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On the Hedging of American Options in Discrete Time Markets with Proportional Transaction Costs

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Abstract

In this note, we consider a general discrete time financial market with proportional transaction costs as in [4], [5], [6] and [10]. We provide a dual formulation for the set of initial endowments which allow to super-hedge some American claim. This extends the results of [1] which was obtained in a model with constant transaction costs and risky assets which evolve on a finite dimensional tree. We also provide fairly general conditions under which the expected formulation in terms of stopping times does not work.

Key words : Transaction cost, American option.

MSC Classification (2000): 91B28, 60G40.

1 Introduction and main result

Set $\mathbb{T} = \{0, \ldots, T\}$ for some $T \in \mathbb{N} \setminus \{0\}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is trivial. Given an integer $d \geq 1$, we denote by $\mathcal{K}$ the set of $\mathcal{C}$-valued processes K such that $\mathbb{R}^d_+ \setminus \{0\} \subset \text{int}(K_t)$ $\mathbb{P}-$a.s. for all $t \in \mathbb{T}$.[1]

[1]Here, we follow [6] and say that a sequence of set-valued mappings $(K_t)_{t \in \mathbb{T}}$ is a $\mathcal{C}$-valued process if there is a countable sequence of $\mathbb{R}^d$-valued $\mathbb{F}$-adapted processes $X^n = (X^n_t)_{t \in \mathbb{T}}$ such that, for every $t \in \mathbb{T}$, $\mathbb{P}-$a.s. only a finite but non-zero number of X^n_t is different from zero and $K_t = \text{cone}\{X^n_t\ ,\ n \in \mathbb{N}\}$. This means that K_t is the polyhedral cone generated by the $\mathbb{P}-$a.s. finite set $\{X^n_t\ ,\ n \in \mathbb{N}\ \text{ and }\ X^n_t \neq 0\}$.

Following the modelization of [5], for a given $K \in \mathcal{K}$ and $x \in \mathbb{R}^d$, we define the process $V^{x,\xi}$ by

$$V_t^{x,\xi} \ := \ x + \sum_{s=0}^{t} \xi_s \ , \ t \in \mathbb{T} \ ,$$

where ξ belongs to

$$\mathcal{A}(K) = \{ \ \xi \in L^0(\mathbb{R}^d;\mathbb{F}) \ \text{ s.t. } \ \xi_t \ \in \ -K_t \qquad \mathbb{P}-\text{a.s.} \quad \text{for all } \ t \in \mathbb{T}\} \ ,$$

and, for a random set $E \subset \mathbb{R}^d$ $\mathbb{P}-$a.s. and $\mathcal{G} \subset \mathcal{F}$, $L^0(E;\mathbb{F})$ (resp. $L^0(E;\mathcal{G})$) is the collection of $\mathbb{F}$-adapted processes (resp. $\mathcal{G}$-measurable variables) with values in E $\mathbb{P}-$a.s.

The financial interpretation is the following: x is the initial endowment in units of the financial assets, ξ_t is the amount of units of assets which is exchanged at t and $-K_t$ is the set of affordable exchanges. We refer to [5] and [6] for a more detailed description.

Therefore,

$$A(x;K) \ := \ \left\{ V^{x,\xi}, \ \xi \in \mathcal{A}(K) \right\}$$

stands for the set of all portfolio processes with initial endowment x, and

$$A_t(x;K) \ := \ \{V_t, \ V \in A(x;K)\}$$

corresponds to the collection of their values at time $t \in \mathbb{T}$.

