

Embeddings into the Pancake Interconnection Network

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Abstract

Owing to its nice properties, the pancake is one of the Cayley graphs that were proposed as alternatives to the hypercube for interconnecting processors in parallel computers. In this paper, we present embeddings of rings, grids and hypercubes into the pancake with constant dilation and congestion. We also extend the results to similar efficient embeddings into the star graph.

1 Introduction

Akers and Krishnamurthy [1] proposed the pancake and the star as alternatives to the hypercube for interconnecting processors in parallel computers. These networks have some nice properties: edge and vertex symmetry (strong symmetry), small degree and diameter, extensibility, high connectivity (robustness), easy routings and broadcasting, etc. To compare favorably with the hypercube, these graphs must also offer good and simple simulations of other interconnection networks. The problem of simulating known networks by the star graph has been extensively studied. For example, Nigam, Sahni and Krishnamurthy [8] consider embeddings of rings and hypercubes in star graphs. Miller, Pritkin, and Sudborough [7] study one-to-one and one-to-many embeddings of hypercubes into Cayley graphs. Jwo, Lakshmi-varahan, and Dhall [6], Qiu, Meijer, and Akl [9, 10] consider embeddings of grids in stars. Bouabdallah, Heydemann, Opatrny and Sotteau [5] present embeddings of complete binary tree into star networks. Azevedo, Bagherzadeh and Latifi [2] propose embeddings of hypercubes in star graphs.

However, embedding interconnection networks into the pancake has received less attention. In this paper, we focus on the problem and present embeddings of rings, grids and hypercubes into the pancake with constant dilation and congestion.

The paper is organized as follows. In the Preliminaries, we state the definitions and the group-theoretic terminology that are used in this paper. Section 3 is devoted to embeddings of rings and arrays into the pancake. In Section 4, we present embeddings of grids into the pancake and the star, and in Section 5, we consider embeddings of hypercubes (binary and generalized hypercubes) into the pancake; the results are extended to the star. The concluding Section 6 briefly outlines possible improvements and open problems.

2 Preliminaries

Following [1, 4], we first present the group-theoretic model used to design and analyze the pancake. Next we define the pancake network itself.

Definition 1 *Let \mathcal{G} be a finite multiplicative group. Let I be the identity in \mathcal{G} and G a set of generators of \mathcal{G} with the following two properties*

- (i) $(\forall g \in G) \quad g^{-1} \in G$;
- (ii) $I \notin G$.

Given (\mathcal{G}, G) , a Cayley graph (V, E) is defined as a simple graph, whose vertex-set and edge-set are

$$V = \mathcal{G} \quad \text{and} \quad E = \{(u, v) \in V \times V \mid u^{-1}v \in G\}.$$

It is easily seen that Cayley graphs (V, E) are finite, connected, undirected, devoid of multiple edges, loop-free, and symmetric. Since interconnection networks may be viewed as an undirected graph, we will use the terms graph and (interconnection) network interchangeably.

Notation. In the remainder of the paper, we use the usual terminology of basic group theory and graph theory. Since we only consider finite groups, the groups are mainly represented as *permutation groups*. The following notation is used:

- Let X and Y be two sets, $X \setminus Y$ denotes the relative complement of the set Y with respect to the set X .
- S_n is the *symmetric group* on n symbols, i.e. on $\{1, \dots, n\}$ for simplicity. The multiplication in S_n is the composition of permutations.

A permutation $\pi \in S_n$ is denoted by $x_1x_2 \cdots x_n$, where we have $\pi(k) = x_k$ for $k = 1, \dots, n$. This representation is referred to as the *standard representation* of a permutation, to distinguish it from other representations that are introduced further. The identity permutation on n symbols is thus $I = 123 \cdots n$.

- τ denotes the *transposition permutation* and $\sigma(\ell, m)$ denotes the *cyclic permutation* of m positions over the first ℓ symbols:

$$\sigma(\ell, m) = (\ell - m + 1) \cdots \ell 1 \cdots (\ell - m)(\ell + 1) \cdots n.$$

- Symbols are denoted by lower case letters, and *blocks* of symbols by upper case letters. No ambiguity may rise from the notation I , since I is (by definition) the unique block of symbols $123 \cdots n$.
- A permutation $\pi = x_1x_2 \cdots x_{i-1}x_ix_{i+1} \cdots x_n$ can be represented by blocks of symbols instead of its symbols themselves. For example, we can write $\pi = AB$, where $A = x_1x_2 \cdots x_{i-1}x_i$ and $B = x_{i+1} \cdots x_n$.

