

Sets of Mutually Unbiased Bases as Arcs in Finite Projective Planes?

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Abstract

This note is a short conceptual elaboration of the conjecture of Saniga *et al* (J. Opt. B: Quantum Semiclass. **6** (2004) L19-L20) by regarding a set of mutually unbiased bases (MUBs) in a d -dimensional Hilbert space as an analogue of an arc in a (finite) projective plane of order d . Complete sets of MUBs thus correspond to $(d+1)$ -arcs, i.e., ovals. In the Desarguesian case, the existence of two principally distinct kinds of ovals for $d = 2^n$ and $n \geq 3$, viz. conics and non-conics, implies the existence of two qualitatively different groups of the complete sets of MUBs for the Hilbert spaces of corresponding dimensions. A principally new class of complete sets of MUBs are those having their analogues in ovals in non-Desarguesian projective planes; the lowest dimension when this happens is $d = 9$.

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It has for a long time been suspected but only recently fully recognized [1–4] that finite (projective and related) geometries may provide us with important clues for solving the problem of the maximum cardinality of MUBs for Hilbert spaces of finite dimensions d . It is well-known [5,6] that this number cannot be greater than $d+1$ and that this limit is reached if d is a power of a prime. Yet, a still unanswered question is if there are non-prime-power values of d for which this bound is attained. On the other hand, the minimum number of MUBs was found to be three for all dimensions $d \geq 2$ [7]. Motivated by these facts, Saniga *et al* [1] have conjectured that the question of the existence of the maximum, or complete, sets of MUBs in a d -dimensional Hilbert space if d differs from a prime power is intricately connected with the problem of whether there exist projective planes whose order d is not a power of a prime. This note aims at getting a deeper insight into this conjecture by introducing particular objects in a finite projective plane, the so-called ovals, which can be viewed as geometrical analogues of complete sets of MUBs.

We shall start with a more general geometrical object of a projective plane, viz. a k -arc – a set of k points, no three of which are collinear [see, e.g., 8,9]. From the definition it immediately follows that $k = 3$ is the minimum cardinality of such an object. If one requires, in addition, that there is at least one tangent (a line meeting it in a single point only) at each of its points, then the maximum cardinality of a k -arc is found to be $d+1$, where d is the order of the projective plane [8,9]; these $(d+1)$ -arcs are called *ovals*. It is striking to observe that such k -arcs in a projective plane of order d and MUBs of a d -dimensional Hilbert space have the *same* cardinality bounds. Can, then, individual MUBs (of a d -dimensional Hilbert space) be simply viewed as points of some abstract projective plane (of order d) so that their basic combinatorial properties are qualitatively encoded in the geometry of k -arcs? A closer inspection of the algebraic geometrical properties of ovals suggests that this may indeed be the case.

To this end in view, we shall first show that every proper (non-composite) conic in $PG(2, d)$, a (Desarguesian) projective plane over the Galois field $GF(d)$, is an oval. A conic is the curve of second order

$$\mathcal{Q} : \sum_{i \leq j} c_{ij} z_i z_j = 0, \quad i, j = 1, 2, 3, \quad (1)$$

where c_{ij} are regarded as fixed quantities and z_i as variables, the so-called homogeneous coordinates of the projective plane. The conic is degenerate (composite) if there exists a change of the coordinate system reducing Eq. (1) into a form of fewer variables; otherwise, the conic is proper (non-degenerate). It is well-known [see, e.g., 8] that the equation of any *proper* conic in $PG(2, d)$ can be brought into the canonical form

$$\tilde{\mathcal{Q}} : z_1 z_2 - z_3^2 = 0. \quad (2)$$

From the last equation it follows that the points of $\tilde{\mathcal{Q}}$ can be parametrized as $\varrho z_i = (\sigma^2, 1, \sigma)$, $\varrho \neq 0$, and this implies that a proper conic in $PG(2, d)$ contains $d+1$ points; the point $(1, 0, 0)$ and d other points specified by the sequences $(\sigma^2, 1, \sigma)$ as the parameter σ runs through the d elements of $GF(d = p^n)$, p being a prime and n a positive integer. Moreover, it can easily be verified that any triple of distinct points of $\tilde{\mathcal{Q}}$ are linearly independent (i.e., not on the same line), as [10]

$$\det \begin{pmatrix} 1 & 0 & 0 \\ \sigma_1^2 & 1 & \sigma_1 \\ \sigma_2^2 & 1 & \sigma_2 \end{pmatrix} = \sigma_2 - \sigma_1 \neq 0 \quad (3)$$

and

$$\det \begin{pmatrix} \sigma_1^2 & 1 & \sigma_1 \\ \sigma_2^2 & 1 & \sigma_2 \\ \sigma_3^2 & 1 & \sigma_3 \end{pmatrix} = (\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1) \neq 0. \quad (4)$$

