ON THE NULL-CONTROLLABILITY OF THE HEAT EQUATION IN UNBOUNDED DOMAINS

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ABSTRACT. We make two remarks about the null-controllability of the heat equation with Dirichlet condition in unbounded domains. Firstly, we give a geometric necessary condition (for interior null-controllability in the Euclidean setting) which implies that one can not go infinitely far away from the control region without tending to the boundary (if any), but also applies when the distance to the control region is bounded. The proof builds on heat kernel estimates. Secondly, we describe a class of null-controllable heat equations on unbounded product domains. Elementary examples include an infinite strip in the plane controlled from one boundary and an infinite rod controlled from an internal infinite rod. The proof combines earlier results on compact manifolds with a new lemma saying that the null-controllability of an abstract control system and its null-controllability cost are not changed by taking its tensor product with a system generated by a non-positive self-adjoint operator.

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1. INTRODUCTION.

1.1. **The problem.** Let M be a smooth connected complete n-dimensional Riemannian manifold with boundary ∂M. When $\partial M \neq \emptyset$, M denotes the interior and $\overline{M} = M \cup \partial M$. Let Δ denote the (negative) Laplacian on M.

Consider a positive control time T and a non-empty open control region Γ of ∂M. Let $\mathbf{1}_{]0,T[\times\Gamma}$ denote the characteristic function of the space-time control region $]0,T[\times\Omega$. The heat equation on M is said to be *null-controllable* in time T by boundary controls on Γ if for all $\phi_0 \in L^2(M)$ there is a control function

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$u \in L^2_{\text{loc}}(\mathbb{R}; L^2(\partial M))$ such that the solution $\phi \in C^0([0,\infty), L^2(M))$ of the mixed Dirichlet-Cauchy problem:

$$(1) \qquad \partial_t \phi - \Delta\phi = 0 \quad \text{in }]0,T[\times M, \quad \phi = \mathbf{1}_{]0,T[\times\Gamma}\, u \quad \text{on }]0,T[\times\partial M,$$

with Cauchy data $\phi = \phi_0$ at $t = 0$, satisfies $\phi = 0$ at $t = T$. The *null-controllability cost* is the best constant, denoted $C_{T,\Gamma}$, in the estimate:

$$\|u\|_{L^2(]0,T[\times\Gamma)} \leqslant C_{T,\Gamma}\|\phi_0\|_{L^2(M)}$$

for all initial data ϕ_0 and control u solving the null-controllability problem described above. The analogous interior null-controllability problem from a non-empty open subset Ω of $\overline{M}$ is also considered:

$$(2) \qquad \begin{aligned} &\partial_t\phi - \Delta\phi = \mathbf{1}_{]0,T[\times\Omega}\, u \text{ on } \mathbb{R}_t \times M, \quad \phi = 0 \text{ on } \mathbb{R}_t \times \partial M, \\ &\phi(0) = \phi_0 \in L^2(M), \ u \in L^2_{\text{loc}}(\mathbb{R}; L^2(M)). \end{aligned}$$

When M is compact (for instance a bounded domain of the Euclidean space), Lebeau and Robbiano have proved (in [LR95] using local Carleman estimates) that, for all T and Γ there is a continuous linear operator $S : L^2(M) \to C_0^\infty(\mathbb{R} \times \partial M)$ such that $u = S\phi_0$ yields the null-controllability of the heat equation (1) on M in time T by boundary controls on Γ. They have also proved the analogous result for (2) which implies that interior null-controllability holds for arbitrary T and Ω. (We refer to [FI96] for a proof of null-controllability for more general parabolic problems using global Carleman estimates.)

The null-controllability of the heat equation when M is an unbounded domain of the Euclidean space is an open problem which Micu and Zuazua have recently underscored in [MZ03]. On the one hand, it is only known to hold when $M \setminus \Omega$ is bounded (cf. [CdMZ01]). On the other hand, its failure can be much more drastic than in the bounded case (when M is the half space and $\Gamma = \partial M$, it is proved in [MZ01a, MZ01b] that initial data whith Fourier coefficients that grow less than any exponential are not null-controllable in any time, whereas there are initial data with exponentially growing Fourier coefficients that are null-controllable).

The geometric aspect of the open problem in [MZ03] is addressed here with examples of null-controllability with unbounded uncontrolled region, and lack thereof including when the distance to the controlled region is finite (cf. th.1.4.iii). The geometric necessary condition in th.1.11 grasps at some notion of "controlling capacity" of a subset that would yield a necessary and sufficient condition for interior null-controllability.

1.2. **Elementary examples.**

Before stating the results in full generality, we give elementary examples.

