OBJECTS THAT WE GIVE OURSELVES

My answer to the question: “Where do mathematical objects come from?” will simply be: mathematical objects are objects we give ourselves.

I am going to argue in favour of such a thesis, but firstly I should try to explain why it is to my mind an answer.

Such an explanation is necessary because I can see perfectly well in what sense my answer is unexpected, as well as the reason why it should be felt this way.

Three possible trajectories

When one asks “Where does X come from?”, he asks for a trajectory, after all. Technically, he only asks for an original location, but one can easily understand that, without being able to specify the trajectory from this original location into the mathematical realm, no one would dare mentioning any particular location. And it is also implicit that there is a kind of force or law which makes the alleged trajectory necessary.

In order to leave pure generality, I will try to picture three kind of answers obeying such an “answer-frame”. The first one will refer to perception, or sensitive abilities, the second one to psychology, and the third one to history.

Let us start with the first answer: mathematical objects come from perceptive objects, sensible objects or data. Mathematical objects have to be understood as a definite elaboration from a sensible basis, would it be through abstraction, synthesis, analysis or idealization. We have some acquaintance with ordinary objects, to which terms of our language refer, either in the manner of proper names or of general classificatory terms. The mathematical object is, so to say, derived from such a commonsense situation and practice: it is nothing but abstraction or idealization operating from and upon this source. At least this could be an answer if we admit first that language has already made the possibility of reference clear, and secondly that we do not have to explain how we move from the reception of sensible multiplicity to its attribution to some unique and well-defined object. Sensible reality becomes the original location but already coped with, efficiently handled by our perceptive and linguistic abilities. And the trajectory is, so to say, empty, since there is something like a qualitative jump from this sensible object to the mathematical one, that we simply have to acknowledge: the mathematical object arises from the sensible one by the instantaneous and magical effect of an epistemological transmutation, which we refer to with words such as idealization or abstraction. Or maybe, we should say in short, to compensate for the lack of a definite trajectory or necessity, that we reach the mathematical object by forgetting whatever particular quality some common object may have, by forgetting all particular constraints a word obeys in virtue of its specific meaning, or by considering the object only as the concept of “instance of some type”. But these “stories” are nothing but disguised paraphrases of the words abstraction and idealization. So I claim that in such an approach, even if we have some familiar and to a certain extent plausible description of the genesis of the mathematical object, we do not really have a trajectory and a necessity connected with this trajectory, simply because there is no common space which the original common sense object and the final mathematical object would both belong to.

The second answer would sound very close to the first one, even if it is philosophically very different. We would still declare that a mathematical object originates in sensation, but we would stop thinking that the category of an object may be considered as given by our practical and linguistic behaviour. We would require from our theory that it gives an account of the way our cognitive abilities lead us from sense data in the form we receive them into theoretical terms at hand for mathematical use. And this would require, for sure, the cognitive explanation of the object as a category. This will imply to answer the following questions: how do we select boundaries for sensible areas, on the basis of which quality scanning, and, moving away from this rather technical approach, how does the general meaning of an "object in the perceptive field" arise. This second answer would therefore pertain to psychology, i.e., in a broader sense, to cognitive science. In that case, the trajectory would ultimately be a processual one, because we would try to describe in an integral way the natural process which leads from the basic work of usual perception to the abstraction more or less magically evoked in the preceding viewpoint. Very likely, such a description would have to be given in a neurophysiological language. And the far-fetched goal would be some neurophysiological account of the meaning of a formal object, of an absolutely undetermined and unqualified object. The evolution trajectory would lead from some neurophysiological data, originally induced by external objects, towards general neurophysiological patterns, or something of that kind. I insist on the last point: such an answer to the question: "Where do mathematical objects come from?" is the kind of answer some cognitive scientists working in the field of mathematical abilities are nowadays looking for.

The third answer would be very different, even if it shares something with the second: its "reductionnist" stance. This answer would simply be: "Mathematical objects come from history and human practice". Only in an unjustified idealistic view, choosing to ignore any relevant context, can one imagine to describe the appearance of mathematical objects at the level of the isolated psychological subject, and on the basis of purely perceptive anhistorical data, will the defender of this last answer argue. Mathematical objects are always the result of some historical elaboration acting upon received mathematical objects. What we have to describe is the way, for example, a mathematician reaches such objects as differentiable manifolds or groups while dealing with some geometrical or equational problems, and any interesting and informed answer will necessarily have to be extracted through the reading and the careful analysis of the works of Riemann and Galois, typically. An explanation will be an explanation in terms of the general features of mathematical practice, with the help of the history of mathematics, or, more exactly, of that kind of history of mathematics in which one tries to catch the general forms and laws of mathematical practice, that is to say, of a rather philosophical history of mathematics.

