$ satisfy an estimate of the form $|\nabla_{t'}^{*\alpha}\Theta'_\beta(t')| \leq C'(\alpha)^{|\beta|+1}$ if $|t'| < r'$, for some constants $C'(\alpha) > 0$, $r' > 0$, and because we have $P_\alpha(\nabla_z^\alpha h, 0) \equiv 0$. We already know that there exist holomorphic functions $\chi_\beta^\alpha(w) = \partial_z^\alpha \varphi'_\beta(w,0)$ in $\{|w| < r\}$ indexed by $\beta \in \mathbb{N}_*^{n-1}$ such that the following formal identity holds:

$$G_\beta^\alpha(\nabla_z^\alpha h(w,0)) = \partial_z^\alpha \varphi'_\beta(w,0) = \chi_\beta^\alpha(w) \quad \text{in } \mathbb{C}[\![w]\!]. \tag{7.17}$$

Now, a direct application of Lemma 6.6 yields the desired estimate:

$$|\partial_z^\alpha \varphi'_\beta(w,0)| \leq C(\alpha)^{|\beta|+1} \quad \text{if} \quad |w| < r(\alpha). \tag{7.15}$$

Thus, we have completed the proof of Lemma 7.2. □

Important remark. When $\alpha \to \infty$, the number $(n+1)\alpha$ of variables in $\nabla_z^\alpha h$ also becomes infinite. Thus, at each step we apply Artin's Theorem in Lemma 6.6, the $r(\alpha)$ may shrink and go to zero as $\alpha \to \infty$.

§8. Convergence of the mapping

8.1. Jump to the second Segre chain. We now complete the final third step by establishing that the power series $h(t)$ is convergent in a neighborhood of 0. Let $\underline{\mathcal{S}}_0^2 = \{\exp w\mathcal{L}(\exp \zeta\underline{\mathcal{L}}(0))\colon |w| < r, |\zeta| < r\}$ be the second conjugate Segre chain [9], or equivalently in our normal coordinates $\underline{\mathcal{S}}_0^2 = \{(w, i\bar\Theta(w,\zeta,0), \zeta, 0)\colon |w| < r, |\zeta| < r\}$. We shall prove that the map h^c is convergent on $\underline{\mathcal{S}}_0^2$. More precisely:

Lemma 8.2. *The formal power series $h(w, i\bar\Theta(\zeta,w,0)) \in \mathbb{C}\{w,\zeta\}^n$ is convergent.*

From this lemma, we see now how to achieve the proof of our Theorem 1.2:

Corollary 8.3. *Then the formal power series $h(w,z) \in \mathbb{C}\{w,z\}^n$ is convergent.*

Proof. We just apply Theorem 4.7, taking into account **(5)** of Lemma 3.2. □

Proof of Lemma 8.2. Thus, we have to show that $h(w, i\bar\Theta(\zeta,w,0)) \in \mathbb{C}\{w,\zeta\}$. To this aim, we consider the conjugate reflection identities (5.2) and (5.9) for various $\beta \in \mathbb{N}_*^{n-1}$ after specifying them over $\underline{\mathcal{S}}_0^2$, *i.e.* after setting $(w,z,\zeta,\xi) := (w, i\bar\Theta(\zeta,w,0), \zeta, 0) \in \mathcal{M}$, which we may write explicitely as follows

$$\begin{cases} \bar f(\zeta,0) \equiv f(w, i\bar\Theta(w,\zeta,0)) - i \displaystyle\sum_{\gamma\in\mathbb{N}_*^{n-1}} \bar g(\zeta,0)^\gamma \; \Theta'_\gamma(h(w, i\bar\Theta(w,\zeta,0)), \\ 0 \equiv [\underline{\mathcal{L}}^\beta \bar f(\zeta,\xi)]_{\xi=0} + i \displaystyle\sum_{\gamma\in\mathbb{N}_*^{n-1}} g(w, i\bar\Theta(w,\zeta,0))^\gamma \; [\underline{\mathcal{L}}^\beta(\bar\Theta'_\gamma(\bar h(\zeta,\xi)))]_{\xi=0}, \end{cases} \tag{8.4}$$

for all $\beta \in \mathbb{N}_*^{n-1}$. Let now $\kappa_0 \in \mathbb{N}_*$ be an integer larger than the supremum of the lengths of some multiindices β^i's, $1 \leq i \leq n-1$, satisfying the determinant property stated in eq. (3.4) of Lemma 3.3, *i.e.* $\kappa_0 \geq \sup_{1\leq i\leq n-1} |\beta^i|$. According to Lemma 7.2, if we consider the equations (8.4) only for a finite number of β's, say for $|\beta| \leq \kappa_0$, there will exist a positive number $r_1 > 0$ with $r_1 \leq r$ and a constant