The simplest (bounded) case to study is when M is a segment and Γ is one of the end points. It is well-known that this problem reduces by spectral analysis to classical results on non-harmonic Fourier series. For further reference, we introduce the optimal fast control cost rate for this problem:

Definition **1.1.** The rate α_* is the smallest positive constant such that for all $\alpha > \alpha_*$ there exists $\gamma > 0$ such that, for all $L > 0$ and $T \in \left]0, \inf(\pi, L)^2\right]$, the null-controllability cost $C_{L,T}$ of the heat equation (1) on the Euclidean interval $M =]0, L[$ (i.e. $\Delta = \partial_x^2$) from $\Gamma = \{0\}$ satisfies: $C_{L,T} \leqslant \gamma \exp(\alpha L^2/T)$.

Computing α_* is an interesting open problem. As proved in [Mil03b],

Theorem 1.2. *The rate α_* defined above satisfies: $1/4 \leqslant \alpha_* \leqslant 2\,(36/37)^2 < 2$.*

The simplest unbounded case where null-controllability holds is probably the following, which extends to an infinite strip the null-controllability from one side of a rectangle proved in [Fat75].

Theorem 1.3. *The heat equation (1) on the infinite strip* $M =]0, L[\times\mathbb{R}$ *of the Euclidean plane (i.e.* $\Delta = \partial_x^2 + \partial_y^2$*) is null-controllable from one side* $\Gamma = \{(x, y) | x = 0, y \in \mathbb{R}\}$ *in any time* $T > 0$*. Moreover, the corresponding null-controllability cost satisfies (with* α_* *as in th.1.2):* $\limsup_{T\to 0} T \ln C_{\Gamma,T} \leqslant \alpha_* L^2$*.*

Here is an example in the usual three dimensional space which illustrates interior null-controllability and lack thereof.

Theorem 1.4. *Consider the heat equation (2) on the infinite rod* $M = S \times \mathbb{R}$ *in the Euclidean space (i.e.* $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$*) where the section* S *is any smooth connected bounded open set of the plane.*

i) It is null-controllable in any time $T > 0$ *from any interior infinite rod* $\Omega = \omega\times\mathbb{R}$ *where the section* ω *is an open non empty subset of* $\overline{S}$*. Moreover, if* ω *contains a neighborhood of the boundary of* S *and* $S\backslash\omega$ *does not contain any segment of length* L*, then the corresponding null-controllability cost satisfies (with* α_* *as in th.1.2):* $\limsup_{T\to 0} T \ln C_{\Omega,T} \leqslant \alpha_* L^2$*.*

ii) It is not null-controllable in any time $T > 0$ *from any interior region* Ω *of finite Lebesgue measure such that* $M \setminus \Omega$ *contains slabs* $S \times [z_1, z_2]$ *of arbitrarily large thickness* $|z_2 - z_1|$*.*

iii) It is not null-controllable in any time $T > 0$ *from the cylindrical interior region* $\Omega = \{(x, y, z) \in M \,|\, x^2 + y^2 < R(z)^2\}$ *if* $(0, 0) \in S$ *and the lower semi-continuous function* $R : \mathbb{R} \to [0,\infty)$ *tends to zero at infinity.*

1.3. **Main results.** A large class of null-controllable heat equations on unbounded domains is generated by the two following theorems concerning respectively boundary and interior controllability. In both theorems, $\tilde{M}$ denotes another smooth complete $\tilde{n}$-dimensional Riemannian manifold and $\tilde{\Delta}$ denotes the corresponding Laplacian.

Theorem 1.5. *Let* γ *denote the subset* $\Gamma \times \tilde{M}$ *of* $\partial(M \times \tilde{M})$*. If the heat equation (1) is null-controllable at cost* $C_{T,\Gamma}$ *then the heat equation:*

$$\partial_t\phi - (\Delta + \tilde{\Delta})\phi = 0 \quad \text{on} \quad \mathbb{R}_t \times M \times \tilde{M}, \quad \phi = \mathbf{1}_\gamma g \quad \text{on} \quad \mathbb{R}_t \times \partial(M \times \tilde{M}),$$
$$\phi(0) = \phi_0 \in L^2(M \times \tilde{M}),\ g \in L^2_{loc}(\mathbb{R}; L^2(\partial(M \times \tilde{M}))),$$

is exactly controllable in any time T *at a cost* $\tilde{C}_{T,\gamma}$ *which is not greater than* $C_{T,\Gamma}$*.*