Even if such answers are more or less the expected ones, we know very well that there is a deep and until now seemingly valuable reason why these answers are not satisfactory. We cannot help thinking that the mathematical object has neither space, nor time, nor psychological, sociological and historical location, and that consequently there exists nothing like a trajectory leading to it from any such location, according to some natural force or necessity. In other words, we cannot help thinking that the mathematical object is ideal, in the sense of the concept of ideality that one can trace back to Plato. And this point is not only a point of faith, it is not only that we are embarrassed by some metaphysical commitment we cannot get rid of. This conception of the mathematical object as ideal inspires various criticisms towards the “plausible” answers first considered.

To the contender of the sensitive origin, we object that the process of abstraction or idealization is very obscure, that being able to name an ordinary object with our words would imply that we already possess the category of mathematical object, or, if there is something more than this in a qualified mathematical object, abstraction and idealization are defined as the process of converting the ordinary object into some mathematical object, or as the process of creating some mathematical object on the basis of the ordinary one. The first hypothesis is hardly acceptable, since all human experience shows that young children with good linguistic and perceptive skills arte far from being able to grasp what mathematical objects are in a satisfactory way and to show it by the appropriate behaviours (subsumption, calculation, deduction). And the second one would lead to the conclusion that idealisation and abstraction convey no explanation, are nothing but scholastic formulations.

To the contender of the psychological origin, we argue in a more or less analogous way that his explanation falls short of what is at stake. For example, the cognitive description of the neurophysiological pattern corresponding to the recognition of a general Gestalt of five objects does not give us the neurophysiological counterpart of the thought of an arbitrary “Brouwer's integer”, of n as constructed by successive adjunctions of unity. And can we specify such an intellectual content in another language than the philosophical or the mathematical one ? Or maybe the mixing of these two languages? What it does mean to possess some mathematical concept is accessed to only in mathematical or philosophical terms, and such an epistemological property belongs to our best understanding of what mathematics are about. To understand the concept of a finite set, for example, has to do with the understanding of the theorem saying that for any map from such a set to itself, to be an injection is equivalent with to be a surjection or to be a bijection, or with the understanding of the general induction principle, probably as much as with the envision of the “kantian” scheme of the skeleton of an “*n*-long” enumeration. In any case, none of these contents seems to be really “natural”, “concrete” and possibly external to mathematics and philosophy.

To the contender of the practical-historical origin, we shall object in the same spirit. Her explanation is maybe very relevant and informative insofar as she explains to us what problems mathematicians in history were facing with, and how they overcame these problems, in a way intimately connected with their general ideological background. And she will argue convincingly that the choice of defining such and such an object or a structure is enlighted by some sort of necessity when we honestly take this context into consideration. Maybe in that way can we understand the introduction of manifolds, infinitesimals, sets, and so on. But these rational reconstructions always have to presuppose mathematical objects with their meaning. This is at least unavoidable in order to formulate what the problem was: this problem is relative to a certain state of knowledge and to certain aims expressed in the language of this state of knowledge. But it is no less unavoidable in order to describe the solution: whatever the motivation found in the historical context may be, what counts as a solution either relates to previous known mathematical objects and previous known techniques leading to the new object or structure, or if the general setting of mathematical objects and structures has been changed, then it has to be explained and understood without reference to the extra-mathematical context. We would not consider as an acceptable way of introducing mathematical contents a discourse which would refer to an economical background or a religious faith, for example. For such reasons we have in the end to concede that mathematical objects come from within mathematics and not from any kind of outer world, even if such a conclusion seems deceptive.

It seems deceptive, and even paradoxical, because it seems to force us to consider that mathematical objects always came and come from the mathematical realm, that is to say, in a way, from themselves: but, if this is the case, there is no “from”, and the question for an origin remains basically unanswered.