$C_1 > 0$ such that each power series $[\underline{\mathcal{L}}^\beta(\bar{\Theta}'_\gamma(\bar{h}(\zeta,\xi)))]_{\xi=0} =: \chi_\gamma^\beta(w,\zeta)$ is holomorphic in the polydisc $\{|w|, |\zeta| < r_1\}$ and satisfies the Cauchy estimate $|\chi_\gamma^\beta(w,\zeta)| \leq C_1^{|\gamma|+1}$ when $|(w,\zeta)| < r_1$. We can now represent the eqs. (8.4) under the brief form

$$s_\beta(w,\zeta,h(w,i\bar{\Theta}(w,\zeta,0))) \equiv 0, \quad |\beta| \leq \kappa_0, \tag{8.5}$$

where the holomorphic functions $s_\beta = s_\beta(w,\zeta,t')$ are simply defined by replacing the terms $[\underline{\mathcal{L}}^\beta(\bar{\Theta}'_\gamma(\bar{h}(\zeta,\xi)))]_{\xi=0}$ by $\chi_\gamma^\beta(w,\zeta)$ in eqs. (8.4), so that the power series s_β converge in the set $\{|w|, |\zeta|, |t'| < r_1\}$. The goal is now to apply Theorem 4.4 to the collection of equations (8.5) in order to deduce that $h(w,i\bar{\Theta}(w,\zeta,0)) \in \mathbb{C}\{w,\zeta\}$.

Remark. As noted in the introduction, another (more powerful) idea would be to apply the Artin Approximation Theorem 4.2 to the equations (8.5) to deduce the existence of a *converging* solution $H(w,\zeta)$ and then to deduce that the reflection function itself converges on the second Segre chain (which is in fact quite easy using Lemma 5.10). This will be achieved in §9 below.

First, we make a precise choice of the $\beta^i \in \mathbb{N}_*^{n-1}$ arising in Lemma 3.3. We set $\underline{\beta}^n = 0$ and, for $1 \leq i \leq n-1$, let $\underline{\beta}^i$ be the infimum of all the multiindices $\beta \in \mathbb{N}_*^{n-1}$ satisfying $\beta > \underline{\beta}^{i+1} > \cdots > \underline{\beta}^n$ for the natural lexicographic order on $\mathbb{N}^{n-1}$, and such that an $(n-i+1)\times(n-i+1)$ minor of the $n \times (n-i+1)$ matrix

$$\mathcal{MAT}_{\beta,\underline{\beta}^{i+1},\dots,\underline{\beta}^n}(t') := \left(\frac{\partial \Theta'_\beta}{\partial t'_j}(t') \quad \frac{\partial \Theta'_{\underline{\beta}^{i+1}}}{\partial t'_j}(t') \quad \cdots \quad \frac{\partial \Theta'_{\underline{\beta}^n}}{\partial t'_j}(t') \right)_{1 \leq j \leq n} \tag{8.6}$$

does not vanish identically as a holomorphic function of $t' \in \mathbb{C}^n$. We thus have $\det\left(\frac{\partial \Theta'_{\underline{\beta}^i}}{\partial t'_j}(t')\right)_{1\leq i,j\leq n} \not\equiv 0$ in $\mathbb{C}\{t'\}$. Concerning the choice of κ_0, we also require that

$$\kappa_0 \geq \inf\{k \in \mathbb{N} \colon \det\left(\frac{\partial \Theta'_{\underline{\beta}^i}}{\partial t'_j}(h(w,i\bar{\Theta}(w,\zeta,0)))\right)_{1\leq i,j\leq n} \not\in \mathfrak{m}(\zeta)^k \ \mathbb{C}[\![w,\zeta]\!]\}, \tag{8.7}$$

