

Another proofs of the geometrical forms of Paley-Wiener theorems for the Dunkl transform and inversion formulas for the Dunkl intertwining operator and for its dual

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Abstract

In this paper we present another proofs of the geometrical forms of Paley-Wiener theorems for the Dunkl transform given in [15], and we prove inversion formulas for the Dunkl intertwining operator V_k and for its dual tV_k and we deduce the expression of the representing distributions of the inverse operators V_k^{-1} and ${}^tV_k^{-1}$.

Keywords : Paley-Wiener theorems ; Inversion Formulas ; Dunkl intertwining operator ; Dual of the Dunkl intertwining operator
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1 Introduction

We consider the differential-difference operators $T_j, j = 1, 2, \dots, d$, on \mathbb{R}^d introduced by C.F.Dunkl in [3]. These operators are very important in pure mathematics and in Physics. They provide a useful tool in the study of special functions with root systems [4,6, 2]. Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in a one dimensional space (see [8,11, 12]).

C.F.Dunkl has proved in [5] that there exists a unique isomorphism V_k from the space of homogeneous polynomials \mathcal{P}_n on \mathbb{R}^d of degree n onto itself satisfying the permutation relations

$$T_j V_k = V_k \frac{\partial}{\partial x_j}, \quad j = 1, 2, \dots, d, \quad (1.1)$$

and

$$V_k(1) = 1. \quad (1.2)$$

This operator is called Dunkl intertwining operator. It has been extended to an isomorphism from $\mathcal{E}(\mathbb{R}^d)$ (the space of C^∞ -functions on \mathbb{R}^d) onto itself satisfying the relations (1.1) and (1.2) (see[14]).

The operator V_k possesses the integral representation

$$\forall x \in \mathbb{R}^d, \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad f \in \mathcal{E}(\mathbb{R}^d), \quad (1.3)$$

where μ_x is a probability measure on \mathbb{R}^d with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [13][14])

We have shown in [14] that for each $x \in \mathbb{R}^d$, there exists a unique distribution η_x in $\mathcal{E}'(\mathbb{R}^d)$ (the space of distributions on \mathbb{R}^d of compact support) with support in $B(0, \|x\|)$ such that

$$V_k^{-1}(f)(x) = \langle \eta_x, f \rangle, f \in \mathcal{E}(\mathbb{R}^d)$$

We have studied also in [14] the transposed operator tV_k of the operator V_k . It has the integral representation

$$\forall y \in \mathbb{R}^d, \quad {}^tV_k(f)(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x). \quad (1.4)$$

where ν_y is a positive measure on \mathbb{R}^d with support in the set $\{x \in \mathbb{R}^d / \|x\| \geq \|y\|\}$ and f in $\mathcal{D}(\mathbb{R}^d)$ (the space of C^∞ -functions on \mathbb{R}^d with compact support).

This operator is called Dual Dunkl intertwining operator.

We have proved in [14] that the operator tV_k is an isomorphism from $\mathcal{D}(\mathbb{R}^d)$ onto itself, satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d, \quad {}^tV_k(T_j f)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y), \quad j = 1, 2, \dots, d. \quad (1.5)$$

Using the operator V_k C.F.Dunkl has defined in [5] the Dunkl kernel K by

$$\forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, \quad K(x, -iz) = V_k(e^{-i\langle \cdot, z \rangle})(x). \quad (1.6)$$

Using this kernel C.F.Dunkl has introduced in [5] a Fourier transform \mathcal{F}_D called Dunkl transform.

In this paper we present another proofs of the geometric forms of Paley-Wiener theorems for the transform \mathcal{F}_D , given in [15] p. 32-33, and we establish the following inversion formulas for the operators V_k^{-1} and ${}^tV_k^{-1}$:

$$\forall x \in \mathbb{R}^d, \quad V_k^{-1}(f)(x) = {}^tV_k(Q(f))(x), \quad f \in \mathcal{D}(\mathbb{R}^d), \quad (1.7)$$

$$\forall x \in \mathbb{R}^d, \quad {}^tV_k^{-1}(f)(x) = V_k(P(f))(x), \quad f \in \mathcal{D}(\mathbb{R}^d), \quad (1.8)$$

where P and Q are pseudo-differential operators on \mathbb{R}^d .

From these relations we deduce the expression of the representing distributions

η_x and Z_x of the inverse operators V_k^{-1} and ${}^tV_k^{-1}$ by using the representing measures μ_x and ν_x of V_k and tV_k . They are given by the following formulas

$$\forall x \in \mathbb{R}^d, \quad \eta_x = {}^tQ(\nu_x), \quad (1.9)$$

$$\forall x \in \mathbb{R}^d, \quad Z_x = {}^tP(\mu_x), \quad (1.10)$$

where tP and tQ are the transposed of the operators P and Q .

The contents of the paper are as follow.

In section two we recall some basic facts from Dunkl's theory, we describe Dunkl operators and the Dunkl kernel.

We introduce in the third section the Dunkl intertwining operator V_k and its dual tV_k and we present their properties.

We define in the fourth section the Dunkl transform introduced in [5] by C.F.Dunkl, and we give the main theorems proved for this transform.

In the fifth section we give another proofs of the geometrical forms of Paley-Wiener theorems for the Dunkl transform. The first proofs of these theorems have been given in [15] p. 32-33. Next we present some applications of the first theorem.

The sixth section is devoted to prove inversion formulas for the Dunkl intertwining operator V_k and for its dual tV_k , and we deduce the expression of the representing distributions of the inverse operators V_k^{-1} and ${}^tV_k^{-1}$.

2 The eigenfunction of the Dunkl operators

In this section we collect some notations and results on Dunkl operators and the Dunkl kernel (see [4,5, 7, 9, 10]).

2.1 Reflection Groups, Root Systems and Multiplicity Functions

We consider \mathbb{R}^d with the euclidean scalar product $\langle \cdot, \cdot \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$. On \mathbb{C}^d , $\|\cdot\|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^d z_j \overline{w_j}$.

For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α , i.e.

$$\sigma_\alpha(x) = x - \left(\frac{2\langle \alpha, x \rangle}{\|\alpha\|^2} \right) \alpha. \quad (2.1)$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a give root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R . All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R; \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \rightarrow \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W . If one

regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W . For abbreviation, we introduce the index

$$\gamma = \gamma(R) = \sum_{\alpha \in R_+} = \sum_{\alpha \in R_+} k(\alpha). \quad (2.2)$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}. \quad (2.3)$$

which is W -invariant and homogeneous of degree 2γ .

For $d = 1$ and $W = \mathbb{Z}_2$, the multiplicity function k is a single parameter denoted $\gamma > 0$ and

$$\forall x \in \mathbb{R}, \quad \omega_k(x) = |x|^{2\gamma}. \quad (2.4)$$

We introduce the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\|x\|^2} \omega_k(x) dx \right)^{-1}. \quad (2.5)$$

which is known for all Coxeter groups W (see [3, 6])

2.2 Dunkl Operators and Dunkl kernel

The Dunkl operators $T_j, j = 1, \dots, d$, on \mathbb{R}^d , associated with the finite reflection group W and multiplicity function k , are given for a function f of class C^1 on \mathbb{R}^d by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}. \quad (2.6)$$

In the case $k = 0$, the $T_j, j = 1, 2, \dots, d$, reduce to the corresponding partial derivatives. In this paper, we will assume throughout that $k \geq 0$ and $\gamma > 0$.

