



The roots of any polynomial equation

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Abstract

We provide a method for solving the roots of the general polynomial equation

$$a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots + a_1 \cdot x + s = 0 \quad (1)$$

To do so, we express x as a powerseries of s , and calculate the first $n-1$ coefficients. We turn the polynomial equation into a differential equation that has the roots as solutions. Then we express the powerseries' coefficients in the first $n-1$ coefficients. Then the variable s is set to a_0 . A free parameter is added to make the series convergent. © 2004 G.A.Uytdewilligen. All rights reserved.

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The method

The method is based on [1]. Let's take the first $n-1$ derivatives of (1) to s . Equate these derivatives to zero.

Then find $\frac{d^i}{ds^i} x(s)$ in terms of $x(s)$ for i from 1 to $n-1$. Now make a new differential equation

$$m_1 \cdot \frac{d^{n-1}}{ds^{n-1}} x(s) + m_2 \cdot \frac{d^{n-2}}{ds^{n-2}} x(s) + \dots + m_n \cdot x(s) + m_{n+1} = 0 \quad (2)$$

and fill in our $\frac{d^i}{ds^i} x(s)$ in (2). Multiply by the denominator of the expression. Now we have a polynomial in $x(s)$ of degree higher than n . Using (1) as property, we simplify this polynomial to the degree of n . Set it equal to (1) and solve $m_1 \dots m_{n+1}$ in terms of s and $a_1 \dots a_n$. Substituting these in (2) gives a differential equation that has the zeros of (1) among its solutions. We then insert

$$x(s) = y(s) - \frac{a_{n-1}}{n \cdot a_n} \quad (3)$$

in (2). Multiplying by the denominator we get a differential equation of the linear form:

$$p_1 \cdot \frac{d^{n-1}}{ds^{n-1}} y(s) + p_2 \cdot \frac{d^{n-2}}{ds^{n-2}} y(s) + \dots + p_n \cdot y(s) = 0 \quad (4)$$

With $p_1(s) \dots p_n(s)$ polynomials in s . If we substitute our powerseries, all the coefficients are determined by the first $n-1$ coefficients. The first coefficients are calculated as follows: A powerseries is filled in in (1).

$$x(s) = \sum_{i=0}^{n-2} b_i \cdot s^i - \frac{a_n}{n \cdot a_{n-1}} \quad (5)$$

and it should be zero for all s . From this, we calculate b_i for i from 0 to $n-2$. b_0 is a root of an $n-1$ degree polynomial and the other b_i are expressed in b_0 . Now a powerseries is inserted in (4):

$$y(s) = \sum_{i=0}^{\infty} b_i \cdot s^i \quad (6)$$

and we get an equation of the form:

$$q_1(i) \cdot c_1 \cdot b_i + q_2(i) \cdot c_2 \cdot b_{i+1} + \dots + q_n(i) \cdot c_n \cdot b_{i+n-1} = 0 \quad (7)$$

where $q_m(i)$ are polynomials in i of degree $n-1$. c_m Are constants.

We define b_{n-1} as the determinant of a matrix A

$$A = \begin{bmatrix} \frac{c_{n-1} \cdot q_{n-1}(0)}{c_n \cdot q_n(0)} & \dots & \frac{c_2 \cdot q_2(0)}{c_n \cdot q_n(0)} & \frac{c_1 \cdot q_1(0)}{c_n \cdot q_n(0)} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & \dots & 0 & 1 & b_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 1 & 0 & 0 & -b_1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1^{n-2} \cdot b_{n-2} \end{bmatrix} \quad (8)$$

and for the rest of the coefficients

$$b_{i+n-1} = \begin{bmatrix} \frac{-c_{n-1} \cdot q_{n-1}(i)}{c_n \cdot q_n(i)} & \frac{-c_{n-2} \cdot q_{n-2}(i)}{c_n \cdot q_n(i)} & \frac{-c_{n-3} \cdot q_{n-3}(i)}{c_n \cdot q_n(i)} & \dots & \frac{-c_1 \cdot q_1(i)}{c_n \cdot q_n(i)} & 0 & 0 & 0 & 0 & \dots \\ -1 & \frac{-c_{n-1} \cdot q_{n-1}(i-1)}{c_n \cdot q_n(i-1)} & \frac{-c_{n-2} \cdot q_{n-2}(i-1)}{c_n \cdot q_n(i-1)} & \dots & \frac{-c_2 \cdot q_2(i-1)}{c_n \cdot q_n(i-1)} & \frac{c_1 \cdot q_1(i-1)}{c_n \cdot q_n(i-1)} & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -1 & \frac{-c_{n-1} \cdot q_{n-1}(1)}{c_n \cdot q_n(1)} & \frac{c_{n-2} \cdot q_{n-2}(1)}{c_n \cdot q_n(1)} & \dots & \frac{c_1 \cdot q_1(1)}{c_n \cdot q_n(1)} & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (9)$$

The series (6) can be proven to be convergent [2] if for a constant E

$$s \leq \frac{1}{E} \cdot |c_n| \quad (10)$$

and if all the absolute values of the coefficients of (1) are chosen smaller than 1. This is done by dividing a polynomial by (more than) the maximum of the absolute values of the coefficients.

To make the series convergent, we transform s to $e \cdot s$. That is, if we insert the powerseries it is not in s but in $e \cdot s$. Writing out the terms of the sum, we find that each term d_i has a factor $s^i \cdot e^{-i+1}$. Setting

$$e = \frac{|c_n|}{E} \tag{12}$$

We still need $s < 1$, which is why we set s to a_0 and $a_0 < 1$.

References

- [1] R. Harley, On the theory of the Transcendental Solution of Algebraic Equations, Quart. Journal of Pure and Applied Math, Vol. 5 p.337. 1862
- [2] E. Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, Inc. p.789. 1993.