It is known from the work of [4], [5] [6] and [10], see also the references therein, that, under mild assumptions, the set $A_T(x;K)$ can be written as

$$\left\{ g \in L^0(\mathbb{R}^d;\mathcal{F}) \ : \ \mathbb{E}\left[Z_T \cdot g - Z_0 \cdot x\right] \leq 0, \ \text{ for all } \ Z \in \mathcal{Z}(K), \ (Z \cdot g)^- \in L^1(\mathbb{R};\mathbb{P}) \right\}$$

where $\mathcal{Z}(K)$ is the set of $(\mathbb{F},\mathbb{P})$-martingales Z such that

$$Z_t \ \in \ K_t^* \qquad \mathbb{P}-\text{a.s.} \quad \text{for all } \ t \in \mathbb{T} \ ,$$

and $K_t^*(\omega)$ denotes the positive polar of $K_t(\omega)$, i.e.

$$K_t^*(\omega) \ := \ \left\{ y \in \mathbb{R}^d \ : \ x \cdot y \geq 0 \ , \ \text{ for all } x \in K_t(\omega) \right\} \ .$$

The operator "$\cdot$" denotes the natural scalar product on $\mathbb{R}^d$ and $L^1(E;\mathbb{P})$ (resp. $L^1(E;\mathbb{F},\mathbb{P})$) is the subset of $\mathbb{P}$-integrable elements of $L^0(E;\mathcal{F})$ (resp. $L^0(E;\mathbb{F})$).

In this paper, we are interested in

$$A^s(x;K) \ := \ \left\{ \vartheta \in L^0(\mathbb{R}^d;\mathbb{F}) \ : \ V - \vartheta \in -\mathcal{A}(K) \text{ for some } V \in A(x;K) \right\} \ ,$$

the set of processes which are dominated by a portfolio in the sense of K: $V_t - \vartheta_t \in K_t$, for all $t \in \mathbb{T}$, $\mathbb{P}$ − a.s. More precisely, our aim is to provide a dual formulation for

$$\Gamma(\vartheta; K) \quad := \quad \left\{ x \in \mathbb{R}^d \ : \ \vartheta \in A^s(x; K) \right\} .$$

In analogy with the standard result for markets without transaction cost, one could expect that $\Gamma(\vartheta; K)$ can admit the dual formulation

$$\Theta(\vartheta; K) = \left\{ x \in \mathbb{R}^d \ : \ \sup_{\tau \in \mathcal{T}(\mathbb{T})} \mathbb{E}\left[Z_\tau \cdot \vartheta_\tau - Z_0 \cdot x \right] \leq 0 \ , \quad \text{for all} \ \ Z \in \mathcal{Z}(K) \right\} \tag{1.1}$$

where $\mathcal{T}(\mathbb{T})$ is the set of all $\mathbb{F}$-stopping times with values in $\mathbb{T}$. However, this characterization does not hold true in general, as shown in the following section. This phenomenon was already pointed out in [1] in a model consisting of one bank account and one risky asset evolving on a finite dimensional tree. In [1], the authors show that a correct dual formulation can be obtained if we replace stopping times by randomized stopping times.

In our general framework, this amounts to introduce a new set of dual variables. For $\tilde{\mathbb{P}} \sim \mathbb{P}$, the associated set of dual variables, $\mathcal{D}(K, \tilde{\mathbb{P}})$, is defined as the collection of process $Z \in L^1(\mathbb{R}^d; \mathbb{F}, \tilde{\mathbb{P}})$ such that

$$Z_t \in K_t^* \ \text{ and } \ \bar{Z}_t \quad := \quad \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{s=t}^{T} Z_s \mid \mathcal{F}_t \right] \in K_t^* \ \ \mathbb{P} - \text{a.s.} \quad \text{for all} \ \ t \in \mathbb{T} \, .$$

In the following, we shall say that a subset B of $L^0(\mathbb{R}^d; \mathbb{F})$ is *closed in measure* if it is closed in probability when identified as a subset of $L^0(\mathbb{R}^{d \times (T+1)}; \mathcal{F})$, i.e.

$$v^n \in B \text{ and } \forall \varepsilon > 0 \quad \lim_{n \to \infty} \mathbb{P}\left[\sum_{t \in \mathbb{T}} \| v_t^n - v_t \| > \varepsilon \right] = 0 \quad \Longrightarrow \quad v \in B \, .$$

We then have the following characterization of $A^s(K) := A^s(0; K)$.