where $\mathfrak{m}(\zeta)$ is the maximal ideal of $\mathbb{C}[\![\zeta]\!]$. We can choose such a finite κ_0, because we know already that the determinant in (8.7) does not vanish identically (this fact can be easily checked, after looking at the composition formula for Jacobians, because, in view of Lemma 3.3, the determinant (8.6) for $(\underline{\beta}^1,\dots,\underline{\beta}^n)$ does not vanish identically and because the determinant $\det\left(\frac{\partial h_j}{\partial t_k}(w,i\bar{\Theta}(w,\zeta,0))\right)_{1\leq j,k\leq n}$ does not vanish identically in view of the invertibility assumption on h and in view of the minimality criterion Lemma 3.2 **(5)**). Thus, after these choices are made, in order to finish the proof by an application of Theorem 4.4, it will suffice to show that:

Lemma 8.8. *There exist $\beta^1,\dots,\beta^{n-1},\beta^n(=0) \in \mathbb{N}^{n-1}$ with $|\beta^i| \leq 2\kappa_0$ such that*

$$\det\left(\frac{\partial s_{\beta^i}}{\partial t'_j}(w,\zeta,h(w,i\bar{\Theta}(w,\zeta,0)))\right)_{1\leq i,j\leq n} \not\equiv 0 \quad in\ \mathbb{C}[\![w,\zeta]\!]. \tag{8.9}$$

Proof. To this aim, we introduce some new power series. We set:

$$R_\beta(w,z,\zeta,t') := \underline{\mathcal{L}}^\beta \bar{f}(\zeta,\xi) + i \sum_{\gamma\in\mathbb{N}_*^{n-1}} \underline{\mathcal{L}}^\beta(\bar{g}(\zeta,\xi)^\gamma)\ \Theta'_\gamma(t'), \tag{8.10}$$

for all $\beta \in \mathbb{N}^{n-1}$, after expanding with respect to (w,z,ζ,ξ) the power series appearing in $\underline{\mathcal{L}}^\beta(\bar g(\zeta,\xi)^\gamma)$, $\underline{\mathcal{L}}^\beta \bar f(\zeta,\xi)$ and after replacing ξ by $z - i\Theta(\zeta,w,z)$, and similarly, we set:

$$S_\beta(w,z,\zeta,t') := \underline{\mathcal{L}}^\beta \bar f(\zeta,\xi) + i \sum_{\gamma\in\mathbb{N}_*^{n-1}} w'^\gamma \ \underline{\mathcal{L}}^\beta(\bar\Theta'_\gamma(\bar h(\zeta,\xi))), \tag{8.11}$$

in coherence with the notation in eq. (8.5) and finally also, we set:

$$T_\beta(w,z,\zeta,t') := -\underline{\omega}_\beta(w,\zeta,\xi) + \Theta'_\beta(t') + \sum_{\gamma\in\mathbb{N}_*^{n-1}} \frac{(\beta+\gamma)!}{\beta!\ \gamma!} \bar g(\zeta,\xi)^\gamma\ \Theta'_{\beta+\gamma}(t'). \tag{8.12}$$

We first remark that, by the very definition of s_β and of S_β, we have

$$\frac{\partial s_\beta}{\partial t'_j}(w,\zeta,h(w,i\bar\Theta(w,\zeta,0))) \equiv \frac{\partial S_\beta}{\partial t'_j}(w,i\bar\Theta(w,\zeta,0),\zeta,h(w,i\bar\Theta(w,\zeta,0))), \tag{8.13}$$

as formal power series, for all $\beta \in \mathbb{N}^{n-1}$, $1 \le j \le n$. Next, let us establish a useful correspondence between the vanishing of the generic ranks of $(R_\beta)_{|\beta|\le 2\kappa_0}$, $(S_\beta)_{|\beta|\le 2\kappa_0}$ and $(T_\beta)_{|\beta|\le 2\kappa_0}$.