Theorem 1.6. *Let* ω *denote the subset* $\Omega \times \tilde{M}$ *of* $M \times \tilde{M}$*. If the heat equation (2) is null-controllable at cost* $C_{T,\Omega}$ *then the heat equation:*

$$\partial_t\phi - (\Delta + \tilde{\Delta})\phi = \mathbf{1}_\omega g \ \text{on} \ \mathbb{R}_t \times M \times \tilde{M}, \quad \phi = 0 \quad \text{on} \quad \mathbb{R}_t \times \partial(M \times \tilde{M}),$$
$$\phi(0) = \phi_0 \in L^2(M \times \tilde{M}),\ g \in L^2_{loc}(\mathbb{R}; L^2(M \times \tilde{M})),$$

is exactly controllable in any time T *at a cost* $\tilde{C}_{T,\omega}$ *which is not greater than* $C_{T,\Omega}$*.*

Remark 1.7. Th.1.4 i) is a particular case of th.1.6 with $M = S$, $\tilde{M} = \mathbb{R}$, inverted Ω and ω, and the cost estimate results from the cost estimate on M proved in [Mil03b]. Th.1.5 and th.1.6 apply, for instance, to any open subset $\tilde{M}$ of the Euclidean space $\mathbb{R}^{\tilde{n}}$. Thanks to the results of [LR95] already mentioned in section 1.1, the conclusions of these theorems hold for arbitrary control regions of a compact M. Then they can be applied recursively, taking the resulting null-controllable product manifold as the new M (the theorems are still valid if M has corners).

Remark 1.8. The case when M is a bounded Euclidean set and $\tilde{M} = (0, \varepsilon)$ with Neumann boundary conditions at both ends has been considered in [dTZ00] with an extra time-dependent potential. When $\varepsilon \to 0$, using global Carleman estimates, it is proved that the cost is uniform (as in th.1.6) and depends on the uniform norm of the potential. Moreover, the limit of the control functions is a control function for the limit problem.

Remark 1.9. The type of boundary conditions are irrelevant to the proof of th.1.5 and th.1.6. These theorems can be combined with th.6.2 in [Mil03a] and th.2.3 in [Mil03b] respectively to obtain bounds on the fast null-controllability cost:

$$\limsup_{T\to 0} T\ln \tilde{C}_{\gamma,T} \leqslant \alpha_* L_\Gamma^2 \quad \text{and} \quad \limsup_{T\to 0} T\ln \tilde{C}_{\omega,T} \leqslant \alpha_* L_\Omega^2$$

for any L_Γ and L_Ω such that every generalized geodesic of length greater than L_Γ passes through Γ at a non-diffractive point, and every generalized geodesic of length greater than L_Ω passes through Ω. We refer readers interested by these bounds to [Mil03b, Mil03a] where more is said about generalized geodesics and the extra geometric assumptions needed to use them.

The last result states a geometric condition which is necessary for the interior null-controllability of the heat equation on an unbounded domain of the Euclidean space. This condition involves the following "distances".

Definition **1.10.** In $\mathbb{R}^n$, the Euclidean distance of points from the origin and the Lebesgue measure of sets are both denoted by $|\cdot|$. Let M be a non-empty open subset of $\mathbb{R}^n$. Let $d : \overline{M}^2 \to \mathbb{R}_+$ denote the distance function on M, i.e. the infimum of lengths of arcs in M with end points x and y (n.b., in terms of Lipschitz potentials: $d(x,y) = \sup_{\psi\in\mathrm{Lip}(\overline{M}),\|\nabla\psi\|_{L^\infty}\leqslant 1}|\psi(x)-\psi(y)|$). The distance of $y\in M$ from the boundary of M is $d_\partial(y) = \inf_{x\in\mathbb{R}^n\setminus M}|x-y|$. The distance of $y\in\overline{M}$ from $\Omega\subset M$ is $d(y,\Omega) = \inf_{x\in\Omega} d(x,y)$. We define the *averaged distance* $\bar{d}_T(y,\Omega)$ of y to Ω with Gaussian weight of variance T by

$$\bar{d}_T(y,\Omega)^2 = -2T\log\left(\int_\Omega \exp\left(-\frac{d(y,x)^2}{2T}\right)dx\right) \geqslant d(y,\Omega)^2 - 2T\log|\Omega| \ .$$

Technically, we shall use the following *bounded distance* of y to ∂M:

$$\underline{d}_T(y,\partial M) = \min\left\{d_\partial(y), T\pi^2 n/4\right\} \ .$$

Theorem 1.11. *Let M be a connected open subset of $\mathbb{R}^n$ and let Ω be an open subset of M. If there are a sequence $\{y_k\}_{k\in\mathbb{N}}$ of points in M, a time $\bar{T}>0$ and a constant $\kappa>1$ such that*

$$\bar{d}_{\bar{T}}(y_k,\Omega)^2 - \kappa\frac{\pi^2 n^2}{4}\left(\frac{\bar{T}}{\underline{d}_{\bar{T}}(y_k,\partial M)}\right)^2 \to +\infty\,,\ \textit{as } k\to+\infty\ , \tag{3}$$

then the heat equation (2) is not null-controllable in any time $T<\bar{T}$. In particular, when Ω has finite Lebesgue measure, if there is a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that $\inf_k d_\partial(y_k)>0$ and $\lim_k d(y_k,\Omega)=\infty$, then the heat equation (2) is not null-controllable in any time T.