Here comes the turning point of that lecture. Such a conclusion will not be considered as deceptive if we are able to cope with the idea that in some cases the quest for an origin cannot or even should not be answered. If we are philosophically prepared to the idea that human rationality shows some ultimate elements which bring some light, convey some explanatory power with respect to all other elements of knowledge, but which we can neither refer to an external origin nor to an external explanatory power. The idea of such contents is a very old and simple one, it is nothing less but the idea of foundations. I claim, coming after a very long line of philosophers, that mathematical objects are part of the foundational setting of human knowledge, and that they cannot be understood in a better way by us than in the foundational way. And this leads to the answer I suggest, concerning the question raised by Marco Panza: mathematical objects are objects that we give ourselves. But we do not give them in an arbitrary way: we follow a very particular and precise way of giving, in which the epistemological potentiality and the general meaning of mathematical objects lies. This is what I'm going to try to explain now.

Giving mathematical objects

We could argue in favour of the idea that mathematical object is given. It would be easy to recall the famous saying attributed to Kronecker, that God gave the natural numbers, human people making what was left. It shows that the concept of givenness is relevant, though the reference to God is disturbing. But, after all, we should also remember that Cantor thought that infinite sets, even if they had to remain unseen by us, at least in their totality, were actually perfectly grasped by God's eye.

Another option will be the reference to contemporary mathematical literature.

I quote the Mac Lane-Moerjdik book of 1992 *Sheaves in Geometry and Logic*, more or less in a random way:

> « Given such a pair of arrows and the two e-m factorisations, we must construct a unique arrow s from m to m' which makes both squares in the diagram of the right below commute » (p. 186)
>
> « In particular, given a second factorization f=e'm' of the same arrow f, this argument, with r=t=1, yields a unique arrow s which is both monic, because m's=m, and epi, because se=e', hence an isomorphism » (p.186)
>
> « Given two subobjects S→A and T→A we can form their intersection as their greatest lower bound (g.l.b.) in SubA simply by taking the pullback, as on the left below » (p. 186)

In the usual textbooks, one can very often find the expression « Given … such that …, (assertion) » . One can also find « Let … be given such that …. Then … ». And even more frequently (to be franck, all the time), we find « Let … be …», in which the reference to givenness seems to be cancelled by the subjunctive form.

In the usual mathematical discourse, we use the right to “summon”, so to say, objects with such and such properties, in order to perform our proofs. Apparently the very fact of calling these objects is the actual way of giving them to us: the expressions such as “given” or “let be given” seem to imply such an interpretation. The expression using a bare subjunctive “Let … be …” shows only our power, without reference to givenness: what we ask, we have.

There are two things that one could say in order to reject my grammatical remark. The first one would be to point out that the objects that the mathematician gives himself, in my cases, are objects of a certain kind bearing such and such property: so, such an act is by no way contemporary of the object birth: we just focus our attention on something which already exists. From a logical point of view, we could say that we are simply using, in a natural deduction framework, the rule of ∃-elimination, which goes through the naming of an arbitrary new individual carrying the property coming after existential quantification.

Before dealing with such an objection, I will give another quotation from Mac Lane-Moerdijk:

> « Thus giving f really amounts to giving the graph in Bx**N** of a many-valued function from B to **N**, consisting of all those pairs (b,n) with n∈g(b). (…)
>
> In the given model S of set theory there is (according to the diagonal argument) no such function f » (p. 278)

In the above mentioned part of the book, which more foundational, we find two occurrences of the *give* radical, with a slightly different use. In the first case, the author considers what would it mean to be equivalent with the givenness of a certain f. In the second case, he mentions some supposedly given model of set theory, inside which the preceding considerations, we can imagine, took place, and which forbids, by the way, the existence of an f of the kind discussed in the first remark.

These uses of the “givenness” seem to consider it as such, and not to be reducible to the proof-trick of ∃-elimination.

But if one really chooses to remain deaf, and never to leave the strictly logical level, then one is going to say that, in the first case, the discourse asserts the logical equivalence of two existential statements, and that, in the second case, it draws some conclusion that would be allowed in any model of set theory, which really makes the discourse a universal one.

We are faced, in a way, with the observable distance between real mathematical discourse and logical norm. The logicist or logic-reductionnist will contend that this distance is psychologically or culturally contingent: the meanings that seem to be expressed in the non logical mathematical rhetoric have no weight and no value, because they play no part in the determination of truth.

I would rather argue that we are pragmatically sure that there is more in these ways of doing and writing mathematics than what the logic-reductionnist supposes. When the mathematician gives objects with such and such properties to herself in order to go further in her proof, she really wants to do "as if" she would have such an object ready-at-hand before her, and the ∃-elimination rule does nothing else than converting into a rule such a common practice. The rule "admits" that noone is able to carry on the proof-task without "positing" the desired object, without mentally entering the world completed by it.