For any block of symbols A , \bar{A} denotes the block obtained by reversing A . For example, if $\rho = x_ix_{i-1} \cdots x_2x_1x_{i+1} \cdots x_n$, we write $\rho = \bar{A}B$, where A and B are the blocks defined above.

Definition 2 [1]

The *pancake network* P_n of dimension n is the Cayley graph (S_n, E) , whose set of generators is

$$G = \{g_i \in S_n \mid g_i = i(i-1) \cdots 321(i+1) \cdots n, i = 2, \dots, n\}.$$

In other words, the $n!$ vertices of P_n are labeled with the permutations on n symbols (of S_n), and any two vertices of P_n , $u = x_1x_2 \cdots x_n$ and $v = y_1y_2 \cdots y_n$, are connected (i.e. $(u, v) \in E$) iff there exists an integer i , $2 \leq i \leq n$, such that $y_j = x_{i-j+1}$ for $j = 1, \dots, i$, and $y_j = x_j$ for $j > i$.

Clearly, there are $(n - 1)$ generators, one for each value of i , $2 \leq i \leq n$, and $|G| = n - 1$. It is easy to show that the Cayley network P_n has $n!$ vertices, each with degree $|G| = n - 1$: P_n is $(n - 1)$ -regular.

When a permutation ρ is obtained from a permutation $\pi = x_1 \cdots x_n$ by applying a generator $g_i \in G$, we write $x_1 \cdots x_i \cdots x_n \rightarrow \underline{x_i \cdots x_1} \cdots x_n$.

P_n can be decomposed into n subpancakes each of dimension $(n - 1)$. Each of the $(n - 1)!$ vertices of each subpancake has a block representation of the form Ai , where $A \in S_{n-1}$ is a “permutations block” on the $(n - 1)$ symbols $\{1, \dots, n\} \setminus \{i\}$, for a given

$i \in \{1, \dots, n\}$ which depends on the considered subpancake. As a consequence, each of the n subpancakes of P_n (one for each value of i , $1 \leq i \leq n$) can be represented with two distinct notations depending on the context:

1. For a given integer i ($1 \leq i \leq n$), $P_{n-1,i}$ denotes the subpancake defined from the above representation, i.e. i is the last symbol of each vertex/permutation of $P_{n-1,i}$.
2. For a given permutation $\pi \in P_{n-1,i}$ ($1 \leq i \leq n$), $P_{n-1}(\pi)$ denotes the subpancake defined from the element π , i.e. π is the representation of a vertex of that subpancake.

A subpancake of dimension k is called a k -pancake.

Definition 3 *The star network of dimension n (or n -star) is the Cayley graph whose set of generators is $G = \{\tau(1,i) \mid i = 1, \dots, n\}$, where $\tau \in S_n$ is the transposition permutation. In other words, the $n!$ vertices of the n -star are labeled with the permutations on n symbols, and each vertex u of the n -star, $u = x_1x_2 \cdots x_n$ is connected to the $n - 1$ vertices v such that $v = x_i x_2 \cdots x_{i-1} x_1 x_{i+1} \cdots x_n$, for $i = 2, \dots, n$.*

Now recall the definition of an embedding of a graph into another graph.

Definition 4 *Given two undirected connected graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ such that $|V_1| \leq |V_2|$, the embedding of H_1 into H_2 maps V_1 into V_2 . The ratio $|V_2|/|V_1|$ is the expansion of the embedding. The dilation of any edge $(x_1, y_1) \in E_1$, is the length of the path $[x_2, y_2]$, where $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$ in the embedding, respectively. The dilation of the embedding is the maximum over all dilations. The congestion of an edge $(x_2, y_2) \in E_2$ is the number of edges $(x_1, y_1) \in E_1$ whose image by the mapping contains (x_2, y_2) . The congestion of the embedding is the maximum over all congestions.*

3 Embeddings of Rings

In this section we consider rings of size $k!$ ($3 \leq k \leq n$). The dilation and the congestion of embeddings of such rings into P_n are shown to be 1.