Remark 1.12. The simple condition in the second part of th.1.11 is enough to prove th.1.4 ii) (consider the points $(0,0,(z_2-z_1)/2)$ of a sequence of slabs $S\times[z_1,z_2]$ in $M\setminus\Omega$ with thickness $|z_2-z_1|$ tending to infinity). Th.1.4 iii) illustrates that it may fail although the finer condition (3) holds. The second term in the geometric condition (3) allows $\{y_k\}_{k\in\mathbb{N}}$ to tend to the boundary of M. To illustrate its usefulness, we give yet another example in rk.3.2.

Remark 1.13. The proof of th.1.11 in sect.3.3 builds on heat kernel estimates. Generalizations to some non-compact manifolds can obviously be obtained using the heat kernel estimates available in the literature (cf. [Zha03] and ref. therein). We consider null-controllability on non-compact manifolds in a forthcoming paper.

2. An abstract lemma on tensor products

In this section, we prove that the cost of null-controllability of an abstract control system is not changed by taking its tensor product with an uncontrolled system generated by a non-positive self-adjoint operator.

2.1. **Abstract setting.**

We first recall the general setting for control systems: admissibility, observability and controllability notions and their duality (cf. [DR77] and [Wei89]).

Let Z and $\mathcal{V}$ be Hilbert spaces. Let $\mathcal{A} : D(\mathcal{A}) \to Z$ be the generator of a strongly continuous group of bounded operators on Z. Let Z_1 denote $D(\mathcal{A})$ with the norm $\|z\|_1 = \|(\mathcal{A} - \beta)z\|$ for some $\beta \notin \sigma(\mathcal{A})$ ($\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$, this norm is equivalent to the graph norm and Z_1 is densely and continuously embedded in Z) and let Z_{-1} be the completion of Z with respect to the norm $\|\zeta\|_{-1} = \|(\mathcal{A}-\beta)^{-1}\zeta\|$. Let Z' denote the dual of Z with respect to the pairing $\langle\cdot,\cdot\rangle$. The dual of $\mathcal{A}$ is a self-adjoint operator $\mathcal{A}'$ on Z'. The dual of Z_1 is the space Z'_{-1} which is the completion of Z' with respect to the norm $\|\zeta\|_{-1} = \|(\mathcal{A}' - \bar{\beta})^{-1}\zeta\|$ and the dual of Z_{-1} is the space Z'_1 which is $D(\mathcal{A}')$ with the norm $\|z\|_1 = \|(\mathcal{A}' - \bar{\beta} z\|$.

Let $\mathcal{C} \in \mathcal{L}(Z_1, \mathcal{V})$ and let $\mathcal{C}' \in \mathcal{L}(\mathcal{V}', Z'_{-1})$ denote its dual. Note that the same theory applies to any $\mathcal{A}$-bounded operator $\mathcal{C}$ with a domain invariant by $(e^{t\mathcal{A}})_{t\geqslant 0}$ since it can be represented by an operator in $\mathcal{L}(Z_1, \mathcal{V})$ (cf. [Wei89]).

We consider the dual observation and control systems with output function v and input function u:

$$\dot{z}(t) = \mathcal{A}z(t), \quad z(0) = z_0 \in Z, \quad v(t) = \mathcal{C}z(t), \tag{4}$$

$$\dot{\zeta}(t) = \mathcal{A}'\zeta(t) + \mathcal{C}'u(t), \quad \zeta(0) = \zeta_0 \in Z', \quad u \in L^2_{\text{loc}}(\mathbb{R}; Z'). \tag{5}$$

We make the following equivalent admissibility assumptions on the observation operator $\mathcal{C}$ and the control operator $\mathcal{C}'$ (cf. [Wei89]): $\forall T > 0$, $\exists K_T > 0$,

$$\forall z_0 \in D(\mathcal{A}), \quad \int_0^T \|\mathcal{C}e^{t\mathcal{A}}z_0\|^2 dt \leqslant K_T \|z_0\|^2, \tag{6}$$

$$\forall u \in L^2(\mathbb{R}; \mathcal{V}'), \quad \|\int_0^T e^{t\mathcal{A}'}\mathcal{C}'u(t)dt\|^2 \leqslant K_T \int_0^T \|u(t)\|^2 dt. \tag{7}$$