For f of class C^1 on \mathbb{R}^d with compact support and g of class C^1 on \mathbb{R}^d we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x) dx = - \int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x) dx, \quad j = 1, 2, \dots, d. \quad (2.7)$$

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x, y) &= y_j u(x, y), \quad j = 1, 2, \dots, d, \\ u(0, y) &= 1, \end{cases} \quad (2.8)$$

admits a unique analytic solution on \mathbb{R}^d , denoted by $K(x, y)$ and called Dunkl kernel.

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

Example

If $d = 1$ and $W = \mathbb{Z}_2$, the Dunkl kernel is given by

$$K(z, t) = j_{\gamma-1/2}(izt) + \frac{zt}{2\gamma+1}j_{\gamma+1/2}(izt), \quad z, t \in \mathbb{C}, \quad (2.9)$$

where for $\alpha' \geq -1/2$, $j_{\alpha'}$ is the normalized Bessel function defined by

$$j_{\alpha}(u) = 2^{\alpha}\Gamma(\alpha+1)\frac{J_{\alpha}(u)}{u^{\alpha}} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(u/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad u \in \mathbb{C}, \quad (2.10)$$

with J_{α} is the Bessel function of first kind and index α (see [5]).

The Dunkl kernel possesses the following properties.

- (i) For $z, t \in \mathbb{C}^d$, we have $K(z, t) = K(t, z)$, $K(z, 0) = 1$, and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.
- (ii) For all $\nu \in \mathbb{Z}_+^d$, $x \in \mathbb{R}^d$, and $z \in \mathbb{C}^d$ we have

$$|D_z^{\nu}K(x, z)| \leq \|x\|^{|\nu|} \exp\left[\max_{w \in W} \langle wx, Rez \rangle\right]. \quad (2.11)$$

In particular

$$|D_z^{\nu}K(x, z)| \leq \|x\|^{|\nu|} \exp[\|x\|\|Rez\|], \quad (2.12)$$

$$|K(x, z)| \leq \exp[\|x\|\|Rez\|] \quad (2.13)$$

and for all $x, y \in \mathbb{R}^d$:

$$|K(ix, y)| \leq 1, \quad (2.14)$$

with

$$D_z^{\nu} = \frac{\partial^{|\nu|}}{\partial^{\nu_1} z_1 \cdots \partial^{\nu_d} z_d} \text{ and } |\nu| = \nu_1 + \cdots + \nu_d.$$

- (iii) For all $x, y \in \mathbb{R}^d$ and $w \in W$ we have

$$K(-ix, y) = \overline{K(ix, y)} \text{ and } K(wx, wy) = K(x, y). \quad (2.15)$$

- (iv) The function $K(x, z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x(y), \quad (2.16)$$

where μ_x is a probability measure on \mathbb{R}^d with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$. Moreover we have

$$Supp\mu_x \cap \{y \in \mathbb{R}^d / \|y\| = \|x\|\} \neq \emptyset, \quad (2.17)$$

(see [13]).

Remark 2.1

When $d = 1$ and $W = \mathbb{Z}_2$, the relation (2.16) is of the form

$$K(x, z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma} \int_{-|x|}^{|x|} (|x| - y)^{\gamma-1} (|x| + y)^\gamma e^{yz} dy. \quad (2.18)$$

Then in this case the measure μ_x is given for all $x \in \mathbb{R} \setminus \{0\}$ by $d\mu_x(y) = \kappa(x, y)dy$ with

$$\kappa(x, y) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma} (|x| - y)^{\gamma-1} (|x| + y)^\gamma 1_{]-|x|, |x|[}(y), \quad (2.19)$$

where $1_{]-|x|, |x|[}$ is the characteristic function of the interval $]-|x|, |x|[$.

We remark that by change of variables, the relation (2.18) takes the following form

$$\forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, \quad K(x, z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\Gamma(\gamma)} \int_{-1}^1 e^{txz} (1-t^2)^{\gamma-1} (1+t) dt, \quad (2.20)$$

3 The Dunkl intertwining operator and its dual

Notation We denote by

- $C(\mathbb{R}^d)$ (resp. $C_c(\mathbb{R}^d)$) the space of continuous functions on \mathbb{R}^d (resp. with compact support).
- $C^p(\mathbb{R}^d)$ (resp. $C_c^p(\mathbb{R}^d)$) the space of functions of class C^p on \mathbb{R}^d (resp. with compact support).
- $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞ -functions on \mathbb{R}^d .
- $\mathcal{D}(\mathbb{R}^d)$ the space of C^∞ -functions on \mathbb{R}^d with compact support.
- $\mathcal{S}(\mathbb{R}^d)$ the space of C^∞ -functions on \mathbb{R}^d which are rapidly decreasing as their derivatives.

We provide these spaces with the classical topology. We consider also the following spaces.

- $\mathcal{E}'(\mathbb{R}^d)$ the space of distributions on \mathbb{R}^d with compact support ; it is the topological dual of $\mathcal{E}(\mathbb{R}^d)$.
- $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions on \mathbb{R}^d ; it is the topological dual of $\mathcal{S}(\mathbb{R}^d)$.

The Dunkl intertwining operator V_k is defined on $\mathcal{C}(\mathbb{R}^d)$ by

$$\forall x \in \mathbb{R}^d, V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad (3.1)$$

where μ_x is the measure given by the relation (2.16) (see [14]).

We have

$$\forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, \quad K(x, z) = V_k(e^{\langle \cdot, z \rangle})(x). \quad (3.2)$$

The operator tV_k satisfying for f in $C_c(\mathbb{R}^d)$ and g in $C(\mathbb{R}^d)$, the relation

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k(g)(x)f(x)\omega_k(x)dx. \quad (3.3)$$

is given by

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \int_{\mathbb{R}^d} f(x)d\nu_y(x), \quad (3.4)$$

where ν_y is a positive measure on \mathbb{R}^d whose support satisfies

$$Supp\nu_y \subset \{x \in \mathbb{R}^d/\|x\| \geq \|y\|\} \text{ and } Supp\nu_y \cap \{x \in \mathbb{R}^d/\|x\| = \|y\|\} \neq \emptyset. \quad (3.5)$$

This operator is called the dual Dunkl intertwining operator (see [14]).

The following theorems give some properties of the operators V_k and tV_k (see [14]).