And a very strong argument in favour of such an understanding of effective mathematical practice is that if we go back to the foundations, we shall see that objects are originally supposed to be “given” in such a way.

Correlative objectivity

Objects contemporary mathematicians are supposed to deal with are, as a matter of fact, sets. In the famous Bourbaki “Théorie des ensembles” treaty, the two expressions "set" and "mathematical object" are said to be synonymous. Sets are now usually introduced – unlike in Bourbaki's treaty – simply by explication of the first order theory ZFC: it is as if the very mention or utterance of the list of axioms had the power of giving birth to an – inconceivably infinite – multiplicity of objects called sets and building the so called universe of sets. In other words: sets are not known in any other way than as constituting, putting together, a collection which satisfies the axioms of ZFC.

So I would simply point out here that there is a specific way of giving objects to ourselves which appears as well accepted today: giving an axiom list, with enough confidence that they don't entail any contradiction together. At least, this procedure seems to be recognized in the case of the basic theories ZFC and PA.

Of course, there is much more to say about this gesture, usually referred to as the axiomatic gesture.

First of all, nobody should forget that ZFC comes after an informal use of set theory, *i. e*. that the cantorian stage is never cancelled. For example, in Jech's volume *Set theory*, just after the book introduces sets by mentioning the axioms, it is recalled that these axioms bring a rigorous framework to the previously familiar intuition of sets and to the pre-axiomatical mathematics of sets. This shows that the axiomatical gesture has some "external" legitimacy.

My second point argues against the pure formalist position that we have no need to think of sets as objects from which mathematics unfold some knowledge. Such a claim is only possible in a philosophical game left to the logician’s arbitrary decision. Working mathematicians not only spontaneously believe, as it is often argued, in the reality of the universe they try to describe, but they also see this universe the way the axiomatics depicts it, and their mathematical thought is nothing else that some deep and complex insight on mathematical objects equipped with their set-theoretical structure. More than this, they have the feeling that these same axioms allow them to act within set-theoretical context or datas. The possibility which was opened to the classical geometer within its geometrical realm – to draw lines and circles for the sake of his goals – is still opened to the common ZFC practitioner: he may for example introduce balls with a certain radius in the middle of some topological reasonment, and such an act is lived as the act of "adding" some object to some objects system presently under consideration. In short: if the strictly formalist stance is of any use or is correlated with anything in the epistemological world, it is not with real mathematics.

I would like to sum up what was my claim in the present section.

I claimed that there is a very well known way for mathematicians to "give themselves objects", and this way is simply the utterance, or stipulation of some first order theory which, so to say, "waits" for some multiplicity satisfying it.

I would like to add that this epistemological fact brings some light to the grammatical examples I was previously trying to put forward. When we are using the "trick" of $\exists$-elimination and "giving" ourselves objects to for the sake of some proof, we are using our formal tools in order to "intervene" into this universe that we globally gave ourselves, we are performing the equivalent of the "construction of concepts" Kant used to speak of. He was describing thanks to this expression a very common and important aspect of mathematical practice. Such an observation has already been made in a more analytical language by Jaako Hintikka, who was also trying to "vindicate" Kant.

Let us point out that the ∃-elimination rule is used to reflect the previously known mathematical way of proving, which deals with the organization and the planning of deduction on the basis of some generic object satisfying a condition. The very fact that our logic works that way, by giving ourselves the "concrete" particularity of the object even if we do not see this particular as a particular -- which was what Kant was referring to as “concept-construction” -- is precisely linked with the fact that, basically, a mathematical object is of this form: the mathematical object is apprehended as satisfying some condition, which means, in kantian terms, as affected by "a difference without concept". We always encounter the mathematical object as something which could be without any prejudice different, other, as far as it is generic under a certain specification. We introduce it in a way adapted to our demonstrative goals in the current mathematical task, while we give ourselves the encompassing framework of all mathematical objects at the foundational origin.

My last comment in this section will show that this way of accessing objects is best described using phenomenology. How can I sum up the husserlian approach to objects? The best would be by emphasizing two points: 1) firstly, phenomenology insists on our basic ability to “point to” some internal or better intentional object, which has not to be externally construed as previously existent (we may very well argue that language testifies this ability, by allowing us to refer to entities that we posit by our very enunciation) ; 2) secondly, phenomenology argues that we know objects as belonging to “objects areas”, and that for each such area some local criterion is given, which specifies under which conditions we recognize an object of the relevant kind as “given”, “introduced”, “experienced” (we may add that for Husserl, these conditions were always supposed to be described in terms of “consciousness settings”).