Proposition 1 *For any integer i , $2 \leq i \leq n - 1$, and any generator $g_i \in G$ as defined in Definition 2, $\underbrace{g_i g_{i+1} \cdots g_i g_{i+1}}_{i+1} = I$.*

Proof. Let a permutation $\pi = x_1 x_2 \cdots x_{i-1} x_i x_{i+1} \cdots x_n = A x_{i+1} B$. Then $\rho = \pi g_i = \bar{A} x_{i+1} B$, and

$$\pi g_i g_{i+1} = \rho g_{i+1} = x_{i+1} A B = x_{i+1} x_1 x_2 \cdots x_{i-1} x_i x_{i+2} \cdots x_n.$$

In other words, $g_i g_{i+1}$ is the cyclic permutation $\sigma(i+1, 1)$, and since $\sigma(i+1, 1)^{i+1} = I$, the result follows. \square

For any permutation π and any sequence of generators $H = h_1, \dots, h_k$, we denote by (π, H) the corresponding sequence of permutations π_0, \dots, π_k such that $\pi_0 = \pi$ and $\pi_i = \pi_{i-1} h_i$, for all $i = 1, 2, \dots, k$.

Definition 5 For $k = 2, \dots, n$, the pancake sequence G_k of order k is the sequence of generators recursively defined as follows:

- (i) $G_2 = g_2$;
- (ii) for $k > 2$, $G_k = \langle G_{k-1}, g_k, G_{k-1}, g_k, \dots, G_{k-1} \rangle$, where G_{k-1} occurs k times in the sequence.

Proposition 2 Given a permutation $\pi \in S_n$, for $k = 3, \dots, n$, (π, G_k) defines a Hamiltonian cycle over the k -pancake containing π . In this Hamiltonian cycle, the vertices of each subpancake have adjacent locations.

Proof. The proof is by induction on k . Since the pancake is vertex transitive, we assume that $\pi = I$.

• **Base** ($k = 3$): Applying the generators of G_3 yields the following sequence of permutations:

$$I = 123 \cdots n \rightarrow \underline{213} \cdots n \rightarrow \underline{312} \cdots n \rightarrow \underline{132} \cdots n \rightarrow \underline{231} \cdots n \rightarrow \underline{321} \cdots n.$$

It is easily verified that all the elements of $P_3(I)$ belong to the sequence and that the last element of the list is connected to the first one through the generator g_3 .

• **Induction step:** Suppose that (π, G_{n-1}) defines a Hamiltonian cycle over $P_{n-1}(\pi)$. We first show that the permutation obtained by applying $\langle G_{n-1}, g_n, \dots, G_{n-1}, g_n \rangle$ (h times) is $(n-h+1)(n-h+2) \cdots n 12 \cdots (n-h)$. The property holds for $h = 1$ since the permutation is obtained from the sequence

$$12 \cdots (n-1)n \rightarrow \underline{(n-1) \cdots 21n} \rightarrow \underline{n12 \cdots (n-1)}.$$

Let us now suppose that the property holds up to h . The next step is then

$$\begin{aligned} & (n-h+1)(n-h+2) \cdots n 12 \cdots (n-h-1)(n-h) \rightarrow \\ & \underline{(n-h-1) \cdots 21n \cdots (n-h+1)(n-h)} \rightarrow \underline{(n-h) \cdots n 12 \cdots (n-h-1)}. \end{aligned}$$

Therefore, after applying $\langle G_{n-1}, g_n, \dots, G_{n-1}, g_n \rangle$ (h times), the permutation corresponds to a vertex of $P_{n-1}(n-h)$. According to the induction step, the next G_{n-1} visits all the vertices of the subpancake $P_{n-1}(n-h)$. Whence the result that all vertices of all the $(n-1)$ -pancakes in P_n are visited. The last visited vertex is $(2 \cdots n1)g_{n-1} = n \cdots 21$, and this permutation is connected to I through g_n : the proof is completed. \square

In the following, we still let $\pi = I$ (w.l.o.g.). The order relation induced by the sequence (I, G_n) on permutations will be referred to as the ordering of the pancake sequence.