Theorem 3.1

- (i) The operator V_k is a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the transmutation relations

$$\forall x \in \mathbb{R}^d, T_j V_k(f)(x) = V_k \left(\frac{\partial}{\partial y_j} f \right) (x), \quad j = 1, 2, \dots, d, \quad f \in \mathcal{E}(\mathbb{R}^d). \quad (3.6)$$

- (ii) For each $x \in \mathbb{R}^d$, there exists a unique distribution η_x in $\mathcal{E}'(\mathbb{R}^d)$ with support in the ball $B(0, \|x\|)$ such that for all f in $\mathcal{E}(\mathbb{R}^d)$ we have

$$V_k^{-1}(f)(x) = \langle \eta_x, f \rangle. \quad (3.7)$$

Moreover

$$\text{Supp} \eta_x \cap \{y \in \mathbb{R}^d / \|y\| = \|x\|\} \neq \emptyset. \quad (3.8)$$

Theorem 3.2

- (i) The operator ${}^t V_k$ is a topological isomorphism from $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) onto itself, satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d, {}^t V(T_j f)(y) = \frac{\partial}{\partial y_j} {}^t V(f)(y), \quad j = 1, 2, \dots, d, \quad f \in \mathcal{D}(\mathbb{R}^d). \quad (3.9)$$

- (ii) For each $y \in \mathbb{R}^d$, there exists a unique distribution Z_y in $\mathcal{S}'(\mathbb{R}^d)$ with support in the set $\{x \in \mathbb{R}^d / \|x\| \geq \|y\|\}$ such that for all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$${}^t V_k^{-1}(f)(y) = \langle Z_y, f \rangle. \quad (3.10)$$

Moreover

$$\text{Supp} Z_y \cap \{x \in \mathbb{R}^d / \|x\| = \|y\|\} \neq \emptyset. \quad (3.11)$$

Example 3.1

When $d = 1$ and $W = \mathbb{Z}_2$, the Dunkl intertwining operator V_k is defined by (3.1) with for all $x \in \mathbb{R} \setminus \{0\}$, $d\mu_x(y) = \kappa(x, y)dy$, where κ given by the relation (2.19).

The dual Dunkl intertwining operator ${}^t V_k$ is defined by (3.4) with $d\nu_y(x) = \kappa(x, y)\omega_k(x)dx$, where κ and ω_k given respectively by the relations (2.19) and (2.4).

Example 3.2

The Dunkl intertwining operator V_t of index $\gamma = \sum_{i=1}^d \alpha_i, \alpha_i > 0$, associated with the reflection group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ on \mathbb{R}^d , is given for all f in $\mathcal{E}(\mathbb{R}^d)$ and for all $x \in \mathbb{R}^d$ by

$$\begin{aligned} V_k(f)(x) &= \prod_{i=1}^d \left(\frac{\Gamma(\alpha_i + 1/2)}{\sqrt{\pi}\Gamma(\alpha_i)} \right) \int_{[-1,1]^d} f(t_1 x_1, t_2 x_2, \dots, t_d x_d) \\ &\quad \times \prod_{i=1}^d (1 - t_i^2)^{\alpha_i - 1} (1 + t_i) dt_1 \cdots dt_d, \end{aligned} \quad (3.12)$$

(see [16])

4 Dunkl transform

In this section we define the Dunkl transform and we give the main results satisfied by this transform (see [5, 9, 10]).

Notation We denote by

- $L_k^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, the space of measurable functions on \mathbb{R}^d such that

$$\begin{aligned} \|f\|_{k,p} &= \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x) dx \right)^{1/p} < +\infty, \quad \text{if } 1 \leq p < +\infty, \\ \|f\|_{k,\infty} &= \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < +\infty. \end{aligned}$$

- $H(\mathbb{C}^d)$ the space of entire functions on \mathbb{C}^d which are rapidly decreasing and of exponential type.

The Dunkl transform of a function f in $\mathcal{D}(\mathbb{R}^d)$ is given by

$$\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x) K(x, -iy) \omega_k(x) dx. \quad (4.1)$$

This transform has the following properties.

- i) For f in $L_k^1(\mathbb{R}^d)$ we have $\|\mathcal{F}_D(f)\|_{k,\infty} \leq \|f\|_{k,1}$.
- (ii) Let f be in $\mathcal{D}(\mathbb{R}^d)$. If $f^-(x) = f(-x)$ and $f_w(x) = f(wx)$ for $x \in \mathbb{R}^d$, $w \in W$, then for all $y \in \mathbb{R}^d$ we have

$$\mathcal{F}_D(f^-)(y) = \overline{\mathcal{F}_D(f)(y)} \quad \text{and} \quad \mathcal{F}_D(f_w)(y) = \mathcal{F}_D(f)(wy). \quad (4.2)$$

- iii) For all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\mathcal{F}_D(f) = \mathcal{F} \circ {}^t V_k(f), \quad (4.3)$$

where \mathcal{F} is the classical Fourier transform on \mathbb{R}^d given by

$$\forall y \in \mathbb{R}^d, \quad \mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x, y \rangle} dx, \quad f \in \mathcal{D}(\mathbb{R}^d), \quad (4.4)$$

The following theorems are proved in [9, 10].

Theorem 4.1. The transform \mathcal{F}_D is a topological isomorphism

- i) from $\mathcal{D}(\mathbb{R}^d)$ onto $\mathbb{H}(\mathbb{C}^d)$,
- ii) from $\mathcal{S}(\mathbb{R}^d)$ onto itself.

The inverse transform is given by

$$\forall x \in \mathbb{R}^d, \quad \mathcal{F}_D^{-1}(h)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} h(y) K(x, iy) \omega_k(y) dy. \quad (4.5)$$

Remark 4.1

An other proof of Theorem 4.1 is given in [15].

Theorem 4.2. Let f be in $L_k^1(\mathbb{R}^d)$ such that the function $\mathcal{F}_D(f)$ belongs to $L_k^1(\mathbb{R}^d)$. Then we have the following inversion formula for the transform \mathcal{F}_D :

$$f(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(y) K(x, iy) \omega_k(y) dy, \quad a.e. \quad (4.6)$$

Theorem 4.3.

i) Plancherel formula for \mathcal{F}_D . For all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2 \omega_x(x) dx = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(y)|^2 \omega_k(y) dy. \quad (4.7)$$

ii) Plancherel Theorem for \mathcal{F}_D . The renormalized Dunkl transform $f \rightarrow 2^{-\gamma-d/2} \times c_k \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$.

5 Another proofs of the geometrical forms of the Paley-Wiener theorems for the Dunkl transform

In this section we present another proofs of the geometrical forms of Paley-Wiener theorems for the transform \mathcal{F}_D , given in [15] p. 32-33.

We define the indicator function I_E of a compact subset E of \mathbb{R}^d by

$$\forall y \in \mathbb{R}^d, I_E(y) = \sup_{x \in E} \langle x, y \rangle,$$

for example, if E is the ball with center 0 and radius R , we have

$$\forall y \in \mathbb{R}^d, I_E(y) = R\|y\|.$$

The indicator of E determines, for each hyperplane direction, the smallest closed half space which contain E . Consequently the Hahn-Banach theorem shows that I_E determines the convex envelope \hat{E} of E and $I_{\hat{E}} = I_E$.

In [15] p. 29 we find other properties of the indicator function I_E .