Lemma 8.14. *The following properties are equivalent:*

(1) $det\left(\frac{\partial R_{\beta^i}}{\partial t'_j}(w,z,\zeta,h(w,z))\right)_{1\le i,j\le n} \equiv 0,\ \forall\ \beta^1,\dots,\beta^n,\ |\beta^1|,\dots,|\beta^n| \le 2\kappa_0.$

(2) $det\left(\frac{\partial S_{\beta^i}}{\partial t'_j}(w,z,\zeta,h(w,z))\right)_{1\le i,j\le n} \equiv 0,\ \forall\ \beta^1,\dots,\beta^n,\ |\beta^1|,\dots,|\beta^n| \le 2\kappa_0.$

(3) $det\left(\frac{\partial T_{\beta^i}}{\partial t'_j}(w,z,\zeta,h(w,z))\right)_{1\le i,j\le n} \equiv 0,\ \forall\ \beta^1,\dots,\beta^n,\ |\beta^1|,\dots,|\beta^n| \le 2\kappa_0.$

End of proof of Lemma 8.8. The proof of Lemma 8.14 will be given just below. To finish the proof of Lemma 8.8, we assume by contradiction that (8.9) is untrue, *i.e.* that **(2)** of Lemma 8.14 holds with $z = i\bar\Theta(w,\zeta,0)$. According to **(3)** of this lemma, we also have that the generic rank of the $n \times \frac{(2\kappa_0+n-1)!}{(2\kappa_0)!(n-1)!}$ matrix

(8.15)
$$\begin{cases} \mathcal{N}_{2\kappa_0}(w,\zeta) := \Bigg(\frac{\partial\Theta'_\beta}{\partial t'_j}(h(w,i\bar\Theta(w,\zeta,0))) + \\ \qquad + \sum_{\gamma\in\mathbb{N}_*^{n-1}} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\ \bar g(\zeta,0)^\gamma\ \frac{\partial\Theta'_{\beta+\gamma}}{\partial t'_j}(h(w,i\bar\Theta(w,\zeta,0)))\Bigg)^{|\beta|\le 2\kappa_0}_{1\le j\le n} \end{cases}$$

is strictly less than n. After making some obvious linear combinations between the columns of $\mathcal{N}_{2\kappa_0}$ with coefficients being formal power series in ζ which are polynomial with respect to the $\bar g_j(\zeta,0) \in \mathfrak{m}(\zeta)$, $1 \le j \le n-1$, we can reduce $\mathcal{N}_{2\kappa_0}$ to the matrix of same formal generic rank

$$\begin{cases} \mathcal{N}^0_{2\kappa_0}(w,\zeta) := \Bigg(\frac{\partial\Theta'_\beta}{\partial t'_j}(h(w,i\bar\Theta(w,\zeta,0))) + \\ + \sum_{|\gamma|\ge 2\kappa_0+1-|\beta|} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\ \bar g(\zeta,0)^\gamma\ \frac{\partial\Theta'_{\beta+\gamma}}{\partial t'_j}(h(w,i\bar\Theta(w,\zeta,0)))\Bigg)^{|\beta|\le 2\kappa_0}_{1\le j\le n} \end{cases}. \tag{8.16}$$

Now, taking the submatrix $\mathcal{N}^0_{\kappa_0}$ of $\mathcal{N}^0_{2\kappa_0}$ for which $|\beta| \leq \kappa_0$, we see that we have reduced $\mathcal{N}^0_{\kappa_0}$ to the simpler form

$$\mathcal{N}^0_{\kappa_0}(w,\zeta) \equiv \mathcal{N}^1_{\kappa_0}(w,\zeta) \text{ mod } (\mathfrak{m}(\zeta)^{\kappa_0+1}\mathcal{M}at_{n\times\frac{(\kappa_0+n-1)!}{(\kappa_0)!(n-1)!}}(\mathbb{C}[\![w,\zeta]\!])), \tag{8.17}$$

where

$$\mathcal{N}^1_{\kappa_0}(w,\zeta) := \left(\frac{\partial \Theta'_\beta}{\partial t'_j}(h(w,i\bar{\Theta}(w,\zeta,0))\right)^{|\beta|\leq\kappa_0}_{1\leq j\leq n}. \tag{8.18}$$