Example. Let $n = 4$. The list of vertices of P_4 ordered by the pancake sequence is: $1234 \rightarrow \underline{2134} \rightarrow \underline{3124} \rightarrow \underline{1324} \rightarrow \underline{2314} \rightarrow \underline{3214} \rightarrow \underline{4123} \rightarrow \underline{1423} \rightarrow \underline{2413} \rightarrow \underline{4213} \rightarrow \underline{1243} \rightarrow \underline{2143} \rightarrow \underline{3412} \rightarrow \underline{4312} \rightarrow \underline{1342} \rightarrow \underline{3142} \rightarrow \underline{4132} \rightarrow \underline{1432} \rightarrow \underline{2341} \rightarrow \underline{3241} \rightarrow \underline{4231} \rightarrow \underline{2431} \rightarrow \underline{3421} \rightarrow \underline{4321}$.

Theorem 1 easily follows.

Theorem 1 *For $k = 3, \dots, n$, the ring of size $k!$ can be embedded into the n -pancake with dilation 1 and congestion 1.*

Proof. Immediate from Proposition 2. Given a Hamiltonian graph of order n , the corresponding ring can be embedded into that graph with dilation and congestion 1. \square

As a consequence of Theorem 1, we also have the

Corollary 1 *For ℓ such that $\ell \leq n!$, the linear array (line) of length ℓ can be embedded into the n -pancake with dilation 1 and congestion 1.*

4 Embeddings of Grids

4.1 Embeddings of $N_1 \times N_2$ Grids

Given any two positive integers N_1 and N_2 , we first consider embeddings of $N_1 \times N_2$ grids with $N_1 N_2 \leq n!$ into P_n and give a negative result.

Proposition 3 *The $N_1 \times N_2$ grid is not a subgraph of the n -pancake.*

Proof. The proof is by contradiction. The 2×2 grid is a subgraph of the $N_1 \times N_2$ grid. Suppose the 2×2 grid were a subgraph of P_n , then there would be two permutations X and Y , and four generators g_i, g_j, g_ℓ, g_k , with $i \neq j, i \neq k, k \neq \ell$, such that $Y = X g_i$ and $Y g_k = X g_j g_\ell$. Hence, $g_i g_k = g_j g_\ell$, which would imply that $j = \ell$ and $k = i$, or $j = i$ and $\ell = k$: a contradiction. \square

Lemma 1 *For any two integers ℓ and m such that $0 \leq m \leq \ell \leq n$, the cyclic permutation $\sigma(\ell, m)$ can always be built with two or three generators of the pancake.*

Proof. Let a permutation $\pi = x_1 \cdots x_{\ell-m} x_{\ell-m+1} \cdots x_\ell x_{\ell+1} \cdots x_n = ABC$, with blocks $A = x_1 \cdots x_{\ell-m}, B = x_{\ell-m+1} \cdots x_\ell$ and $C = x_{\ell+1} \cdots x_n$. Then, $\pi \sigma(\ell, m) = x_{\ell-m+1} \cdots x_\ell x_1 \cdots x_{\ell-m} x_{\ell+1} \cdots x_n = BAC$, and we have the following path joining π to $\pi \sigma(\ell, m)$: $\pi = ABC \rightarrow \bar{A}BC \rightarrow \bar{B}AC \rightarrow BAC = \pi \sigma(\ell, m)$. The length of this path is 3 when $1 < m < \ell - 1$, and it is 2 whenever $m = 1$ or $m = \ell - 1$. \square

Now from Lemma 1 we present an embedding of the $n \times (n - 1)!$ grid in P_n with constant dilation.

Theorem 2 *The $n \times (n - 1)!$ grid can be embedded in the n -pancake with dilation 7.*

Proof. The first row of the grid is represented by the first $(n - 1)$ -pancake ordered from the pancake sequence. For $0 \leq j \leq (n - 1)! - 1$, let π_j be the vertex of the pancake corresponding to the node $(0, j)$ on the grid. A node (i, j) , with $i \neq 0$, is represented by $\pi_j \sigma(n, i)$. Now, considering two adjacent nodes on the grid, let us compute the distance between those vertices of the pancake that represent them.