Hence, a proper conic of $PG(2, d)$ is indeed an oval. The converse statement is, however, true for d odd only; for d even and greater than four there also exist ovals which are *not* conics [8–11]. In order to see this explicitly, it suffices to recall that all the tangents to a proper conic \mathcal{Q} of $PG(2, d = 2^n)$ are concurrent, i.e., pass via one and the same point, called the nucleus [8–11]. So, the conic \mathcal{Q} together with its nucleus form a $(d+2)$ -arc. Deleting from this $(d+2)$ -arc a point belonging to \mathcal{Q} leaves us with an oval which shares $d = 2^n$ points with \mathcal{Q} . Taking into account that a proper conic is uniquely specified by *five* of its points, it then follows that such an oval cannot be a conic if $n \geq 3$; for, indeed, if it were then it would have with \mathcal{Q} more than five points in common and would thus coincide with it, a contradiction.

Let us rephrase these findings in terms of the above-introduced MUBs – k -arcs analogy. We see that whilst for any $d = p^n$ there exist complete sets (c-sets for short) of MUBs having their counterparts in proper conics, $d = 2^n$ with $n \geq 3$ also feature c-sets whose analogues are ovals which are not conics. In other words, our analogy implies that MUBs do not behave the same way in odd and even (power-of-prime) dimensions. And this is, indeed, the property that at the *number theoretical* level has been known since the seminal work of Wootters and Fields [5, see also 7], being there intimately linked with the fact that so-called Weil sums

$$\left| \sum_{k \in GF(p^n)} e^{\frac{2\pi i}{p} \text{Tr}(mk^2+nk)} \right|, \quad (5)$$

with $m, n \in GF(p^n)$ and the absolute trace operator “Tr” defined as

$$\text{Tr}(\eta) \equiv \eta + \eta^p + \eta^{p^2} + \dots + \eta^{p^{n-1}}, \quad \eta \in GF(p^n), \quad (6)$$

are non-zero (and equal to $\sqrt{p^n}$) for all $p > 2$, playing thus a key role for proving the mutual unbiasedness in these cases, but vanish for $p = 2$ [see, e.g., 12]. In the light of our analogy, this difference acquires a qualitatively new, and more refined, algebraic-geometrical contents/footing. Remarkably, this refinement concerns especially even (2^n) dimensions, as we shall demonstrate next.

In the example above, we constructed a particular kind of an oval by adjoining to a proper conic its nucleus and then removing a point of the conic; such an oval, called a pointed-conic, was shown to be inequivalent to a conic for $n \geq 3$. However, for $n \geq 4$ there exists still another type of non-conic ovals, termed irregular ones, that cannot be constructed this way [see, e.g., 8,11,13]. This intriguing hierarchy of oval’s types is succinctly summarized in the following table:

n	1	2	3	≥ 4
ordinary conic	yes	yes	yes	yes
pointed-conic	no	no	yes	yes
irregular oval	no	no	no	yes

Pursuing our analogy to the extreme, one observes that whereas $d = 2$ and $d = 4$ can accommodate only one kind of c-sets of MUBs, viz. those present also in odd dimensions and having their

counterparts in ordinary conics, $d = 8$ should already feature two different types and Hilbert spaces of $d \geq 16$ should be endowed with as many as three qualitatively different kinds of such sets. So, if this analogy holds, a new MUBs' physics is to be expected to emerge at the three-qubit level and become fully manifested for four- and higher-order-qubit states/configurations.

Finally, we shall briefly address the non-Desarguesian case. We start with an observation that the definition of an oval is expressed in purely combinatorial terms and so it equally well applies to finite *non*-Desarguesian planes. These planes, however, do not admit coordinatization in terms of any Galois field [14–16]; hence, the c -sets of MUBs corresponding to ovals in such planes must fundamentally differ from “Desarguesian” sets. The lowest order for which non-Desarguesian planes were found to exist is $d = 9$, and there are even three distinct kinds of them; this means that it is also two-qutrit states whose properties merit a careful inspection.¹ The most tantalizing aspect of this analogy is, however, the case where d is composite (i.e., not a prime power) because such projective planes, if they exist, must necessarily be non-Desarguesian [14,15]. So, if there exist c -sets of MUBs for d composite, their properties cannot be described in terms of *fields*; instead, one has to employ a more abstract concept, that of (planar) *ternary rings*, as these are proper systems for charting non-Desarguesian projective planes [15,16]. And this is perhaps the most serious implication of our approach and a serious challenge for further geometrically-oriented explorations of MUBs, especially given an important role that MUBs start playing in current quantum cryptographic schemes/protocols and quantum information theory in general.

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¹It is a really intriguing fact to realize here that the two smallest non-trivial dimensions our approach singles out, viz. $d = 8 = 2^3$ and $d = 9 = 3^2$, are precisely those (product dimensions) where the so-called *unextendible product bases* (UPBs) first appear [see, e.g., 17,18]. This indicates that our oval geometries may underlie a wider spectrum of finite-dimensional quantum structures than sole MUBs.