5.1 The Dunkl transform of functions

Theorem 5.1. Let E be a W -invariant compact convex set of \mathbb{R}^d and f an entire function on \mathbb{C}^d . Then f is the Dunkl transform of a function in $\mathcal{D}(\mathbb{R}^d)$ with support in E , if and only if for all $q \in \mathbb{N}$ there exists a positive constant C_q such that

$$\forall z \in \mathbb{C}^d, |f(z)| \leq C_q (1 + \|z\|)^{-q} e^{I_E(Imz)} \quad (5.1)$$

Proof

The necessity condition of the theorem is proved in [9] corollary 4.10, p. 156. We prove now the sufficiency condition.

The upper bounds (5.1) imply that the function φ given by

$$\forall x \in \mathbb{R}^d, \varphi(x) = \int_{\mathbb{R}^d} f(y)K(iy, x)\omega_k(y)dy, \quad (5.2)$$

is a C^∞ -function on \mathbb{R}^d .

We consider the function

$$\forall x \in \mathbb{R}^d, \phi(x) = \int_{\mathbb{R}^d} f(y)K(iy, x)\frac{\omega_k(y)}{(1+\|y\|^2)^p}dy, \quad (5.3)$$

with $p \in \mathbb{N}$ such that $p > \gamma + \frac{d}{2} + 1$.

This function is of class C^∞ on \mathbb{R}^d and we have

$$\forall x \in \mathbb{R}^d, (I - \Delta_k)^p \phi(x) = \varphi(x), \quad (5.4)$$

where $\Delta_k = \sum_{j=1}^d T_j^2$ is the Dunkl Laplacian. As E is W -invariant then to show that $\text{Supp}\varphi \subset E$, it is sufficient to prove that the support of ϕ is contained in E . The relation (5.3) can also be written in the form

$$\forall x \in \mathbb{R}^d, \phi(x) = \int_{\mathbb{R}^d} f(y)K(iy, x)M_k(y)dy, \quad (5.5)$$

where $M_k(y) = \frac{\omega_k(y)}{(1+\|y\|^2)^p}$.

From (2.3) we have

$$\int_{\mathbb{R}^d} M_k(y)dy < +\infty. \quad (5.6)$$

We put

$$m_k(z) = m_k(z_1, \dots, z_d) = \int_{\mathbb{R}^d} \frac{M_k(y_1, \dots, y_d)}{(y_1 - z_1) \cdots (y_d - z_d)} dy_1 \cdots dy_d, \quad (5.7)$$

and for $a_j \geq 0, j = 1, 2, \dots, d$,

$$m_k^{a_1, \dots, a_d}(z) = \int_{-a_1}^{a_1} \cdots \int_{-a_d}^{a_d} \frac{M_k(y_1, \dots, y_d)}{(y_1 - z_1) \cdots (y_d - z_d)} dy_1 \cdots dy_d. \quad (5.8)$$

These functions are respectively holomorphic on $\mathbb{C}^d \setminus \mathbb{R}^d$ and $\mathbb{C}^d \setminus (\prod_{j=1}^d [-a_j, a_j])$, and for all $z \in \mathbb{C}^d \setminus \mathbb{R}^d$ we have

$$\lim_{a_1, \dots, a_d \rightarrow +\infty} m_k^{a_1, \dots, a_d}(z) = m_k(z). \quad (5.9)$$

Using Riemann sums, the function $m_k^{a_1, \dots, a_d}$ can also be written in the form

$$m_k^{a_1, \dots, a_d}(z) = \lim_{n_1, \dots, n_d \rightarrow +\infty} \left(\prod_{j=1}^d \frac{2a_j}{n_j} \right) \times \sum_{r_1=0}^{n_1} \dots \sum_{r_d=0}^{n_d} \frac{M_k(-a_1 + r_1 \frac{2a_1}{n_1}, \dots, -a_d + r_d \frac{2a_d}{n_d})}{\prod_{j=1}^d (-a_j + r_j \frac{2a_j}{n_j} - R_j)}. \quad (5.10)$$

For $j = 1, 2, \dots, d$, let Γ_j the quasi-rectangular path in \mathbb{C} determined by the points $z_j = -R_j - ib_j, z_j = -R_j + i\eta_j, z_j = +R_j + i\eta_j, z_j = R_j + ib_j$, with $R_j > 0, 0 < b_j < \frac{a_j}{n_j}, \eta_j > 0$, the half circles $C_{r_j}, r_j = 1, 2, \dots, n_j$, of center $(-a_j + r_j \frac{2a_j}{n_j})$ and radius $\frac{a_j}{n_j}$ and the segments $\{z_j = x_j + ib_j, x_j \in [-R_j, -a_j(1 + \frac{1}{n_j}) + b_j]\}$ $\{z_j = x_j + ib_j, x_j \in [a_j(1 + \frac{1}{n_j}) - b_j, R_j]\}$.

From Cauchy theorem we have

$$I_{a_1, \dots, a_d} = \int_{\Gamma_1} \dots \int_{\Gamma_d} f(z)K(iz, x)m_k^{a_1, \dots, a_d}(z)dz = 0. \quad (5.11)$$

On the other hand from (5.8) and (5.10) we have

$$\begin{aligned} I_{(a_1, \dots, a_d)} &= \int_{-a_1}^{a_1} \dots \int_{-a_d}^{a_d} M_k(y_1, \dots, y_d) \int_{\Gamma_1} \dots \int_{\Gamma_d} \frac{f(z)K(iz, x)}{\prod_{j=1}^d (y_j - z_j)} dz dy_1, \dots, dy_d, \\ &= \lim_{\substack{n_1 \rightarrow +\infty \\ n_d \rightarrow +\infty}} \left(\prod_{j=1}^d \frac{2a_j}{n_j} \right) \sum_{r_1=0}^{n_1} \dots \sum_{r_d=0}^{n_d} M_k(-a_1 + r_1 \frac{2a_1}{n_1}, \dots, -a_d + r_d \frac{2a_d}{n_d}) \\ &\quad \times \int_{\Gamma_1} \dots \int_{\Gamma_d} \frac{f(z_1, \dots, z_d)K(iz_1, \dots, iz_d, x)}{\prod_{j=1}^d (-a_j + r_j \frac{2a_j}{n_j} - z_j)} dz_1 \dots dz_d. \end{aligned} \quad (5.12)$$