We can therefore picture what we are going to call “correlative objectivity” in phenomenological terms: the rule is that we consider a multiplicity of objects to be given once we have made explicit a list of first order axioms, written in a first order idiom. The multiplicity is “seen” as a collection of objects collectively satisfying the axioms. This way of introducing objects has a lot to do with fiction: we are not far from using our linguistic ability to evoke some horse with a human head. This procedure is aptly connected with the phenomenological concept of intentionality (pointing to things defined in terms of our pointing to them). But it is a very peculiar and modern way of making fictions, which we could characterize as a “collective” way of making fictions. Objects which are introduced through correlative intentionality are “fictionized” as building a collection satisfying first order conditions which usually have universal content. It is possible to argue that we also find this kind of intentionality in science fiction (I’m thinking here of Ursula Le Guin’s *The Left Hand of Darkness,* for example), but on the whole I think we should recognize that mathematics have recently discovered and systematized with Hilbert a very exciting new intentional way.

There are two more features in objects that are phenomenologically given by some intentional form: the particular intentional style of the givenness is connected to some intuitive type, and the way of asserting truths about the concerned objects is so to say prescribed by the intentional way.

The first point motivates the disagreement between standard analytical philosophy and husserlian phenomenological approach, because any reference to something like intuition would mean psychologism, that is to say, evil. I will argue that it is very important, in order to understand and practice the contemporary mathematical game 1) to accept to “see” multiplicities as “shaped” by a list of axioms – in a metaphorical way, but also with the help of some vague and concrete images, like the image of the potatoe for the general set; 2) to be ready to deal with them as a kind of world, in the context of which the axioms offer us some gestures that are allowed (like the gesture of “building” the power set, to take an easy example). I can globally define this attitude towards sets as the intuitional element accompanying the intentional way of correlative objectivity. Such an intuition is partly negative: we have to be prepared to accept many multiplicities satisfying our axioms, the intended objective landscape cannot be fully determined, at least for the interesting cases (involving infinity).

The second point gave rise to intra-phenomenological criticism, stemming from anti-objectivist and anti-transcendantalist trends of phenomenology: following a group of authors, phenomenology should get rid of the idea of scientific truth altogether. I argue that in our case, the connection with a way of saying the truth is so essential that it would be heavily irrespectuous of the intentional way under our scrutiny to neglect it. When we give ourselves objects in the "correlative way", we assume that the only way of saying the truth concerning such objects is to deduce theorems about them in reference to the axiomatic basis. This point, very clearly, makes a new and important difference between ordinary fiction and contemporary mathematical practice of correlative objectivity. But in order to put forward another, and even more important difference, we must go to the alternative way of "giving ourselves objects": the constructive one.

Constructive objectivity

Constructive objectivity was originally described by Brouwer in terms of the mental process dividing the present in two opposite parts, the *before* and the *after*, this process being supposed to be iterated in any possible way. Since then, we got some practice and some experience in introducing the same kind of pre-formal objectivity. We now proceed without any explicit reference to a mental process, simply by mentioning some basic primitive objects and offering some perfectly clear building rules: it is understood that we are going to consider as a regular object of a defined class any object that has been built on the basis of primitive objects and with the use of the building rules, *and only such an object*. So, the integer numbers may be introduced that way, by the following specifications:

i) $ is an integer number;

ii) if N is an integer number, then N$ is a integer number;

iii) nothing else is an integer number.

But we also introduce formulas of prepositional calculus in the same way, by this new list of specifications:

i) a propositional variable is a formula;

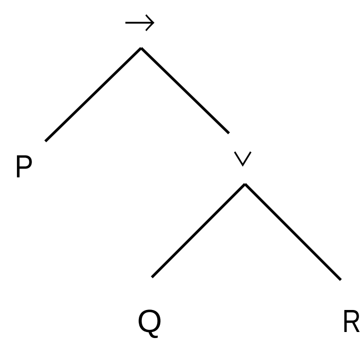
ii) if F is a formula, then $\neg F$ is a formula;

iii) if F and G are formulas, then $F\wedge G$, $F\vee G$, $F\rightarrow G$ and $F\leftrightarrow G$ are formulas;

iv) nothing else is a formula.

There are a lot of objects, all of them most interesting in contemporary logic and discrete mathematics, that may be introduced that way, without any reference to set-theoretical objectivity, independently of our formal setting used for mathematical practice in general. Trees, terms, formulas, demonstrations are specified that way.