But

$$\det\left(\frac{\partial \Theta'_{\underline{\beta}^i}}{\partial t'_j}(h(w,i\bar{\Theta}(w,\zeta,0)))\right)_{1\leq i,j\leq n} \not\equiv 0 \text{ in } \mathbb{C}[\![w,\zeta]\!] \text{ (mod } \mathfrak{m}(\zeta)^{\kappa_0+1}), \tag{8.19}$$

by the choice of the $\underline{\beta}^i$'s and of κ_0, which is the desired contradiction. □

Proof of Lemma 8.14. The equivalence **(1)** $\Longleftrightarrow$ **(3)** follows by an inspection of the proof of Lemma 5.3: to pass from the system $R_\beta = 0$, $|\beta| \leq 2\kappa_0$, to the system $T_\beta = 0$, $|\beta| \leq 2\kappa_0$, we have only use in the proof some linear combinations with coefficients in $\mathbb{C}[\![\zeta,\xi]\!]$. The equivalence **(1)** $\Longleftrightarrow$ **(2)** is related with the substance of Lemma 5.10. Indeed, in the relation $r'(t',\tau') \equiv \alpha'(t',\tau')\,\bar{r}'(\tau',t')$, with $\alpha'(0,0) = -1$, insert first $\tau' := \bar{h}(\tau)$ to get $r'(t',\bar{h}(\tau)) \equiv \alpha'(t',\bar{h}(\tau))\,\bar{r}'(\bar{h}(\tau),t')$ and then differentiate by the operator $\underline{\mathcal{L}}^\beta$ to obtain

$$\underline{\mathcal{L}}^\beta r'(t',\bar{h}(\tau)) \equiv \alpha'(t',\bar{h}(\tau))\;\underline{\mathcal{L}}^\beta\bar{r}'(\bar{h}(\tau),t') + \sum_{\gamma\leq\beta,\gamma\neq\beta} {\alpha'}^\beta_\gamma(t',t,\tau)\;\underline{\mathcal{L}}^\gamma\bar{r}'(\bar{h}(\tau),t'). \tag{8.20}$$

In our notations, $R_\beta(w,z,\zeta,t') = \underline{\mathcal{L}}^\beta r'(t',\bar{h}(\tau))$ and $S_\beta(w,z,\zeta,t') = \underline{\mathcal{L}}^\beta\bar{r}'(\bar{h}(\tau),t')$, after replacing ξ by $z - i\Theta(\zeta,w,z)$. We deduce

$$\left\{\begin{aligned} &\frac{\partial R_\beta}{\partial t'_j}(w,z,\zeta,h(w,z)) = \alpha'(h(w,z),\bar{h}(\tau))\frac{\partial S_\beta}{\partial t'_j}(w,z,\zeta,h(w,z)) + \\ &+ \sum_{\gamma\leq\beta,\gamma\neq\beta} {\alpha'}^\beta_\gamma(t,\tau,h(w,z))\;\frac{\partial S_\gamma}{\partial t'_j}(w,z,\zeta,h(w,z)). \end{aligned}\right. \tag{8.21}$$

Equation (8.21) shows that the terms $\frac{\partial R_\beta}{\partial t'_j}(w,z,\zeta,h(w,z))$ are trigonal linear combinations of the terms $\frac{\partial S_\gamma}{\partial t'_j}(w,z,\zeta,h(w,z))$, $\gamma \leq \beta$, with nonzero diagonal coefficients. This completes the proofs of Lemmas 8.14 and 8.2 and completes finally our proof of Theorem 1.2. □

Remark. Once we know that $h(w,z) \in \mathbb{C}\{w,z\}$, we deduce that the reflection function associated with the formal equivalenve of Theorem 1.2 is convergent, i.e. that $\mathcal{R}'_h(w,z,\bar{\lambda},\bar{\mu}) \in \mathbb{C}\{w,z,\bar{\lambda},\bar{\mu}\}$.

§9. Convergence of the reflection function

Using the conjugate reflection identities (5.9), we shall observe that we may prove the following more general statement (*see* [12], where the first proof of it was provided differently). Notice also that a second proof of Theorem 1.2 can be derived from Theorem 9.1 by applying afterward Lemma 3.3 and Theorem 4.4.