We apply residus theorem to the integrals of the second member of this relation. As from (2.13) and (5.1) the integrals on the segments $\{z_j = -R_j + it_j, t_j \in [b_j, \eta_j]\}$ and $\{z_j = R_j + it_j, t_j \in [b_j, R_j]\}, j = 1, 2, \dots, d$, tend to zero when $R_1, \dots, R_d \rightarrow +\infty$. Then if we make $R_1, R_2, \dots, R_d \rightarrow +\infty$, we deduce from (5.11) that

$$\begin{aligned} &\int_{-\infty}^{-a_1 - b_1} \dots \int_{-\infty}^{-a_d - b_d} f(y + ib)K(i(y + ib), x)m_k^{a_1, \dots, a_d}(y + ib)dy + \\ &\int_{a_1 + b_1}^{\infty} \dots \int_{a_d + b_d}^{\infty} f(y + ib)K(i(y + ib), x)m_k^{a_1, \dots, a_d}(y + ib)dy + \\ &\int_{-a_1}^{a_1} \dots \int_{-a_d}^{a_d} f(y)K(iy, x) \frac{\omega_k(y)}{(1 + \|y\|^2)^p} dy - \\ &\frac{1}{(i\pi)^d} \int_{\mathbb{R}^d} f(y + i\eta)K(i(y + i\eta), x)m_k^{a_1, \dots, a_d}(y + i\eta)dy = 0, \end{aligned} \quad (5.13)$$

where $b = (b_1, b_2, \dots, b_d)$, $\eta = (\eta_1, \eta_2, \dots, \eta_d)$ and $y = (y_1, y_2, \dots, y_d)$. But from (5.1) (2.13) and (5.6) we deduce that the two first integrals tend to zero when

$a_1, \dots, a_d \rightarrow +\infty$. Thus by making $a_1, \dots, a_d \rightarrow +\infty$ in (5.13) we obtain

$$\forall x \in \mathbb{R}^d, \phi(x) = \frac{1}{(i\pi)^d} \int_{\mathbb{R}} f(y + i\eta) K(i(y + i\eta), x) m_k(y + i\eta) dy.$$

Using (5.1) and (2.13) we deduce that there exists a positive constant M such that

$$\forall x \in \mathbb{R}^d, |\phi(x)| \leq e^{I_E(\eta) - \langle x, \eta \rangle} M \int_{\mathbb{R}^d} \frac{|m_k(y + i\eta)|}{(1 + \|y\|^2)^p} dy, \quad (5.14)$$

If $x \notin E$, there exists η_0 such that $I_E(\eta_0) - \langle x, \eta_0 \rangle < 0$. By taking $\eta = s\eta_0$, with $s \geq 1$, in (5.14), there exists a positive constant M_0 such that

$$|\phi(x)| \leq M_0 e^{s(I_E(\eta_0) - \langle x, \eta_0 \rangle)} e^{s(I_E(\eta_0) - \langle x, \eta_0 \rangle)}.$$

But the second member of this inequality tend to zero when s go to the infinity. Then the support of ϕ is contained in E .

As the support of φ is contained in E , then from Theorem 4.2 we deduce that

$$\forall y \in \mathbb{R}^d, f(y) = \mathcal{F}_D(\varphi)(y).$$

Remark 5.1

Theorem 5.1 shows that Theorem 4.1 i) which is Theorem 5.2 i) of [15], can be proved without using Theorem 4.4 of [10], chap.3 p. 58.

Corollary 5.1. Let E be a W -invariant compact convex set of \mathbb{R}^d . Then for all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\text{Supp } f \subset E \iff {}^tV_k(f) \subset E. \quad (5.15)$$

Proof

We deduce the result from the relation

$$\mathcal{F}_D(f) = \mathcal{F} \circ {}^tV_k(f), \quad f \in \mathcal{D}(\mathbb{R}^d),$$

the Paley-Wiener theorem for the classical Fourier transform \mathcal{F} (see [1] Theorem 2.6, p. 17) and Theorem 5.1.

Corollary 5.2. Let E be a W -invariant compact convex set of \mathbb{R}^d . Then for all $x \in E$, the support of the distribution η_x given by the relation (3.7) is contained in E .

Proof

From (3.3) for all g in $\mathcal{D}(\mathbb{R}^d)$ with support in the complementary E^c of E and f in $\mathcal{D}(\mathbb{R}^d)$ with support in E , we have

$$\int_{\mathbb{R}^d} V_k^{-1}(g)(x) f(x) dx = \int_{\mathbb{R}^d} g(y) {}^tV_k^{-1}(f)(y) \omega_k(y) dy. \quad (5.16)$$

But from (5.15) we have

$$\text{Supp } f \subset E \Rightarrow \text{Supp } {}^tV_k^{-1}(f) \subset E.$$

Thus

$$\int_{\mathbb{R}^d} V_k^{-1}(g)(x)f(x)dx = 0.$$

This relation implies

$$\forall x \in E, V_k^{-1}(g)(x) = 0.$$

But from (3.7) we have

$$\forall x \in E, V_k^{-1}(g)(x) = \langle \eta_x, g \rangle.$$

Thus the support of η_x is contained in E .

Remark 5.2

Corollary 5.2 is Propostion 6.3 of [15] p. 30. The proof of this Corollary constitutes another proof of this proposition.

In the following we give an ameliorated version of the proof of Proposition 6.3.

Let $x \in E$ and $\varepsilon \in]0, 1]$. We consider the functions f_x and f_x^ε given by

$$\forall y \in \mathbb{R}^d, f_x(y) = \frac{e^{-i\langle x, y \rangle}}{(1 + \|y\|^2)^p}, \quad (5.17)$$

and

$$\forall y \in \mathbb{R}^d, f_x^\varepsilon(y) = f_x(y)\mathcal{F}_D(\mathcal{V}_\varepsilon)(y), \quad (5.18)$$

with $p \in \mathbb{N}$, such that $p > \gamma + \frac{d}{2} + 1$, and \mathcal{V}_ε the function defined by

$$\forall x \in \mathbb{R}^d, \mathcal{V}_\varepsilon(x) = \frac{1}{\varepsilon^{2\gamma+d}} \tilde{\mathcal{V}}\left(\frac{\|x\|}{\varepsilon}\right), \quad (5.19)$$

where \mathcal{V} is a radical, positive function in $\mathcal{D}(\mathbb{R}^d)$, with support in the ball of center 0 and radius 1, satisfying $\int_{\mathbb{R}^d} \mathcal{V}(x)\omega_k(x)dx = 1$, and $\tilde{\mathcal{V}}$ the function on $[0, +\infty[$ given by $\mathcal{V}(x) = \tilde{\mathcal{V}}(\|x\|)$.

As the functions f_x and f_x^ε belong to $(L_k^1 \cap L_k^2)(\mathbb{R}^d)$, then from Theorem 4.3 ii), the functions F_x and F_x^ε defined by

$$\forall t \in \mathbb{R}^d, F_x(t) = \int_{\mathbb{R}^d} f_x(t)K(iy, x)\omega_k(y)dy, \quad (5.20)$$

$$\forall t \in \mathbb{R}^d, F_x^\varepsilon(t) = \int_{\mathbb{R}^d} f_x^\varepsilon(t)K(iy, x)\omega_k(y)dy, \quad (5.21)$$

are in $C(\mathbb{R}^d)$ and from the relation (2.14) and the fact that there exists $M > 0$ such that

$$\forall y \in \mathbb{R}^d, |\mathcal{F}_D(\mathcal{V}_\varepsilon)(y) - 1| \leq \varepsilon M \|y\|^2, \quad (5.22)$$

we deduce that for all $t \in \mathbb{R}^d$:

$$\lim_{\varepsilon \rightarrow 0} F_x^\varepsilon(t) = F_x(t). \quad (5.23)$$

The relation (5.21) can also be written in the form

$$\forall t \in \mathbb{R}^d, F_x^\varepsilon(t) = \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} \mathcal{F}_D(\mathcal{V}_\varepsilon)(y) K(iy, t) M_k(y) dy,$$

where $M_k(y) = \frac{\omega_k(y)}{(1+\|y\|^2)^p}$. Using the method applied in the proof of Theorem 5.1, which is the same method applied also in the proof of Proposition 6.3, we deduce that

$$\forall t \in \mathbb{R}^d, |F_x^\varepsilon(t)| \leq e^{(I_{E+B_\varepsilon}(\eta) - \langle t, \eta \rangle)} \int_{\mathbb{R}^d} \frac{|m_k(u + i\eta)|}{(1 + \|u\|^2)^p} du,$$

where B_ε is the ball of center 0 and radius ε .