With this assumption, the output map $z_0 \mapsto v$ from $D(\mathcal{A})$ to $L^2_{\text{loc}}(\mathbb{R}; \mathcal{V})$ has a continuous extension to Z. The equations (4) and (5) have unique solutions $z \in C(\mathbb{R}, Z)$ and $\zeta \in C(\mathbb{R}, Z')$ defined by:

$$z(t) = e^{t\mathcal{A}}z_0, \quad \zeta(t) = e^{t\mathcal{A}'}\zeta(0) + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}u(s)ds. \tag{8}$$

The following dual notions of observability and controllability are equivalent (cf. [DR77]).

Definition 2.1. The system (4) is *final observable* in time $T > 0$ at cost $\kappa_T > 0$ if the following observation inequality holds: $\forall z_0 \in Z, \quad \|z(T)\|^2 \leqslant \kappa_T^2 \int_0^T \|v(t)\|^2 dt$. The system (5) is *null-controllable* in time $T > 0$ at cost $\kappa_T > 0$ if for all ζ_0 in Z', there is a u in $L^2(\mathbb{R}; \mathcal{V}')$ such that $\zeta(T) = 0$ and $\int_0^T \|u(t)\|^2 dt \leqslant \kappa_T^2 \|\zeta_0\|^2$. The *null-controllability cost* for (5) in time T is the smallest constant in the latter inequality (equivalently in the former observation inequality), still denoted κ_T. When (5) is not null-controllable in time T, we set $\kappa_T = +\infty$.

2.2. **Tensor products.** Now, we introduce the specific tensor product structure of the abstract control systems (5) under consideration here. Let X, Y, V be separable Hilbert spaces and I denote the identity operator on each of them. Let $A : D(A) \to X$ and $B : D(B) \to Y$ be generators of strongly continuous semigroups of bounded operators on X and Y. Let $C \in \mathcal{L}(X_1, V)$ be admissible for the control system:

$$\dot{\xi}(t) = A'\xi(t) + C'u(t), \quad \xi(0) = \xi_0 \in X', \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}; V'). \tag{9}$$

Let $X \overline{\otimes} Y$ and $V \overline{\otimes} Y$ denote the closure of the algebraic tensor products $X \otimes Y$ and $V \otimes Y$ for the natural Hilbert norms. The operator $C \otimes I : D(C) \otimes Y \to V \overline{\otimes} Y$ is densely defined on $X \overline{\otimes} Y$. The operator $A \otimes I + I \otimes B$ defined on the algebraic $D(A) \otimes D(B)$ is closable and its closure, denoted $A + B$, generates a strongly continuous semigroup of bounded operators on $X \overline{\otimes} Y$.

Lemma 2.2. *Let $Z = X \overline{\otimes} Y$, $\mathcal{V} = V \overline{\otimes} Y$, $\mathcal{A} = A + B$ and $\mathcal{C} = C \otimes I$. If B is a non-positive self-adjoint operator, then, for all $T > 0$, the null-controllability cost κ_T for (5) is lower then the null-controllability cost k_T for (9) in the same time T.*

Proof. We may assume that k_T is finite. By definition it satisfies:

$$\forall x \in X, \quad \|e^{TA}\|^2 \leqslant k_T^2 \int_0^T \|Ce^{tA}\|^2 dt. \tag{10}$$

We have to prove that:

$$\forall z \in X \overline{\otimes} Y \quad \mathcal{E} := \|e^{T(A+B)}z\|^2 \leqslant k_T^2 \int_0^T \|(C \otimes I)e^{t(A+B)}z\|^2 dt =: \mathcal{O}. \tag{11}$$

As explained in the proof of lem. 7.1 in [Mil04]:

$$\forall t \geqslant 0, \quad e^{t(A+B)} = e^{tA} \otimes e^{tB} \ . \tag{12}$$

Applying the spectral theorem for unbounded self-adjoint operators on separable Hilbert spaces to $B \leqslant 0$ (cf. th. VIII.4 in [RS79]), yields a measure space $(M, \mathcal{M}, \mu)$ with finite measure μ, a measurable function $b : M \to (-\infty, 0]$ and a unitary operator $U : Y \to L^2(M, d\mu)$ such that:

$$\forall y \in Y, \quad \|e^{tB}y\|^2 = \int_M e^{2tb(m)}|Uy(m)|^2 \mu(dm) \ . \tag{13}$$