Theorem 1.1 *Assume that $A^s(K)$ is closed in measure and that the no-arbitrage condition*

$$NA(K) \ : \quad A_T(0; K) \cap L^0(\mathbb{R}_+^d; \mathcal{F}) \quad = \quad \{0\}$$

holds. Then, the following assertions are equivalent :

(i) $\vartheta \in A^s(K)$

(ii) for all $\tilde{\mathbb{P}} \sim \mathbb{P}$ *and* $Z \in \mathcal{D}(K, \tilde{\mathbb{P}})$ *such that* $(\vartheta \cdot Z)^- \in L^1(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$ *we have*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t\right] \quad \leq \quad 0 \, .$$

(iii) for some $\tilde{\mathbb{P}} \sim \mathbb{P}$ *we have*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t\right] \quad \leq \quad 0$$

for all $Z \in \mathcal{D}(K, \tilde{\mathbb{P}})$ *such that* $(\vartheta \cdot Z)^- \in L^1(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$.

Since $A^s(0; K) = A^s(x; K) - x$, this immediately provides a dual formulation for $\Gamma(\vartheta; K)$.

Corollary 1.1 *Let the conditions of Theorem 1.1 hold. Then, for all* $\vartheta \in L^0(\mathbb{R}^d; \mathbb{F})$,

$$\Gamma(\vartheta; K) = \left\{ x \in \mathbb{R}^d \ : \ \mathbb{E}\left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t\right] \leq \bar{Z}_0 \cdot x \ \ \forall \, Z \in \mathcal{D}(K; \mathbb{P}), \ (Z \cdot \vartheta)^- \in L^1(\mathbb{R}; \mathbb{F}, \mathbb{P}) \right\}.$$

Remark 1.1 The integrability conditions on $(Z \cdot \vartheta)^-$ are trivially satisfied if there is some $\mathbb{R}^d$-valued constant c such that $\vartheta_t + c \in K_t$ $\mathbb{P}-$a.s. for all $t \in \mathbb{T}$, i.e. the *liquidation value* of ϑ is uniformly bounded from below. Indeed, in that case $Z_t \cdot (\vartheta_t + c) \geq 0$ $\mathbb{P}-$a.s. for all $Z_t \in L^0(K_t^*; \mathcal{F})$.

Following the approach of [5] and [6] the closure property of $A^s(0; K)$ can be obtained under the general assumption

$$\xi \in \mathcal{A}(K) \text{ and } \sum_{t \in \mathbb{T}} \xi_t = 0 \ \ \mathbb{P}-\text{a.s.} \quad \Longrightarrow \quad \xi_t \in K_t^0 \ \ \mathbb{P}-\text{a.s.} \quad \text{for all } t \in \mathbb{T} \tag{1.2}$$

where $K^0 = (K_t^0)_{t \in \mathbb{T}}$ is defined by $K_t^0 = K_t \cap (-K_t)$ for $t \in \mathbb{T}$.

Proposition 1.1 *Assume that* (1.2) *holds, then* $A^s(K)$ *is closed in measure.*

Remark 1.2 1. In the case of efficient frictions, i.e. $K_t^0 = \{0\}, \ \forall t \in \mathbb{T}$, it is shown in [5] that the assumption (1.2) is a consequence of the *strict no-arbitrage* property

$$NA^s(K) : \ A_t(0; K) \cap L^0(K_t; \mathcal{F}_t) \quad \subset \quad L^0(K_t^0; \mathcal{F}_t) \ \text{ for all } \ t \in \mathbb{T} \, .$$

2. In the case where K_t^0 may not be trivial, (1.2) holds under the *robust no-arbitrage* condition introduced by [10] and further studied by [6],

$$NA^r(K) : \ NA(\tilde{K}) \text{ holds for some } \tilde{K} \in \mathcal{K} \text{ which dominates } K,$$

where $\tilde{K}$ dominates K if $K_t \setminus K_t^0 \ \subset \ \mathrm{ri}(\tilde{K}_t) \quad \mathbb{P}-\text{a.s.} \quad \text{for all} \ \ t \in \mathbb{T}$.
3. It is shown in [7] that the condition $K_t^0 = \{0\}$ in 1. can be replaced by the weaker one: $L^0(K_t^0; \mathcal{F}_{t-1}) \subset L^0(K_{t-1}^0; \mathcal{F}_{t-1})$ for all $1 \leq t \leq T$. See also [8].