As for the proof of Theorem 5.1, we deduce that the support of F_x^ε is contained in $E + B_\varepsilon$.

From this result and (5.23) we show that

$$\text{Supp} F_x \subset E.$$

By applying now the remainder of the proof given in [15] p. 32, we deduce that $\text{Supp} \eta_x \subset E$.

5.2 The Dunkl transform of distributions

Notation. We denote by $\mathcal{H}(\mathbb{C}^d)$ the space of entire functions on \mathbb{C}^d which are slowly increasing and of exponential type.

The Dunkl transform of a distribution S in $\mathcal{E}'(\mathbb{R}^d)$ is defined by

$$\forall y \in \mathbb{R}^d, \mathcal{F}_D(S)(y) = \langle S_x, K(-iy, x) \rangle. \quad (5.24)$$

Remark 5.3

When S is given by a function g in $\mathcal{D}(\mathbb{R}^d)$, and denoted by T_g , the relation (5.24) coincides with (4.1), because T_g is defined by

$$\langle T_g, \varphi \rangle = \int_{\mathbb{R}^d} g(x) \varphi(x) \omega_k(x) dx, \varphi \in \mathcal{E}(\mathbb{R}^d). \quad (5.25)$$

The following theorem is proved in [15].

Theorem 5.2. The transform \mathcal{F}_D is a topological isomorphism from $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{H}(\mathbb{C}^d)$.

We give now the geometrical form of the Paley-Wiener theorem for distributions.

Theorem 5.3. Let E be a W -invariant compact convex set of \mathbb{R}^d and f an entire function on \mathbb{C}^d . Then f is the Dunkl transform of a distribution in $\mathcal{E}'(\mathbb{R}^d)$ with support in E if and only if there exist a positive constant C and $N \in \mathbb{N}$ such that

$$\forall z \in \mathbb{C}^d, |f(z)| \leq C(1 + \|z\|^2)^N e^{I_E(Imz)}. \quad (5.26)$$

Proof

- Necessity condition

We consider a distribution S in $\mathcal{E}'(\mathbb{R}^d)$ with support in E .

Let \mathcal{X} be in $\mathcal{D}(\mathbb{R}^d)$ equal to 1 in a neighbourhood of E , and θ in $\mathcal{E}(\mathbb{R})$ such that

$$\theta(t) = \begin{cases} 1, & \text{if } t \leq 1, \\ 0, & \text{if } t > 2. \end{cases}$$

We put $\eta = Imz, z \in \mathbb{C}^d$.

We denote by ψ_z the function defined on \mathbb{R}^d by

$$\psi_z(x) = \chi(x)K(-ix, z)\theta(\langle x, \eta \rangle - I_E(\eta)).$$

This function belongs to $\mathcal{D}(\mathbb{R}^d)$ and it is equal to $K(-ix, z)$ in a neighbourhood of E . Thus

$$\forall z \in \mathbb{C}^d, \mathcal{F}_D(S)(z) = \langle S_x, \psi_z(x) \rangle. \quad (5.27)$$

As S is with compact support, then it is of finite order N . Then there exists a positive constant C_0 such that

$$\forall z \in \mathbb{C}^d, |\mathcal{F}_D(S)(z)| \leq C_0 \sum_{|p| \leq N} \sup_{x \in \mathbb{R}^d} |D^p \psi_z(x)|. \quad (5.28)$$

Using Leibniz rule, we obtain

$$\forall x \in \mathbb{R}^d, D^p \psi_z(x) = \sum_{q+r+s=p} \frac{p!}{q!r!s!} D^q \mathcal{X}(x) D^r K(-ix, z) D^s \theta(\langle x, \eta \rangle - I_E(\eta)). \quad (5.29)$$

We have

$$\forall x \in \mathbb{R}^d, |D^q \mathcal{X}(x)| \leq \text{const.},$$

and

$$\forall x \in \mathbb{R}^d, |D^s \theta(\langle x, \eta \rangle - I_E(\eta))| \leq \text{const.} \|\eta\|^{|s|}.$$

On the other hand from (2.11) and the fact that E is W -invariant we have

$$\forall x \in \mathbb{R}^d, |D^r K(-ix, z)| \leq \|z\|^{|r|} e^{\langle x, \eta \rangle}.$$

Using these inequities and (5.29) we deduce that there exists a positive constant C_1 such that

$$\forall x \in \mathbb{R}^d, |D^p \psi_z(x)| \leq C_1 (1 + \|z\|^2)^N e^{\langle x, \eta \rangle}.$$

From this relation and (5.28) we obtain

$$\forall z \in \mathbb{C}^d, |\mathcal{F}_D(S)(z)| \leq C_2 (1 + \|z\|^2)^N \sup e^{\langle x, \eta \rangle}, \quad (5.30)$$

where C_2 is a positive constant, and the supremum is calculated for

$$\langle x, \eta \rangle \leq I_E(\eta) + 2.$$

But this inequality implies

$$\sup e^{\langle x, \eta \rangle} \leq e^2 e^{I_E(\eta)}. \quad (5.31)$$

From (5.30) and (5.31) we deduce that there exists a positive constant C such that

$$\forall z \in \mathbb{C}^d, |\mathcal{F}_D(S)(z)| \leq C(1 + \|z\|^2)^N e^{I_E(\eta)}. \quad (5.32)$$

- Sufficiency condition

Let f be an entire function on \mathbb{C}^d satisfying the condition (5.26). We consider the functions g and $g_\varepsilon, \varepsilon \in]0, 1]$, given by

$$\forall y \in \mathbb{R}^d, g(y) = \frac{f(y)}{(1 + \|y\|^2)^{p+N}}, \quad (5.33)$$

and

$$\forall y \in \mathbb{R}^d, g_\varepsilon(y) = g(y) \mathcal{F}_D(\mathcal{V}_\varepsilon)(y), \quad (5.34)$$

with $p \in \mathbb{N}$, such that $p > \gamma + \frac{d}{2} + 1$, and \mathcal{V}_ε the function defined by (5.19).