- Two nodes $(0, j)$ and $(0, j + 1)$ are represented by two adjacent vertices of the pancake.
- Two nodes (i, j) and $(i + 1, j)$ are represented by the two vertices $X = \pi_j \sigma(n, i)$ and $Y = \pi_j \sigma(n, i + 1) = X \sigma(n, 1)$. According to Lemma 1, the distance between X and Y is 2.
- Two nodes (i, j) and $(i, j + 1)$ are represented by the two vertices $Y_1 = \pi_j \sigma(n, i)$ and $Y_2 = \pi_{j+1} \sigma(n, i)$. Then, for a generator $g_k, \pi_{j+1} = \pi_j g_k$. To compute the distance between Y_1 and Y_2 , two distinct cases (and two subcases) may arise:
 - **First case:** $\pi_j = ABC, \pi_{j+1} = \bar{A}BC$ and $Y_1 = CAB$. Then, $Y_2 = C\bar{A}B$, and a path joining Y_1 to Y_2 is $CAB \rightarrow \bar{A}\bar{C}B \rightarrow A\bar{C}B \rightarrow C\bar{A}B$. The distance from Y_1 to Y_2 is 3.

– **Second case:** $\pi_j = ABC$, $\pi_{j+1} = \bar{B}\bar{A}\bar{C}$, and $Y_1 = BCA$. To obtain Y_2 , two subcases must be considered.

- **First subcase:** $Y_2 = \bar{B}_1\bar{A}\bar{C}\bar{B}_2$. In this subcase, a path from Y_1 to Y_2 is $Y_1 = B_1B_2CA \rightarrow \bar{B}_2\bar{B}_1CA \rightarrow B_2\bar{B}_1CA \rightarrow \bar{A}\bar{C}B_1\bar{B}_2 \rightarrow CAB_1\bar{B}_2 \rightarrow \bar{C}AB_1\bar{B}_2 \rightarrow \bar{B}_1\bar{A}\bar{C}\bar{B}_2 = Y_2$. The distance from Y_1 to Y_2 is 6.
- **Second subcase:** $Y_2 = \bar{A}_1\bar{C}\bar{B}\bar{A}_2$. In this last subcase a path from Y_1 to Y_2 is $Y_1 = BCA_1A_2 \rightarrow \bar{A}_2\bar{A}_1\bar{C}\bar{B} \rightarrow A_2\bar{A}_1\bar{C}\bar{B} \rightarrow BCA_1\bar{A}_2 \rightarrow \bar{C}\bar{B}A_1\bar{A}_2 \rightarrow C\bar{B}A_1\bar{A}_2 \rightarrow B\bar{C}A_1\bar{A}_2 \rightarrow \bar{A}_1\bar{C}\bar{B}\bar{A}_2 = Y_2$.

This last configuration yields a (maximal) distance 7 from Y_1 to Y_2 . □

The same method applies to the $(n + (n - 2) + (n - 3) + \dots + (p + 1)) \times p!$ grid, for $p = 2, \dots, n - 1$; the following Theorem 3 shows that this grid can be embedded into P_n with constant dilation. Note that the term $(n - 1)$ is omitted in the definition of the grid. Indeed, by Proposition 3, we already know that the $(n + (n - 1) + (n - 2) + \dots + (p + 1)) \times p!$ grid ($2 \leq p \leq n - 1$) is not a subgraph of P_n .

Theorem 3 *For $p = 2, \dots, n - 1$, the $(n + (n - 2) + (n - 3) + \dots + (p + 1)) \times p!$ grid can be embedded in the n -pancake with dilation 4.*

Proof. The first row of the mesh is represented by the first p -pancake. The next $(n - 1)$ rows are obtained by applying the cyclic permutations $\sigma(n, i)$ to the first row. The next $(n - 2)$ rows are obtained by applying the cyclic permutations $\sigma(n - 1, i)$ to the first row, etc. Finally, the last p rows are obtained by applying the cyclic permutations $\sigma(p + 1, i)$ to the first row. The only new adjacent nodes to consider are $X\sigma(k, k - 1)$ and $X\sigma(k - 1, 1)$. Let $X = x_1Ax_{k-1}x_kB$. Then, $X\sigma(k, k - 1) = Ax_{k-1}x_kx_1B$ and $X\sigma(k - 1, 1) = x_{k-1}x_1Ax_kB$. A path joining these two vertices is

$$\begin{aligned} Ax_{k-1}x_kx_1B &\rightarrow x_kx_{k-1}\bar{A}x_1B \rightarrow x_1Ax_{k-1}x_kB \\ &\rightarrow \bar{A}x_1x_{k-1}x_kB \rightarrow x_{k-1}x_1Ax_kB. \end{aligned}$$

The distance between the two vertices is thus 4 and the proof follows. □

4.2 Embeddings of n -Grids

For embedding n -grids into P_n , a new representation of permutations is first introduced. A permutation π may be represented as $\pi = a_2a_3 \dots a_n$, where a_i is the number of symbols less than i that are located at the left of i in the standard representation of π .