Since X is separable, there is a unique isomorphism from $X \overline{\otimes} L^2(M, d\mu)$ to $L^2(M, d\mu; X)$ so that $x \otimes f(m) \mapsto f(m)x$ (cf. th. II.10 in [RS79]). We denote by $\mathcal{U} : X \overline{\otimes} Y \to L^2(M, d\mu; X)$ the composition of this isomorphism with $I \otimes U$. Similarly, there is a unique isomorphism from $V \overline{\otimes} L^2(M, d\mu)$ to $L^2(M, d\mu; V)$ so that $v \otimes f(m) \mapsto f(m)v$. We denote by $\mathcal{V} : V \overline{\otimes} Y \to L^2(M, d\mu; V)$ the composition of this isomorphism with $I \otimes U$. By decomposing into an orthonormal basis of X, (13) implies:

$$\forall z \in X \overline{\otimes} Y, \quad \|(I \otimes e^{tB})z\|^2 = \int_M e^{2tb(m)}|\mathcal{U}z(m)|^2 \mu(dm) \tag{14}$$

$$\forall w \in V \overline{\otimes} Y, \quad \|(I \otimes e^{tB})w\|^2 = \int_M e^{2tb(m)}|\mathcal{V}w(m)|^2 \mu(dm) \ . \tag{15}$$

Let $z \in X \overline{\otimes} Y$. Applying (10) to $\mathcal{U}z(m)$ for fixed $m \in M$ and integrating yields:

$$\int_M \|e^{TA}\mathcal{U}z(m)\|^2 e^{2tb(m)} \mu(dm) \leqslant k_T^2 \int_M e^{2Tb(m)} \int_0^T \|Ce^{tA}\mathcal{U}z(m)\|^2 dt \, \mu(dm) \ .$$

Since $e^{TA}\mathcal{U}z = \mathcal{U}(e^{TA} \otimes I)z$, (14) and (12) imply that the left hand side is $\mathcal{E}$ defined in (11). Using Fubini's theorem and $b \leqslant 0$ to bound the right hand side from above yields:

$$\mathcal{E} \leqslant k_T^2 \int_0^T \int_M e^{2tb(m)} \|Ce^{tA}\mathcal{U}z(m)\|^2 \mu(dm)\, dt \ .$$

Since $Ce^{tA}\mathcal{U}z = \mathcal{V}(Ce^{tA} \otimes I)z$, (15) and (12) imply that the right hand side is $\mathcal{O}$ defined in (11), which completes the proof of (11). □

2.3. **Proof of th.1.3, th.1.5 and th.1.6.** The first part of th.1.3 is a particular case of th.1.5. The second part is an estimate on the null-controllability cost which results from def.1.1 and lem.2.2 with $X = L^2(0,L)$, $Y = L^2(\mathbb{R})$, $Z = \mathbb{R}$, $A = \partial_x^2$, $D(A) = H^2(0,L) \cap H_0^1(0,L)$, $B = \partial_y^2$, $D(B) = H^2(\mathbb{R})$, $Cf = \partial_x f_{\rceil x=0}$. The reader balking at the abstraction of lem.2.2 can prove it in this particular case using the Fourier transform on the real line in the y variable where the spectral theorem was used (then μ is the Lebesgue measure and $b(m) = -|m|^2$) and a discrete Fourier decomposition on the interval in the x variable.

Th.1.5 and th.1.6 are direct applications of lem.2.2 with $X = L^2(M)$, $Y = L^2(\tilde{M})$, $A = \Delta$, $D(A) = H^2(M) \cap H_0^1(M)$, $B = \tilde{\Delta}$, $D(B) = H^2(\tilde{M}) \cap H_0^1(\tilde{M})$. Th.1.5 corresponds to $Z = L^2(\Gamma)$ and $Cf = \partial_\nu f_{\rceil \Gamma}$ where ∂_ν denotes the exterior Neumann vector field on ∂M. Th.1.6 corresponds to $Z = L^2(\Omega)$ and $Cf = f_{\rceil \Omega}$.

3. Geometric necessary condition.

In this section, we prove th.1.11. Henceforth, the domain of the Laplacian is $D(\Delta) = H^2(M) \cap H_0^1(M)$. Since controllability and observability in def.2.1 are equivalent, the heat equation (2) is null-controllable in time T if and only if there is a $C_{\Omega,T} > 0$ such that

$$(16) \qquad \forall f_0 \in L^2(M), \quad \int_M |e^{T\Delta} f_0|^2 dx \leqslant C_{\Omega,T} \int_0^T \int_M |e^{t\Delta} f_0|^2 dx\, dt \ .$$

As for th.2.1 in [Mil03b] where the null-controllability cost $C_{\Omega,T}$ (on a compact M) was bounded from below as $T \to 0$, the strategy is to choose the initial datum f_0 to be an approximation of the Dirac mass δ_y at some $y \in M$ which is as far from Ω as possible. Therefore both proofs build on heat kernel estimates. But here we need estimates which are uniform on M for compact times and we use the finer notion of averaged distance of y to Ω (cf. def.1.10).