2 Counter examples

In this section, we first show that the duality relation

$$\mathbf{D}(K): \ \Gamma(\vartheta; K) = \Theta(\vartheta; K) \ \text{ for all } \ \vartheta \in L^0(\mathbb{R}^d; \mathbb{F})$$

does not hold for a large class of $\mathcal{C}$-valued process $K \in \mathcal{K}$. For $x \in \mathbb{R}^d$, let us define

$$c_t(x) \quad := \quad \min \{ c \in \mathbb{R} \ : \ c\mathbf{1}_1 - x \in K_t \} \ . \tag{2.1}$$

In financial terms, $c_t(x)$ is the minimal amount, in terms of the first asset, necessary to dominate x in the sense of K_t at time t. If the first asset is interpreted as a numeraire, it corresponds to the *constitution value* of x in terms of this numeraire. Here, $\mathbf{1}_1$ stands for the $\mathbb{R}^d$ vector $(1, 0, \ldots, 0)$.

Proposition 2.1 *If there exists $x \in \mathbb{R}^d$ such that*

(i) $y - c_0(x)\mathbf{1}_1 \in K_0^0 \Rightarrow y - x \in K_0^0$ *or* $\mathbb{P}[y - x \in K_1] < 1$

(ii) $x - c_0(x)\mathbf{1}_1 \notin K_0$.

Then, there exists ϑ such that $\Theta(\vartheta; K) \neq \Gamma(\vartheta; K)$, i.e. $\mathbf{D}(K)$ is not satisfied.

The proof is postponed to the end of the section.

Remark 2.1 Condition (ii) means that there are directions with efficient frictions at time 0. Condition (i) has the following interpretation. If a portfolio y is equivalent to the *constitution value* of x then it dominates x in the sense of K_0. However, since x and y have the same constitution value, $c_0(x) = c_0(y)$, it can not be *too large*. In particular, if it is not equivalent to x, then it can not dominate x component by component. In that case, we assume that there is randomness enough so that the probability that y still dominates x at time 1 is less than 1.

Remark 2.2 1. If $K_0^0 = \{0\}$ and $x \neq c_0(x)\mathbf{1}_1$ then (ii) holds since by definition we already have $c_0(x)\mathbf{1}_1 - x \in K_0$. If we also assume that $\mathbb{P}[c_1(x) > c_0(x)] > 0$ then (i) is satisfied too.

Example 2.1 *1. Efficient frictions:* consider the following cones

$$K_t \quad = \quad \left\{ (x^1, x^2) \in \mathbb{R}^2 \ : x^1 + (1 + \lambda_t)x^2 \geq 0 \ , \ x^1 + (1 - \mu_t)x^2 \geq 0 \right\} ,$$

where $t \in \mathbb{T} := \{0, 1\}$, $\lambda_0 < \lambda_1$ and $\mu_0, \mu_1 \in (0, 1)$. Observe that $K_0^0 = \{0\}$. For $x = (0, 1)$, $c_0(x) = 1 + \lambda_0 < c_1(x) = 1 + \lambda_1$. Then, the conditions of the remark above hold so that $\mathbf{D}(K)$ is not true.

2. Partial frictions: consider the preceding case where we add an asset which has no transaction cost with the first one, i.e.

$$K_t \;=\; \left\{(x^1,x^2,x^3)\in\mathbb{R}^3 \;:\, x^1+(1+\lambda_t)x^2+x^3\geq 0\;,\; x^1+(1-\mu_t)x^2+x^3\geq 0\right\}.$$

We put $x=(0,1,0)$ so that assumption (ii) holds. We now check (i). It is clear that if $y-c_0(x)\mathbf{1}_1\in K_0^0$ then $y-x\notin K_0^0$. Observe that $y=(y^1,0,y^3)$ with $y^1+y^3=c_0(x)$, so $y^1+(-1)(1+\lambda_1)+y^3<0$ which implies that $y-x\notin K_1$.