Objects of this kind are objects we have to build, which makes as clear as possible that we give them to ourselves. But, as in the previous case, the situation is still phenomenological: by the same gesture which gives us objects of such and such constructive class, we gain the ability to see them in a specific way, to see them as we built them, so to say. I am supposed to see the formula $P\rightarrow(Q\vee R)$ "under" its propositional structure, that is to say, more or less, as the tree



My ability to "read" constructive objectivity with its structure is an essential prerequisite for my actually performing constructive mathematics. This point becomes even more constraining if we consider now what the accepted ways of knowing such objects are. On the one hand, we have what we could call after Hilbert *Uebersicht*: we are allowed to draw a conclusion which forces itself upon us at the level of the very vision of such tree-structures. But this way of knowing only works with finite examples, of small size we should add. The genuine mathematical knowledge-tool for constructive classes, gaining for us universally quantified statements, is the tool of "induction on the building of the object ". We prove that every primitive object of the class has property H, and that, if some objects have the property H, then the object built from them, following any of the given rules, also has property H, and we may then conclude that every object of the class has property H. We legitimate such induction-proofs by referring to the imagined construction-tree, and we argue that every leaf has property H, and that the property H, considered as a kind of colour, diffuses each time we apply one of the rules higher and higher along the tree-nodes, until we reach the top which finally becomes coloured also.

So we consider objects of this kind as correctly given precisely because we keep good control of the objects introduced that way: we "see" them in a satisfactory way, and we are able to convert this way of seeing in a way of knowing. Following Husserl, this is how knowledge has to initiate its course.

At this stage, we have described the two phenomenological basis of mathematical practice: the two different ways the mathematician is able to "give herself objects". This could be our last word with respect to the question raised by this conference. But I think we should also try to deal with another issue, which is of considerable importance from the standpoint of philosophy of mathematics: we should try to understand how mathematics deals with these two basic intuitions or phenomenological insights, how mathematics deals with its twofold foundations, twofold envisioning, twofold proving. In the following section, we try to say something about that.

Correlative and constructive intentionality together

We should say first that the constructive way of seizing objects and of knowing them is nowadays accepted as the ultimate way through which all human knowledge is built: there is nothing in which we agree more and in a more secure way than this constructive knowledge. I may say this without any hesitation because, as we all know, to know that something has been correctly proved in some formal system is to recognize the "formula" as a regular object of the constructive class of "theorems". So there is not a mathematical result, be it a result of constructive mathematics, of logic or of formal mathematics, which does not rely upon "constructive certainty". This already establishes some connection between our two basic mathematical games: you cannot play the current axiomatic game of correlative objectivity without referring to the other game.

But there is another connection between the phenomenological mode of giving objects to ourselves in the "correlative way" and the competing mode of giving objects to ourselves in the "constructive way", and this connexion is related to the question of the infinite. I come now to this second point.

It is well known that, in ZFC theory, infinite comes with the so called "axiom of the infinite", which reads

$\exists x\,(\varnothing \in x) \wedge (\forall y\ y \in x \rightarrow y \cup \{y\} \in x)$.

This axiom says that there is a set which includes every object of the constructive class of "ordfin", whose names are defined by the following instructions:

i) $\varnothing$ is the name of an ordfin

ii) If S is the name of an ordfin, then the character chain $S \cup \{S\}$ is also the name of an ordfin

iii) Nothing is the name of an ordfin except what appears as such in the light of i) and ii)

In other words, set theory, like any first order theory, allows us to build the so called "terms", dealing with local individuals and functional constants. If we accept to play the intentional game of correlative objectivity, there is, in any correlative multiplicity stipulated by the axiom list, one well defined object for every such term, enjoying the property of being directly known by the theory, instead of "called" by the axioms in order to fulfil them collectively.

What we really mean, in a set theoretical context, by an infinite set, is therefore a set able to include the simplest open class which we may build, following constructive intentionality, at the level of the term names for objects.

In more philosophical terms: it belongs to the meaning of what we aim at as infinite in a set theoretical context that such an infinite contains as actual the supposedly achieved series implicitly set up by constructive recursion on names. The infinite is neither the qualitative concept of what escapes every determination, like in the tradition of negative theology (philosophically taken up again by Heidegger), nor is it the qualitative concept of what counts as a basis or foundation for itself, as in the tradition of positive theology (philosophically taken up again by the classical metaphysics of Spinoza or Leibniz): the infinite is the actually intentioned multiplicity which has the ability to include every open constructive class. Such an infinite is determined through its quantitative relation to the finite, as Kant very clearly diagnoses it in the case of space's infinity, although this relation is not a compositional assignment (*i. e.* we cannot give a number to the infinite, nor express its size with our finite or constructive resources).