3.1. **Heat kernel estimates.** Let $K_M(t,x,y)$ denote the Dirichlet heat kernel on M (i.e. the fundamental solution "$e^{t\Delta}\delta_y(x)$"). We recall some well-known facts about it. The heat kernel on M satisfies the following upper bound (cf. th.3.2.7 in [Dav89]): $\forall \varepsilon \in]0,1[$, $\exists a_\varepsilon > 0$ s.t.

$$(17) \qquad \forall t > 0, \forall x, y \in M, \quad K_M(t,x,y) \leqslant a_\varepsilon t^{-n/2} \exp\left(-\frac{d(x,y)^2}{4(1+\varepsilon)t}\right) \ .$$

Let C be a bounded open subset of M. Let $(\lambda_j)_{j\in\mathbb{N}^*}$ be a nondecreasing sequence of nonnegative real numbers and $(e_j)_{j\in\mathbb{N}^*}$ be an orthonormal basis of $L^2(M)$ such that e_j is an eigenfunction of the Dirichlet Laplacian on C with eigenvalue $-\lambda_j$. By the maximum principle, the heat kernel on M satisfies the lower bound:

$$(18) \qquad \forall t > 0, \forall x, y \in C, \quad K_M(t,x,y) \geqslant K_C(t,x,y) = \sum_j e^{-t\lambda_j} e_j(y) e_j(x) \ .$$

From these pointwise bounds on the heat kernel, we deduce bounds for the L^2 norms appearing in (16). Def.1.10 and (17) imply

$$(19) \qquad \int_{T_1}^{T_2} \int_\Omega |K_M(t,x,y))|^2 dx\, dt \leqslant a_\varepsilon^2 \frac{T_2 - T_1}{T_1^n} \exp\left(-\frac{\bar{d}_{(1+\varepsilon)T_2}(y,\Omega)^2}{2(1+\varepsilon)T_2}\right) \ .$$

If $C \subset M$ is an n-dimensional cube with center y and half diagonal length d, i.e. with edge length $c = 2d/\sqrt{n}$, then the first eigenvalue and eigenfunction of the Dirichlet Laplacian on C are $\lambda_1 = n\left(\frac{\pi}{2c}\right)^2$ and $e_1(x) = c^{-n/2}\prod_{m=1}^{n}\cos\left(\frac{\pi(x_m - y_m)}{2c}\right)$. Therefore, (18) imply

(20)
$$\int_M |K_M(t,x,y))|^2 dx \geqslant \int_C |K_C(t,x,y))|^2 dx \geqslant e^{-2\lambda_1 t}|e_1(y)|^2 = \frac{n^{n/2}}{(2d)^n}\exp\left(-\frac{\pi^2 n^2 t}{8d^2}\right).$$

Remark 3.1. We tried without tangible improvement to deduce L^2 lower bounds on the heat kernel from the uniform pointwise lower bounds available in the literature (cf. [vdB90]) instead of deducing it from the more basic fact (18).

3.2. **Proof of th.1.11.** Let $\{y_k\}_{k\in\mathbb{N}}$, $\bar{T}$ and κ satisfy the geometric condition (3). By contradiction, assume that the heat equation (2) is null-controllable in some time $T < \bar{T}$, i.e. the observability inequality (16) holds for some $C_{\Omega,T}$. Let $\varepsilon \in]0,1[$, $\varepsilon < \kappa - 1$, and let $\kappa' = \kappa(1+\varepsilon)^{-1} > 1$. Let $\alpha > 0$ be such that $\bar{T} = (1+\alpha)(1+\varepsilon)T$ and let $\underline{T} = (1+\alpha)T$. Since $\underline{d}_T/T$ is non-increasing, (3) implies

$$s_k := \frac{\bar{d}_{\bar{T}}(y_k,\Omega)^2}{2\bar{T}} - \kappa'\frac{\pi^2 n^2 \underline{T}}{8\underline{d}_{\underline{T}}(y_k,\partial M)^2} \to +\infty\,, \text{ as } k \to +\infty\,. \tag{21}$$