Example. Let $n = 5$. The permutation 12345 is represented by 1234, and the permutation 54321 is represented by 0000. Similarly, the permutation 42153 is represented by 0203.

The map $S_n \longrightarrow \{\pi = a_2, \dots, a_n \mid 0 \leq a_i \leq i - 1\}$ is obviously one-one, and it is used in this subsection to embed the $2 \times 3 \times \dots \times (n - 1) \times n$ grid into P_n .

Theorem 4 *The $2 \times 3 \times \dots \times (n - 1) \times n$ grid can be embedded into the n -pancake with dilation 6.*

Proof. Let two vertices X and Y on the grid be labeled with the two permutations $a_2 \dots a_{i-1} \alpha a_{i+1} \dots a_n$ and $a_2 \dots a_{i-1} \beta a_{i+1} \dots a_n$, respectively. X and Y are connected on the grid iff $\alpha = \beta + 1$ or $\alpha = \beta - 1$. Let us find the distance from X to Y in the pancake.

We may assume w.l.o.g. that $\alpha = \beta + 1$. Let $X = A x_k B i C$, where x_k is the α th symbol $< i$ and all symbols in B are $> i$. Consider the permutation $A i B x_k C = b_2 \dots b_n$ and compare the a_j s and the b_j s.

First, for all symbols in A and C , $b_j = a_j$. Next, for each symbol j in B , a new symbol that is smaller than j is located on the left of j : it is the symbol i . Similarly, a new symbol that is smaller than j is located on the right of j : it is the symbol x_k . Hence, $b_j = a_j + 1 - 1 = a_j$. New symbols located on the left of x_k are larger than x_k ; they are either the symbol i or any symbol $j > i$, and hence, $b_{x_k} = a_{x_k}$. There is only one new symbol smaller than i located on the right of i : it is the symbol x_k , other symbols are larger than i . Hence, $b_i = a_i - 1 = \alpha - 1 = \beta$.

Therefore, $A i B x_k C = Y$, and a path joining X to Y is

$$\begin{aligned} X = A x_k B i C &\rightarrow x_k \bar{A} B i C \rightarrow i \bar{B} A x_k C \rightarrow B i A x_k C \\ &\rightarrow \bar{B} i A x_k C \rightarrow \bar{A} i B x_k C \rightarrow A i B x_k C = Y. \end{aligned}$$

The distance from X to Y is thus 6, and the dilation follows. \square

The following corollary is easily derived.

Corollary 2 *The binary hypercube Q_n can be embedded in the n -pancake with dilation 6.*

Proof. Q_n is a subgraph of the $2 \times 3 \times \dots \times (n - 1) \times n$ grid. \square

The same method applies to embed the n -grid into the star graph (see Definition 3).

Theorem 5 *The $2 \times 3 \times \dots \times (n-1) \times n$ grid can be embedded into the n -star with dilation 3.*

Proof. The above representation of permutations is used again. Consider two permutations $X = aAxByC$ and $Y = aAyBxC$, and compute the distance from X to Y within the n -star. A path joining X to Y is

$$X = aAxByC \rightarrow yAxBaC \rightarrow xAyBaC \rightarrow aAyBxC = Y.$$

The distance from X to Y is thus 3, and the dilation follows. \square

5 Embeddings of the Generalized Hypercube

The $2 \times 3 \times \dots \times (n-1) \times n$ *generalized hypercube (GHC)* [3] is the graph (V, E) whose vertices are labeled with the permutations x_2, \dots, x_n , where $0 \leq x_i \leq i-1$. Any two vertices $u, v \in V$ are connected iff their labels differ in only one position: i.e. there is an edge $(u, v) \in E$ between the two vertices $u = x_2, \dots, x_{i-1}, \alpha, x_{i+1}, \dots, x_n$ and $v = x_2, \dots, x_{i-1}, \beta, x_{i+1}, \dots, x_n$ iff $\alpha \neq \beta$ for some symbols α and β from x_2 onwards.