If we come back now to the intentional gesture stipulating the ZFC axioms and bringing the vision of a “world” fulfilling these axioms, we understand that this world has to do with the more restricted and more controlled area of constructive objects, that we give ourselves in another way: it is supposed to include all objects of this area, usually through some coding. The relative and partial definiteness we usually attribute to the ZFC universe, pictured by the big V-drawing of the cumulative hierarchy of ranks, refers all the time to the constructive clearness of the earlier stages: when we think, understand and to a certain point see what the continuum is, we mean that we see through the basic constructive way what the denumerable is, and mean that we try to extend and generalize the constructive building of subsets, in order to achieve some grasping of “the class of every subset of **N**”. This gives a very strong reason to separate the ZFC fiction from every common literary fiction, even if this literary fiction was being formulated in the guise of first order collective formulation: ZFC fiction is designed to encompass the core of mathematical objectivity related to the minimal and basic skill of constructive intentionality. That is why mathematicians often and willingly see the ZFC universe as unique and perfectly characterized, despite all the relativizing logical information which is ready-at-hand (which would in principle prohibit the use of *the* in preceding sentence!).

Our views and fregean tradition

I shall conclude this paper with some remarks concerning the philosophical difference between the conception that was defended here and what the fregean tradition is able to inspire (or, more precisely, prescribes).

It is well known that, according to the fregean view, any reference to intuition is already reprehensible: such a reference counts as a case of the naughty sin of psychologism. We are not allowed to pretend that we would enjoy some intuitive relation to any category of objects, because intuition may not belong to anything but the level of "representation", which is the bad level, to be distinguished from the good level of "meaning": representation is only built with this personal material, memories, images and so on, which cannot be shared, and for that reason, has to be excluded from the game of truth and science.

This is why, in particular, the intentional conceptions of phenomenology are rejected: intentionality, would the fregean philosopher say, makes the objects depend on representation, intuition, of this awful realm of psychology. And such views make truth itself relative, this is the ultimate argument against any philosophical view of this kind. When Putnam, recently, tried to develop his conception of internal realism, which we, european philosophers, are tempted to recognize as a new formulation of intentional ideas in the context of analytical philosophy, he was immediately charged with this accusation of unescapable relativism.

As a matter of fact, we have the feeling that, for the Frege-Russell tradition, any strong reference to objects is already suspect: objects have no other status as that of meaning of terms arising in the analysis of true sentences. We have to begin with the presupposition of true sentences, and try to formulate some philosophical judgments describing this assumed truth as an "objective" truth, independent of human will or feeling or representation. But such a truth may not rely too heavily upon objects, which would appear as relativizing it, more or less in the way Kant claims that the reference to some object as its cause or aim makes any practical maxime "pathological", un-categorical, unable of universality. Objects are always on the verge of being criticized by the moving of truth, bringing some new sets of terms arising from the associated "content of possible judgment". Or, if objects are allowed to come under consideration, they should be unquestionable objects, already known and recognized as such, well connected, in an unproblematic way, to terms referring to them.

What I will simply argue against this fregean viewpoint is that it makes the task of giving any philosophical account of mathematics strictly impossible, or, to be more precise and honest, it forces the fregean philosopher of mathematics to strange and not far from schizophrenic positions.

Because there is nothing more pervasive in mathematical practice than objects. Mathematicians understand their own practice as finding, discovering, knowing, describing, changing, and elaborating specific objects. In everyday mathematics there is a general setting of accepted objects, termed as sets, and it is an important matter to take care of what is a genuine and legal object and what is not (for example: if G is a group, the class of groups H being goals of at least one morphism whose domain is G is not a set). The universe of set, as we said, is considered in a panoramic way through the inversed cone of ranks. We investigate its local or fine structure by identifying, if necessary, objects under consideration as constructive objects. The common testimony of mathematicians would be that they value these objects as more stable and consistent than objects of any other type. When the working mathematician works, what he is doing gets described in the most truthful way as a kind of formal practice inside set theoretical realm, in which he handles some definite objects. The French mathematician Moshe Flato writes in a popular essay about mathematics that a mathematician does not totally regard logicians as members of the same team, even though logic has been organized in a technical way, because they lack what he calls "le sens de l'objet" ("the feeling for the object", I would translate).