On the contrary, we can also show that $\mathbf{D}(K)$ does not *only hold* in the case where $K_t=K_t^0+\mathbb{R}^d_+$ $\mathbb{P}-$a.s., i.e. there is no transaction costs.

Proposition 2.2 *There exists $(\Omega,\mathcal{F},\mathbb{P})$ and $K\in\mathcal{R}$ such that $NA(K)$ holds, $K_t^0=\{0\}$ for all t, and such that for all $\vartheta\in L^0(\mathbb{R}^d;\mathbb{F})$ we have $\Theta(\vartheta;K)=\Gamma(\vartheta;K)$.*

Proof. We take Ω trivial, i.e. $|\Omega|=1$ with $\mathcal{F}_0=\mathcal{F}_T=\{\Omega,\emptyset\}$, and put $K=K_0$ constant. Then, $x\in\Theta(\vartheta;K)$ reads $\sup_{Z_t\in K_t^*} Z_t\cdot(\vartheta_t-x)\leq 0$, i.e. $x-\vartheta\in K_t$ for all $t\in\mathbb{T}$. □

Proof of Proposition 2.1: Let x be such that $(i)-(ii)$ are satisfied. We consider the asset ϑ defined by $\vartheta_t=c_0(x)\mathbf{1}_1\mathbb{I}_{t=0}+x\mathbb{I}_{t>0}$. From the martingale property of Z,

$$\begin{aligned}\sup_{\tau\in\mathcal{T}(\mathbb{T})}\mathbb{E}\left[Z_\tau\cdot\vartheta_\tau-Z_0\cdot(c_0(x)\mathbf{1}_1)\right] &= \sup_{\tau\in\mathcal{T}(\mathbb{T})}\mathbb{E}\left[Z_\tau\cdot(x-c_0(x)\mathbf{1}_1)\mathbb{I}_{\tau>0}\right]\\ &= \max\left\{0\;;\;Z_0\cdot(x-c_0(x)\mathbf{1}_1)\right\}\end{aligned}$$

which is non positive by (2.1). Hence, $c_0(x)\mathbf{1}_1\in\Theta(\vartheta;K)$. If $\mathbf{D}(K)$ holds, then there exists a portfolio $V\in A(c_0(x)\mathbf{1}_1;K)$ such that $V_0-c_0(x)\mathbf{1}_1\in K_0$ and therefore $V_0-c_0(x)\mathbf{1}_1\in K_0^0$. By (i) there is two cases. If $V_0-x\in K_0^0$, then $x-c_0(x)\mathbf{1}_1\in K_0^0\subset K_0$ which is a contradiction of (ii). If $\mathbb{P}\left[V_0-x\in K_1\right]<1$, we can not have $V_1-x=V_0+\xi_1-x\in K_1$ $\mathbb{P}-$a.s. with $\xi_1\in -K_1$ $\mathbb{P}-$a.s. □

3 Proofs

In this section, we first provide the proof of Theorem 1.1. It follows from standard arguments based on the Hahn-Banach separation theorem. For ease of notations, we simply write $A(K)$ and $A^s(K)$ in place of $A(0;K)$ and $A^s(0;K)$. We denote by $\mathcal{L}^0$ the set of $\mathbb{F}$-adapted processes with values in $\mathbb{R}^d$ and by $\mathcal{L}^1(\tilde{\mathbb{P}})$ (resp. $\mathcal{L}^\infty$) the subset of these elements which are $\tilde{\mathbb{P}}$-integrable, $\tilde{\mathbb{P}}\sim\mathbb{P}$, (resp. bounded). Observe that $\mathcal{L}^0$ (resp. $\mathcal{L}^\infty$) can be identified as a subset of $L^0(\mathbb{R}^{d\times(T+1)};\mathcal{F})$ (resp. $L^\infty(\mathbb{R}^{d\times(T+1)};\mathcal{F})$, the set of bounded elements of $L^0(\mathbb{R}^{d\times(T+1)};\mathcal{F})$).