Embedding the GHC into the n -pancake could be performed via the representation of permutations defined in the previous section. Unfortunately, the resulting dilation is $O(n)$, i.e. the dilation would then have the same order of magnitude as the diameter of the pancake. Consequently, a more suited representation of permutations must be used.

A permutation $\pi = x_1 \dots x_n$ is represented by $a_2 \dots a_n$ with the following rule **(R)**:

for $k = n$ **to** 2 (step -1) **do**

$$a_k \leftarrow x_k - 1 ; x_k \leftrightarrow k \quad (\text{i.e. exchange symbols } x_k \text{ and } k \text{ in permutation } \pi).$$

Example. Let $n = 8$ and $X = 27351864$. Applying rule **(R)** step by step yields

$$\begin{aligned} a_8 &= 4 - 1 = 3, & \text{and } Y_1 &= 27351468, \\ a_7 &= 6 - 1 = 5, & \text{and } Y_2 &= 26351478, \\ a_6 &= 4 - 1 = 3, & \text{and } Y_3 &= 24351678, \\ a_5 &= 1 - 1 = 0, & \text{and } Y_4 &= 24315678, \\ a_4 &= 1 - 1 = 0, & \text{and } Y_5 &= 21345678, \\ a_3 &= 3 - 1 = 2, & \text{and } Y_6 &= 21345678, \\ a_2 &= 1 - 1 = 0, & \text{and } Y_7 &= 12345678. \end{aligned}$$

The representation of X is 0200353.

Proposition 4 *The representation given above defines a one-one mapping between the n -pancake and the $2 \times 3 \times \dots \times (n-1) \times n$ generalized hypercube.*

Proof. Let $a_2, \dots, a_p, p, \dots, n-1$ ($2 \leq p \leq n-1$) denote the above representation of a permutation. The proof is by induction on p .

• **Base:** Let $p = 2$ and $X = x_1 x_2 3 4 \dots (n-1)n$ be a permutation. In that case, $\{x_1, x_2\} = \{1, 2\}$. If $x_1 = 1$, from rule **(R)** we have $a_i = i - 1$ for each value of i (in particular, $a_2 = 1$). Similarly, if $x_1 = 2$ we have $a_2 = 0$. Hence, the property holds for $p = 2$.

• **Induction step:** Given p ($2 \leq p \leq n-1$), suppose the map from the set of permutations $\{x_1 \dots x_p (p+1) \dots n\}$ onto the subgraph of the GHC defined by $\{a_2, \dots, a_p, p, \dots, n-1\}$ is one-to-one. Let us prove that the property also holds for $(p+1)$.

Consider the permutations $X = x_1 \dots x_p x_{p+1} (p+2) \dots n$, and notice that the symbol $(p+1)$ belongs to the set $\{x_1, \dots, x_p x_{p+1}\}$. According to rule **(R)**, the representation of X is constructed step by step, from $i = n$ down to $i = 2$, by performing $a_i = x_i - 1$ and exchanging symbols x_i and i within X . Therefore, x_{p+1} and $(p+1)$ are exchanged in the representation of X and a_{p+1} takes all the values $0, 1, \dots, p$.

Since X is of the form $x_1 \dots x_p (p+1) \dots n$, and according to the induction step, the values a_i ($1 < i < p+1$) cover the whole set $\{0, \dots, i-1\}$. Whence the property holds for $(p+1)$. \square

Now, let $X = a_2 \dots a_{i-1} \alpha a_{i+1} \dots a_n$ and $Y = a_2 \dots a_{i-1} \beta a_{i+1} \dots a_n$ be the above representations of two permutations. To find the distance from X to Y within the pancake P_n we need Lemma 2 first.