All of this would be, I guess, simply categorized by the fregean philosopher as psychological arguments, invoking personal representations, living experiences, at best collective delusions. On the contrary, the phenomenologist recognizes all these elements as expressions of some specific intentional way, which is mathematical intentionality. He understands these subjective or psychological elements as having nothing to do with intimacy or private knowledge, but rather as part of a specific intersubjective and shared structure which is intentionality, and which would be nothing if it would not be also given as a living experience.

And what we can argue against the fregean philosopher is that even his preferred game or area of telling the truth following the right logical forms has been identified as ruled by (a kind of) intuition. Nowadays, our public standard for what counts as correct logical inference is derivation inside some formal system; and this in turn is characterized as showing, by exhibiting the corresponding tree-structure, that some object is part of some constructive class (the class of theorems for such and such formal system). But such a recognition has nothing to do with any strictly logical concept, belonging to the philosophical tradition of logic, discourse and truth: it does not depend on notions such as *concept*, *negation*, *universal*/*particular*, *name*, and so on. It only refers to the game of construction following recursive rules, a game that we play with our basic sensorial abilities while sharing the feeling of its genericity and legitimacy. The common practice of building objects following rules, of seeing and recognizing them under their tree-structure, is at the same time a writing game, a speech game and a consciousness game (this latter aspect being historically the first which was seen by Brouwer). All of them are played together and without feeling their difference, as it has to be if they are supposed to keep their foundational value: such a threefold game has become universally recognized as the basis of any rational trade, as the practical concrete core of logic and mathematics themselves. And I think that I showed in this paper that I am entitled to describe this core as the phenomenological core of constructive intentionality.

My second and last point against fregean tradition will refer to the very concept of epistemology. How can we really produce any epistemological discourse if we are not allowed to raise the question of objects? If we may not ask about the givenness of objects? Classical, post-kantian epistemology holds the tenet that the possible truth of any scientific production depends basically upon the way objects are defined, of the way some regular access is decided for them: such points have to be made clear before we are in a position to evaluate the basic atomic sentences of this particular scientific attempt as "empirically" true or false. And as a matter of fact, the answers to that preliminary problem, for classical epistemology, is in most cases that objects arise as conventional arrangement of sense data, that the objectual basis of any scientifical theory has no immediate empirical validity because it is not already accessed to by ordinary discourse through available terms.

Every analysis of that kind may appear as forbidden by a fregean prohibition. We may not describe how objects are given to us because ultimately, this cannot be done in any other way but by referring to our living experiences, by enrolling some unacceptable psychologism. But how are we going to judge the performance of science in that case? And what I call "to judge", in that context, means to understand, to celebrate some scientific achievement as different of some other one, or as encompassing it, or as deepening its insight on an alleged reality, and so on.

The problem is that if we refuse any (universal) psychology to scientific subjects, we are lead to refuse the same kind of psychology to scientific theories as well. So the only thing a fregean can say is that Newtonian mechanics was true but is not any more, or that, today, quantum mechanics and general relativity theory are both true without been synthesised. But there are huge identity differences between these three theories, which arise from the fact that they do not picture the world, or better the story of something that moves, in the same way, and which is strongly connected with their very different way of using mathematics. These differences can be described as intentional differences: these theories do not give themselves objects in the same way. The classical, post-kantian epistemology aims basically at some understanding of such differences, and claims that we need to understand the intentional presupposition of every scientific area, the way the objects are decided to be aimed at, under such and such (universal) subjective conditions, following the standpoint of the concerned area.

Fregean tradition seems to allow only logical analysis of scientific truths accepted as such. It has to be conceded that philosophers working along these lines were able to settle some interesting questions, and to bring forth some fine discussions (concerning the unity of science or the under-determination of theories for example). But, in most cases, it was at the price of not paying any interest to the actual content of scientific theories any more, and first of all to their mathematical content. For these philosophical considerations, it does not seem to be relevant that Newtonian mechanics may be formulated at the level of the tangent or cotangent bundle of the configuration variety, that general relativity deals with some differentiable manifold along with a local metrics, or that quantum mechanics chooses as a framework for the deterministic description of virtuality a Hilbert space.

If we want to describe in philosophically interesting ways what science does, the way it improves, deepens and revolutionates its own achievements, we must accept the question of the givenness of objects, which was understood in this paper as the foundational question laying behind the question “Where do mathematical objects come from?", which was the theme of this conference.