Proposition 3.1 *Let the conditions of Theorem 1.1 hold. Then, for all $\tilde{\mathbb{P}} \sim \mathbb{P}$, there is some $Z \in \mathcal{D}(K; \tilde{\mathbb{P}}) \cap \mathcal{L}^\infty$ such that*

$$\sup_{\vartheta \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\sum_{t \in \mathbb{T}} Z_t \cdot \vartheta_t \right] \leq 0 .$$

Proof. Since $A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})$ is closed in $L^1(\mathbb{R}^{d\times(T+1)}; \mathcal{F}, \tilde{\mathbb{P}})$ (when identified with a subset of $L^1(\mathbb{R}^{d\times(T+1)}; \mathcal{F}, \tilde{\mathbb{P}})$) and convex, it follows from the Hahn-Banach separation theorem, $NA(K)$ and the fact that $A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})$ is a cone, that there is some $\eta = (\eta_t)_{t\in\mathbb{T}} \in L^\infty(\mathbb{R}^{d\times(T+1)}; \mathcal{F})$ such that

$$\sup_{\vartheta \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\sum_{t \in \mathbb{T}} \eta_t \cdot \vartheta_t \right] \leq 0 . \tag{3.1}$$

By possibly replacing η_t by $\mathbb{E}[\eta_t \mid \mathcal{F}_t]$, we can assume that η is $\mathbb{F}$-adapted. Fix some arbitrary $\xi \in \mathcal{A}(K) \cap \mathcal{L}^\infty$, so that $V^{0,\xi} \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})$. Since

$$\sum_{t \in \mathbb{T}} \eta_t \cdot V_t^{0,\xi} = \sum_{t \in \mathbb{T}} \xi_t \cdot \left(\sum_{s=t}^{T} \eta_s \right)$$

we deduce from the above inequality that

$$\sup_{\xi \in \mathcal{A}(K) \cap \mathcal{L}^\infty} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\sum_{t \in \mathbb{T}} \bar{\eta}_t \cdot \xi_t \right] \leq 0 ,$$

where we defined

$$\bar{\eta}_t := \mathbb{E}^{\tilde{\mathbb{P}}} \left[\sum_{s=t}^{T} \eta_s \mid \mathcal{F}_t \right] \quad t \in \mathbb{T} .$$

This shows that $\bar{\eta}_t \in K_t^* \ \mathbb{P}-$ a.s. for all $t \in \mathbb{T}$. For an arbitrary bounded element ξ_t in $L^0(K_t; \mathcal{F}_t)$, the process $V_s^{0,\xi} = -\mathbb{1}_{s=t}\xi_t$, $s \in \mathbb{T}$, belongs to $A^s(K)$. In view of (3.1), this implies that $\eta_t \in K_t^* \ \mathbb{P}-$ a.s. □

Proposition 3.2 *Let the conditions of Theorem 1.1 hold. Fix $\tilde{\mathbb{P}} \sim \mathbb{P}$ and $\vartheta \in \mathcal{L}^1(\tilde{\mathbb{P}})$. If*

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t \right] \leq 0$$

for all $Z \in \mathcal{D}(K, \tilde{\mathbb{P}})$ such that $\vartheta \cdot Z \in \mathcal{L}^1(\tilde{\mathbb{P}})$, then $\vartheta \in A^s(K)$.