Let $k \in \mathbb{N}$ and let $f_0(x) = K_M(\alpha T, x, y_k)$ so that $e^{t\Delta}f_0(x) = K_M(\alpha T + t, x, y_k)$. Plugging into (16) the upper bound (19) with $T_1 = \alpha T$ and $T_2 = \underline{T}$ and the lower bound (20) for the cube C with center y_k and half diagonal length $d = \underline{d}_{\underline{T}}(y_k,\partial M)$ (this is just the optimal choice for d) yields:

$$\frac{n^{n/2}}{(2\underline{d}_{\underline{T}}(y_k,\partial M))^2}\exp\left(-\frac{\pi^2 n^2\underline{T}}{8\underline{d}_{\underline{T}}(y_k,\partial M)^2}\right) \leqslant C_{\Omega,T}\frac{a_\varepsilon^2}{\alpha^n T^{n-1}}\exp\left(-\frac{\bar{d}_{\bar{T}}(y_k,\Omega)^2}{2\bar{T}}\right)\,.$$

Since $\kappa' > 1$, we deduce that there is an $s > 0$ independent of k such that $\ln C_{\Omega,T} \geqslant s_k - s$ and $\lim_k s_k = +\infty$ as in (21). This contradicts the existence of $C_{\Omega,T}$ and completes the proof of th.1.11.

3.3. **Proof of th.1.4 iii) and another example.** To prove that the geometric condition (3) holds for M and Ω defined in th.1.4 iii), we consider a sequence $m_k = (0,0,z_k) \in M$ with $\lim_k z_k = +\infty$. Since S is bounded, we may assume that R is bounded. Let $G_T(z) = \exp(-z^2/(2T))$ and let $D(z)$ denote the disk with center $(0,0)$ and radius $R(z)$. We have:

$$\begin{aligned} I_k &:= \int_\Omega G_T(d(m_k,m))dm = \int_{\mathbb{R}}\exp\left(-\frac{(z-z_k)^2}{2T}\right)\int_{D(z)}\exp\left(-\frac{x^2+y^2}{2T}\right)dx\,dy\,dz \\ &\leqslant \int_{\mathbb{R}}\pi R(z)^2 G(z-z_k)dz = \pi R^2 * G_T(z_k) \to 0\,, \text{ as } k \to +\infty\,, \end{aligned}$$

since $G_T \in L^1(\mathbb{R})$, $R^2 \in L^\infty(\mathbb{R})$ and $\lim_{|z|\to\infty} R(z) = 0$. Therefore, by def.1.10, $\bar{d}_{\bar{T}}(m_k,\Omega)^2 = -2T\ln I_k \to +\infty$ and, since $(0,0) \in S$, $\underline{d}_{\bar{T}}(m_k,\partial M)^2 \geqslant d_\partial(m_k)^2 = \inf_{(x,y)\in\mathbb{R}^2\setminus S}(x^2+y^2) > 0$. Hence (3) holds for the sequence $\{m_k\}_{k\in\mathbb{N}}$ with any $\bar{T}$ and κ, which completes the proof of th.1.4 iii).

Remark 3.2. To illustrate the usefulness of the second term in the geometric condition (3), we give an example close to th.1.4 ii) where (3) is satisfied by a sequence $\{m_k\}_{k\in\mathbb{N}}$ tending to the boundary of M.

Consider the shrinking rod $M = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 < R(|z|)^2\}$ where the continuous non-increasing function $R : [0,\infty) \to]0,\infty)$ tends to zero at infinity. The heat equation (2) is not null-controllable in any time $T > 0$ from any interior region Ω of finite Lebesgue measure such that $M \setminus \Omega$ contains a sequence of slabs $S_k := \{(x,y,z) \in \mathbb{R}^2 \times [0,\infty) \,|\, x^2+y^2 < R(z)^2, |z-z_k| \leqslant d_k\}$ satisfying

$$\exists \kappa' > 1, \quad d_k^2 - \kappa' \frac{\pi^2 n^2}{4} \left(\frac{T}{R(z_k+d_k)}\right)^2 \to +\infty\,, \text{ as } k \to +\infty\ .$$

Indeed $m_k = (0,0,z_k)$ satisfies $d_\partial(m_k) \geqslant R(z_k+d_k)$ for $d_k \geqslant \|R\|_{L^\infty}$, and $d(m_k,\Omega) \geqslant d_k$. Hence $\{m_k\}$ satisfies (3) for any $\kappa \in]1,\kappa'[$ and $\bar{T} = \sqrt{\kappa'/\kappa}T > T$. In particular, if $\lim_{z\to+\infty} zR(z) = +\infty$ (i.e. M does not shrink too fast) then the heat equation (2) is not null-controllable in any time T from any bounded Ω.

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