Lemma 2 *Let two permutations X and Y denoted $a_2 \dots a_{i-1} \alpha a_{i+1} \dots a_n$ and $a_2 \dots a_{i-1} \beta a_{i+1} \dots a_n$, respectively. Their standard representations differ in at most three positions, i.e. $X = AxByCzD$ and $Y = AzBxCyD$.*

Proof. Consider two permutations W and Z , such that $W = AxBpC$, where the symbol p is located at position p , with $p > x$, and $Z = ApBxC$. Let us compare each of the respective representations of W and Z .

In each representation, the values of a_i corresponding to C are equal; the values of a_p are $(p-1)$ and $x-1$, and for $i < p$, a_i is obtained from rule **(R)**. The values of the a_i s within each representation of W and Z are equal to the symbols in two permutations π_W and π_Z (respectively), each obtained by applying rule **(R)**. Now, this construction of π_W and π_Z yields $\pi_W = \pi_Z = ExFp(p+1) \dots n$, where E and F are two blocks of symbols in $\{1, \dots, x\} \setminus \{x, p, \dots, n\}$. Hence, the representations of W and Z differ in one position only and, for a given X , there are $(p-1)$ such Y s. The standard representations of two such Y s differ in three positions, and the proof follows. \square

The following Theorem 6 derives from Lemma 2.

Theorem 6 *The $2 \times 3 \times \cdots \times (n - 1) \times n$ generalized hypercube can be embedded into the n -pancake with dilation 8.*

Proof. Let two permutations X and Y , whose representations differ in one position only. According to Lemma 2, each of their standard representations differs in at most three positions, i.e. $X = AxByCzD$ and $Y = AzBxCyD$. A path joining X to Y is thus

$$\begin{aligned} X = AxByCzD &\rightarrow z\bar{C}y\bar{B}x\bar{A}D \rightarrow xByCz\bar{A}D \rightarrow \bar{B}xyCz\bar{A}D \rightarrow \\ &BxyCz\bar{A}D \rightarrow \bar{C}yx\bar{B}z\bar{A}D \rightarrow Cyx\bar{B}z\bar{A}D \rightarrow y\bar{C}x\bar{B}z\bar{A}D \rightarrow AzBxCyD = Y, \end{aligned}$$

and the dilation follows. \square

Corollary 3 *Let $d = 1 + \lceil \lg 3 \rceil + \cdots + \lceil \lg(n - 1) \rceil + \lceil \lg n \rceil$ be the dimension of the binary hypercube Q_d . Q_d can be embedded into the n -pancake with dilation 8.*

Proof. Q_d is a subgraph of the $2 \times 3 \times \cdots \times (n - 1) \times n$ generalized hypercube. \square

The latter representation of permutations yields an embedding of the GHC into the star graph.

Theorem 7 *The $2 \times 3 \times \cdots \times (n - 1) \times n$ generalized hypercube can be embedded into the star graph of dimension n with dilation 4.*

Proof. Again, the above representation of permutations is used. Let two permutations X and Y , wherein at most three symbols have not the same location, i.e. $X = aAxByCzD$ and $Y = aAzBxCyD$. A path joining X to Y in the n -star is thus

$$\begin{aligned} X = aAxByCzD &\rightarrow yAxBaCzD \rightarrow zAxBaCyD \\ &\rightarrow xAzBaCyD \rightarrow aAzBxCyD = Y, \end{aligned}$$

and the result follows. \square

This last theorem improves on the result presented in [8]. Indeed, Nigam *et al.* show that the *binary* hypercube can be embedded into the n -star with dilation 4. Since the binary hypercube is a subgraph of the GHC, Theorem 7 generalizes that result.

6 Conclusion

We presented embeddings of rings, grids, and hypercubes into the pancake interconnection network. All embeddings have constant dilations, and some of them lead to similar results into the star graph. Possible improvements on the above results are twofold.

1. The embedding capabilities offered by the pancake interconnection network are very restrictive. In the present paper, the only embeddings of grids that are considered have size $n \times (n-1)!$ and $(n + (n-2) + (n-3) + \dots + (p+1)) \times p!$, for $p = 2, \dots, n-1$. Finding embeddings of $N_1 \times N_2$ grids for all pairs (N_1, N_2) such that $N_1 N_2 \leq n!$ would be a much more general result.
2. Some embeddings presented in the paper have congestion $O(n)$. A class of problems of the following kind remains open: find embeddings of the same interconnection networks with constant dilation and congestion, or else, show that such embeddings do not exist.

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