Proof. Since $A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})$ is closed and convex, if $\vartheta \notin A^s(K)$, we can find some $\eta = (\eta_t)_{t\in\mathbb{T}} \in L^\infty(\mathbb{R}^{d\times(T+1)};\mathcal{F})$ such that

$$\sup_{\tilde{\vartheta}\in A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T} \tilde{\vartheta}_t \cdot \eta_t\right] \;<\; \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T} \vartheta_t \cdot \eta_t\right] .$$

The same arguments as in the proof of Proposition 3.1 then shows that we can choose $\eta \in \mathcal{D}(K,\tilde{\mathbb{P}})$ which leads to a contradiction. □

Proof of Theorem 1.1 1. In view Proposition 3.2, the implication (ii) $\Rightarrow$ (i) is obtained by considering $\tilde{\mathbb{P}}$ with density with respect to $\mathbb{P}$ defined by $H/\mathbb{E}[H]$ with $H := \exp(-\sum_{t\in\mathbb{T}}\|\vartheta_t\|)$.
2. It is clear that (ii) implies (iii) while the reverse implication follows from the fact that $Z \in \mathcal{D}(K,\mathbb{P})$ if and only if $\tilde{H}Z \in \mathcal{D}(K,\tilde{\mathbb{P}})$ where $\tilde{H}_t := \mathbb{E}\left[d\tilde{\mathbb{P}}/d\mathbb{P} \mid \mathcal{F}_t\right]$.
3. The last implication (i) $\Rightarrow$ (ii) is trivial. Indeed, recall that, for $\xi \in \mathcal{A}(K)$,

$$\mathbb{E}\left[\sum_{t\in\mathbb{T}} Z_t \cdot V_t^{0,\xi}\right] \;=\; \mathbb{E}\left[\sum_{t\in\mathbb{T}} \bar{Z}_t \cdot \xi_t\right] .$$

Since $\bar{Z}_t \in L^0(K_t^*;\mathcal{F}_t)$ and $\xi_t \in L^0(-K_t;\mathcal{F}_t)$, the last term is non-positive. Moreover, $V_t^{0,\xi} - \vartheta_t \in L^0(K_t;\mathcal{F}_t)$ implies $Z_t \cdot V_t^{0,\xi} \geq Z_t \cdot \vartheta_t$. □

We now provide the proof of Proposition 1.1. The following Lemma can be found in [3].

Lemma 3.1 *Set $\mathcal{G} \subset \mathcal{F}$ and $E \subset \mathbb{R}^d$. Let $(\eta^n)_{n\geq 1}$ be a sequence in $L^0(E;\mathcal{G})$. Set $\tilde{\Omega} := \{\liminf_{n\to\infty}\|\eta^n\| < \infty\}$. Then, there is an increasing sequence of random variables $(\tau(n))_{n\geq 1}$ in $L^0(\mathbb{N};\mathcal{G})$ such that $\tau(n) \to \infty$ $\mathbb{P}-$a.s. and , for each $\omega \in \tilde{\Omega}$, $\eta^{\tau(n)}(\omega)$ converges to some $\eta^*(\omega)$ with $\eta^* \in L^0(E;\mathcal{G})$.*

Proof of Lemma 1.1. We use an inductive argument. For $t \in \mathbb{T}$, we denote by Σ_t the set of processes $\vartheta \in \mathcal{L}^0$ such that

$$\exists\, \xi \in \mathcal{A}(K) \;\text{ s.t. }\; \sum_{s=t}^{\tau} \xi_s - \vartheta_\tau \;\in\; K_\tau \;\, \mathbb{P}-\text{a.s.} \;\text{ for all }\; t \leq \tau \leq T .$$

Clearly, Σ_T is closed in measure. Assume that Σ_{t+1} is closed and let ϑ^n be a sequence in Σ_t such that $\vartheta^n_s \to \vartheta_s$ $\mathbb{P}-$a.s. for $t \leq s \leq T$. Let $\xi^n \in \mathcal{A}(K)$ be such that

$$\sum_{s=t}^{\tau} \xi^n_s - \vartheta^n_\tau \;\in\; K_\tau \;\, \mathbb{P}-\text{a.s.} \;\text{ for all }\; t \leq \tau \leq T .$$

Set $\tilde{\Omega} := \{\liminf_{n\to\infty} \|\xi_t^n\| < \infty\}$. Since $\tilde{\Omega} \in \mathcal{F}_t$, we can work separately on $\tilde{\Omega}$ and $\tilde{\Omega}^c$.