Theorem 9.1. *Let $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ be a formal invertible CR mapping between two real analytic hypersurfaces in $\mathbb{C}^n$ and assume that (M,p) is minimal. Then the reflection mapping $\mathcal{R}'_h$ is convergent, i.e.*

$$\mathcal{R}'_h(t,\bar{\nu}') \in \mathbb{C}\{t,\bar{\nu}'\}. \tag{9.2}$$

Proof. We come back to the conjugate reflection identities (5.9) and we put for the arguments $(w,z,\zeta,\xi) := (w, i\bar{\Theta}(w,\zeta,0),\zeta,0)$. By Lemma 7.2, we know that all the terms $\underline{\mathcal{L}}^\beta(\bar{\Theta}'_\gamma(\bar{h}))$ with these arguments are convergent power series in (w,ζ). The same holds for the terms $\underline{\mathcal{L}}^\beta(\bar{f})$. We thus get the equations (8.5) where the term $h(w,i\bar{\Theta}(w,\zeta,0))$ is only formal. Since the equations $s_\beta(w,\zeta,t')$ are analytic, we can apply Artin's approximation theorem. Consequently, there exists a convergent power series $H(w,\zeta) \in \mathbb{C}\{w,\zeta\}^n$ satisfying

$$s_\beta(w,\zeta,H(w,\zeta)) \equiv 0, \qquad \forall\, \beta \in \mathbb{N}^{n-1}. \tag{9.3}$$

Here, we consider the complete list of equations $s_\beta = 0$ for all β. Then using calculations similar to (8.20), (8.21), namely by applying the CR derivations $\underline{\mathcal{L}}^\beta$ to the identity $r'(t',\bar{h}(\tau)) \equiv \alpha'(t',\bar{h}(\tau))\,\bar{r}'(\bar{h}(\tau),t')$, we deduce that H satisfies the first family of reflection identities, namely

$$\underline{\mathcal{L}}^\beta \bar{f} \equiv -i \sum_{\gamma\in\mathbb{N}^{n-1}_*} \underline{\mathcal{L}}^\beta(\bar{g}^\gamma)\,\Theta'_\gamma(H). \tag{9.4}$$

Here, the arguments of $\underline{\mathcal{L}}$ and of $\bar{h}$ are $(w,i\bar{\Theta}(w,\zeta,0),\zeta,0)$ and the arguments of H are (w,ζ). Applying the calculation of Lemma 5.3 to (9.4) and comparing to (5.4), we deduce that for all $\beta\in\mathbb{N}^{n-1}$, we have

$$\begin{aligned}\underline{\omega}_\beta &\equiv \Theta'_\beta(h) + \sum_{\gamma\in\mathbb{N}^{n-1}_*} \frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}^\gamma\,\Theta'_{\beta+\gamma}(h) \equiv \\ &\equiv \Theta'_\beta(H) + \sum_{\gamma\in\mathbb{N}^{n-1}_*} \frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}^\gamma\,\Theta'_{\beta+\gamma}(H).\end{aligned} \tag{9.5}$$

Here, the terms $\underline{\omega}_\beta$ are formal power series of (w,ζ) which depend on the jets of $\bar{h}$, as shown by Lemma 5.3. As the infinite system (9.5) is trigonal, it is formally invertible, and here, for this precise system, the inverse is easy to compute and we get the simple formula

$$\Theta'_\beta(h) \equiv \underline{\omega}_\beta + \sum_{\gamma\in\mathbb{N}^{n-1}_*} (-1)^\gamma\,\frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}^\gamma\,\underline{\omega}_{\beta+\gamma} \equiv \Theta'_\beta(H). \tag{9.6}$$

We deduce that

$$\Theta'_\beta(h(w,i\bar{\Theta}(w,\zeta,0))) \equiv \Theta'_\beta(H(w,\zeta)) \in \mathbb{C}\{w,\zeta\}, \tag{9.7}$$

which shows that all the components of the reflection mapping are convergent on the second Segre chain. It remains to apply the theorem of Gabrielov (Eakin-Harris) to deduce that $\Theta'_\beta(h(t))$ is convergent for all β. The final Cauchy estimate follows as in Lemma 6.4. The proof of Theorem 9.1 is complete. □

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