As the functions g and g_ε belong to $(L_k^1 \cap L_k^2)(\mathbb{R}^d)$, then from Theorem 4.3 ii) the functions G and G_ε defined by

$$\forall x \in \mathbb{R}^d, G(x) = \int_{\mathbb{R}^d} g(y) K(iy, x) \omega_k(y) dy,$$

$$\forall x \in \mathbb{R}^d, G_\varepsilon(x) = \int_{\mathbb{R}^d} g_\varepsilon(y) K(iy, x) \omega_k(y) dy,$$

are in $C(\mathbb{R}^d)$ and from (2.14) and (5.22), for all $x \in \mathbb{R}^d$, we obtain

$$\lim_{\varepsilon \rightarrow 0} G_\varepsilon(x) = G(x). \quad (5.35)$$

By making the same proof as for the Remark 5.1, we deduce that the support of G_ε is contained in $E + B_\varepsilon$, where B_ε is the ball of center 0 and radius ε . From (5.35) we deduce that

$$\text{Supp } G \subset E.$$

We put

$$S = (I - \Delta_k)^{p+N} G.$$

where Δ_k is the Dunkl Laplacian.

Then S is a distribution in $\mathcal{E}'(\mathbb{R}^d)$ with support in E . Moreover, we have

$$\forall y \in \mathbb{R}^d, \mathcal{F}_D(S)(y) = \int_{\mathbb{R}^d} (1 + \|y\|^2)^{p+N} G(x) K(-iy, x) \omega_k(x) dx.$$

Using Theorem 4.3 ii) and (5.33) we obtain

$$\forall y \in \mathbb{R}^d, \mathcal{F}_D(S)(y) = f(y).$$

This completes the proof of the Theorem.

6 Inversion formulas for the Dunkl intertwining operator and for its dual

6.1 The pseudo-differential operators P and Q

We consider the pseudo-differential operators P and Q defined on $\mathcal{D}(\mathbb{R}^d)$ by

$$\forall x \in \mathbb{R}^d, P(f)(x) = \frac{2^{2\gamma}}{\pi^d c_k^2} \mathcal{F}^{-1}[\omega_k \mathcal{F}(f)](x). \quad (6.1)$$

$$\forall x \in \mathbb{R}^d, Q(f)(x) = \frac{2^{2\gamma}}{\pi^d c_k^2} \mathcal{F}_D^{-1}[\omega_k \mathcal{F}_D(f)](x). \quad (6.2)$$

Proposition 6.1. For all f in $\mathcal{D}(\mathbb{R}^d)$ the function $P(f)$ and $Q(f)$ are of class C^∞ on \mathbb{R}^d and we have

$$\frac{\partial}{\partial x_j} P(f)(x) = P\left(\frac{\partial}{\partial \xi_j} f\right)(x), j = 1, 2, \dots, d, \quad (6.3)$$

$$T_j Q(f)(x) = Q(T_j f)(x), j = 1, 2, \dots, d, \quad (6.4)$$

Proof

We deduce the results by derivation under the integral sign, by using (2.7), (2.8) and the relations

$$\forall y \in \mathbb{R}^d, iy_j \mathcal{F}(f)(y) = \mathcal{F}\left(\frac{\partial}{\partial \xi_j} f\right)(y), \quad (6.5)$$

$$\forall y \in \mathbb{R}^d, iy_j \mathcal{F}_D(f)(y) = \mathcal{F}_D(T_j f)(y). \quad (6.6)$$

Proposition 6.2. Let E be a W -invariant compact convex set of \mathbb{R}^d . Then for all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$Supp f \subset E \Rightarrow Supp P(f) \subset E \text{ and } Supp Q(f) \subset E. \quad (6.7)$$

Proof

We obtain (6.7) by using the Paley-Wiener theorem for the Fourier transform \mathcal{F} (see [1] Theorem 2.6, p. 17), Theorem 5.1, and by applying the methode used in the proof of Theorem 5.1.

Proposition 6.3. We suppose that $k(\alpha) \in \mathbb{N}$ for all $\alpha \in R_+$. Then for all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$P(f) = \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \dots + \alpha_d \frac{\partial}{\partial \xi_d} \right)^{2k(\alpha)} \right] f, \quad (6.8)$$

$$Q(f) = \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} (\alpha_1 T_1 + \dots + \alpha_d T_d)^{2k(\alpha)} \right] f. \quad (6.9)$$

Proof

As $k(\alpha) \in \mathbb{N}$ for all $\alpha \in R_+$, then for all $f \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\forall y \in \mathbb{R}^d, \omega_k(y)\mathcal{F}(f)(y) = \prod_{\alpha \in R_+} (\langle \alpha, y \rangle)^{2k(\alpha)} \mathcal{F}(f)(y), \quad (6.10)$$

and

$$\forall y \in \mathbb{R}^d, \omega_k(y)\mathcal{F}_D(f)(y) = \prod_{\alpha \in R_+} (\langle \alpha, y \rangle)^{2k(\alpha)} \mathcal{F}_D(f)(y). \quad (6.11)$$

But

$$\forall y \in \mathbb{R}^d, \langle \alpha, y \rangle \mathcal{F}(f)(y) = \mathcal{F} \left[-i \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d} \right) f \right] (y), \quad (6.12)$$

and using (2.7), (2.8) we deduce that

$$\forall y \in \mathbb{R}^d, \langle \alpha, y \rangle \mathcal{F}_D(f)(y) = \mathcal{F}_D[-i(\alpha_1 T_1 + \cdots + \alpha_d T_d)f](y). \quad (6.13)$$

From (6.10), (6.11) and (6.12), (6.13) we obtain

$$\forall y \in \mathbb{R}^d, \omega_k(y)\mathcal{F}(f)(y) = \mathcal{F} \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} (\alpha_1 \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d})^{2k(\alpha)} f \right] (y),$$

and

$$\forall y \in \mathbb{R}^d, \omega_k(y)\mathcal{F}_D(f)(y) = \mathcal{F}_D \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} (\alpha_1 T_1 + \cdots + \alpha_d T_d)^{2k(\alpha)} f \right] (y).$$

These relations, the inversion formula for the Fourier transform \mathcal{F} , and Theorem 4.2 imply (6.8), (6.9).

Theorem 6.1. For all f in $\mathcal{D}(\mathbb{R}^d)$ we have the following transmutation relation

$$\forall x \in \mathbb{R}^d, P({}^t V_k(f))(x) = {}^t V_k(Q(f))(x). \quad (6.14)$$

Proof

From Proposition 6.2 and the relations (6.1), (4.3) we have

$$\begin{aligned} \forall x \in \mathbb{R}^d, P({}^t V_k(f))(x) &= \frac{2^{2\gamma}}{\pi^d c_k^2} \mathcal{F}^{-1} \left[\omega_k \mathcal{F} o {}^t V_k(f) \right] (x), \\ &= \frac{2^{2\gamma}}{\pi^d c_k^2} {}^t V_k \left\{ \mathcal{F}_D[\omega_k \mathcal{F}_D(f)] \right\} (x). \end{aligned}$$

Then (6.2) implies

$$\forall x \in \mathbb{R}^d, P({}^t V_k(f))(x) = {}^t V_k(Q(f))(x).$$

6.2 Inversion formulas for the Dunkl intertwining operator and for its dual

In this subsection we give inversion formulas for the operators V_k and tV_k and we deduce the expressions of the representing distributions of the operators V_k^{-1} and ${}^tV_k^{-1}$.