1. If $\mathbb{P}\left[\tilde{\Omega}\right] = 1$, after possibly passing to a random sequence (see Lemma 3.1), we can assume that ξ_t^n converges $\mathbb{P}-$ a.s. to some $\xi_t \in L^0(-K_t;\mathcal{F}_t)$. Since

$$\sum_{s=t+1}^{\tau} \xi_s^n - (\vartheta_\tau^n - \xi_t^n) \quad \in \quad K_\tau \;\; \mathbb{P}-\text{a.s.} \quad \text{for all} \;\; t+1 \le \tau \le T \; ,$$

and Σ_{t+1} is closed, we can find some $\tilde{\xi} \in \mathcal{A}(K)$ such that

$$\sum_{s=t+1}^{\tau} \tilde{\xi}_s - (\vartheta_\tau - \xi_t) \quad \in \quad K_\tau \;\; \mathbb{P}-\text{a.s.} \quad \text{for all} \;\; t+1 \le \tau \le T \; .$$

This shows that $\vartheta \in \Sigma_t$.

2. If $\mathbb{P}\left[\tilde{\Omega}\right] < 1$, then we can assume without loss of generality that $\mathbb{P}\left[\tilde{\Omega}\right] = 0$. Following line by line the proof of Lemma 2 in [6] and using the K_s's closure property, we can find some $\hat{\xi} \in \mathcal{A}(K)$ with $\|\hat{\xi}_t\| = 1$ such that

$$\kappa_\tau := \sum_{s=t}^{\tau} \hat{\xi}_s \quad \in \quad K_\tau \;\; \mathbb{P}-\text{a.s.} \quad \text{for all} \;\; t \le \tau \le T \; .$$

By (1.2), we must have $\hat{\xi}_\tau - \kappa_\tau \in K_\tau^0 \;\; \mathbb{P}-$ a.s. $\forall t \le \tau \le T$. Since $\hat{\xi}_\tau$ and $-\kappa_\tau$ $\in -K_\tau \;\; \mathbb{P}-$ a.s., we deduce that

$$\hat{\xi}_\tau \in K_\tau^0 \;\; \text{and} \;\; \kappa_\tau = \sum_{s=t}^{\tau} \hat{\xi}_s \quad \in \quad K_\tau^0 \;\; \mathbb{P}-\text{a.s.} \quad \text{for all} \;\; t \le \tau \le T \; . \tag{3.2}$$

Since $\|\hat{\xi}_t\| = 1$, there is a partition of $\tilde{\Omega}$ into disjoint subsets $\Gamma_i \in \mathcal{F}_t$ such that $\Gamma_i \subset \{(\hat{\xi}_t)^i \neq 0\}$ for $i = 1, \ldots, d$. We then define

$$\check{\xi}_s^n \quad = \quad \sum_{i=1}^{d} \left(\xi_s^n - \beta_t^{n,i} \hat{\xi}_s\right) \mathbb{1}_{\Gamma_i} \quad s \in \mathbb{T}$$

with $\beta_t^{n,i} = (\xi_t^n)^i / (\hat{\xi}_t)^i$ on Γ_i, $i = 1, \ldots, d$. In view of (3.2) and definition of ξ^n, we have

$$\sum_{s=t}^{\tau} \check{\xi}_s^n - \vartheta_\tau^n \quad \in \quad K_\tau \;\; \mathbb{P}-\text{a.s.} \quad \text{for all} \;\; t \le \tau \le T \; ,$$

since $K_\tau - K_\tau^0 \subset K_\tau$, $\tau \in \mathbb{T}$. We can then proceed as in [6] and obtain the required result by repeating the above argument with $(\check{\xi}^n)_{n\ge1}$ instead of $(\xi^n)_{n\ge1}$ and by iterating this procedure a finite number of times. □

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