Theorem 6.2. For all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) = V_k(P(f))(x). \quad (6.15)$$

Proof

From Theorem 3.2 ii), for f in $\mathcal{D}(\mathbb{R}^d)$ the function ${}^tV_k^{-1}(f)$ belongs to $\mathcal{D}(\mathbb{R}^d)$. Then from Theorem 4.2 we have

$$\forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} K(iy, x) \mathcal{F}_D({}^tV_k^{-1}(f))(y) \omega_k(y) dy. \quad (6.16)$$

But from the relations (4.3), (3.1), we have

$$\forall y \in \mathbb{R}^d, \mathcal{F}_D({}^tV_k^{-1}(f))(y) = \mathcal{F}(f)(y),$$

and

$$\forall y \in \mathbb{R}^d, K(iy, x) = \mathcal{F}(\check{\mu}_x)(y),$$

where $\check{\mu}_x$ is the probability measure given by

$$\int_{\mathbb{R}^d} f(t) d\check{\mu}_x(t) = \int_{\mathbb{R}^d} f(-t) d\mu_x(t), f \in C(\mathbb{R}^d).$$

Thus (6.16) can also be written in the form

$$\forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}(\check{\mu}_x)(y) \omega_k(y) \mathcal{F}(f) dy. \quad (6.17)$$

But from Proposition 6.2 and (6.1) we have

$$\forall y \in \mathbb{R}^d, \omega_k(y) \mathcal{F}(f)(y) = \frac{2^{2\gamma}}{\pi^d c_k^2} \mathcal{F}(P(f))(y).$$

Then by using (6.17) and the properties of the Fourier transform \mathcal{F} we obtain

$$\begin{aligned} \forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(\check{\mu}(y) \mathcal{F}(P(f)))(y) dy, \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(\check{\mu}_x * P(f))(y) dy, \end{aligned}$$

where $*$ is the classical convolution product on \mathbb{R}^d of a measure and a function. By using Proposition 6.2, the fact that $\check{\mu}_x$ is a probability measure on \mathbb{R}^d and the inversion formula for the Fourier transform \mathcal{F} , we deduce that

$$\forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) = \check{\mu}_x * P(f)(0).$$

But from (3.1) we have

$$\check{\mu}_x * P(f)(0) = \int_{\mathbb{R}^d} P(f)(-t) d\check{\mu}_x(t) = V_k(P(f))(x).$$

Thus

$$\forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) = V_k(P(f))(x).$$

Corollary 6.1. For all f in $\mathcal{D}(\mathbb{R}^d)$ we have the following relation

$$\forall x \in \mathbb{R}^d, V_k(P^2(f))(x) = QV_k(P(f))(x). \quad (6.18)$$

Proof

We deduce (6.18) from Theorem 6.2, 6.3.

Theorem 6.4. For all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\forall x \in \mathbb{R}^d, V_k^{-1}(f)(x) = {}^tV_k(Q(f))(x). \quad (6.19)$$

Proof

We obtain the result by using Proposition 6.3 and Theorems 6.1, 6.2.

Notation

We denote by $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions on \mathbb{R}^d . It is the topological dual of $\mathcal{D}(\mathbb{R}^d)$.

Definition 6.1. We define the transposed operators tP and tQ of the operators P and Q on $\mathcal{D}'(\mathbb{R}^d)$ by

$$\langle {}^tP(S), f \rangle = \langle S, P(f) \rangle, f \in \mathcal{D}(\mathbb{R}^d), \quad (6.20)$$

$$\langle {}^tQ(S), f \rangle = \langle S, Q(f) \rangle, f \in \mathcal{D}(\mathbb{R}^d). \quad (6.21)$$

Proposition 6.4. We suppose that $k(\alpha) \in \mathbb{N}$ for all $\alpha \in R_+$. Then for all $S \in \mathcal{D}'(\mathbb{R}^d)$ we have

$${}^tP(S) = \left[\prod_{\alpha \in R_+} \left(\alpha \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d} \right)^{2k(\alpha)} \right] S, \quad (6.22)$$

$${}^tQ(S) = \left[\prod_{\alpha \in R_+} (\alpha T_1 + \cdots + \alpha_d T_d)^{2k(\alpha)} \right] S, \quad (6.23)$$

where $T_j, j = 1, 2, \dots, d$, are the Dunkl operators defined on $\mathcal{D}'(\mathbb{R}^d)$ by

$$\langle T_j S, f \rangle = -\langle S, T_j f \rangle, f \in \mathcal{D}(\mathbb{R}^d).$$

Theorem 6.5. The representing distributions η_x and Z_x of the inverse of the Dunkl intertwining operator and of its dual, are given by

$$\forall x \in \mathbb{R}^d, \eta_x = {}^tQ(\nu_x), \quad (6.24)$$

and

$$\forall x \in \mathbb{R}^d, Z_x = {}^tP(\mu_x). \quad (6.25)$$

where μ_x and ν_x are the representing measures of the Dunkl intertwining operator V_k and of its dual tV_k .

Proof

- From (3.4), for all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\forall x \in \mathbb{R}^d, {}^tV_k(Q(f))(x) = \langle \nu_x, Q(f) \rangle = \langle {}^tQ(\nu_x), f \rangle. \quad (6.26)$$

On the other hand from (3.7) :

$$\forall x \in \mathbb{R}^d, V_k^{-1}(f)(x) = \langle \eta_x, f \rangle.$$

We obtain (6.24) from this relation, (6.26) and (6.19).

- By using (3.1), for all f in $\mathcal{D}(\mathbb{R}^d)$ the relation (6.15) can also be written in the form

$$\forall x \in \mathbb{R}^d, {}^tV_k^{-1}(f)(x) = \langle \mu_x, P(f) \rangle = \langle {}^tP(\mu_x), f \rangle. \quad (6.27)$$

But from (3.10) we have

$$\forall x \in \mathbb{R}^d, {}^tV^{-1}(f)(x) = \langle Z_x, f \rangle.$$

We deduce (6.25) from this relation and (6.27).

Remark 6.1

When $k(\alpha) \in \mathbb{N}$ for all $\alpha \in R_+$, we have

$$\forall x \in \mathbb{R}^d, \eta_x = \left[\prod_{\alpha \in R_+} (\alpha_1 T_1 + \dots + \alpha_d T_d)^{2k(\alpha)} \right] (\nu_x), \quad (6.28)$$

and

$$\forall x \in \mathbb{R}^d, Z_x = \left[\prod_{\alpha \in R_+} \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \dots + \alpha_d \frac{\partial}{\partial \xi_d} \right)^{2k(\alpha)} \right] (\mu_x). \quad (6.29)$$

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