

Modalities of mathematical objectivation

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In a recent paper, where they resume the philosophical import of the neo-logicist program, B. Hale and C. Wright characterize as conservative those responses to Benacerraf's Dilemma[cf. [5],]

which take [...] the Dilemma head-on, unrepentently maintaining both that pure mathematics is correctly construed at syntactic face-value and that, so construed, it represents, at least for the greater part, a body of *a priori* knowledge.

They then distinguish “intuituional” from “intellectual” conservative responses in the following way[cf. [5], 104]:

It may be proposed, first, that an epistemology of mathematics should reckon with a special faculty—traditionally ‘intuition’—which enables an awareness of systems of abstract, and specially mathematical objects and of their characteristic properties, broadly as ordinary sense perception makes us aware of ordinary concrete objects and their properties. Or it may be proposed that access to the objects of pure mathematics is afforded by our general abilities of reason and understanding.

As an example of an intuituional conservative response they discuss Parsons' theory of intuition. As examples of intellectual conservative responses, they firstly discuss Shapiro's *ante rem* structuralism and then present their own neo-Fregean Platonism.

1

It is not my aim here to discuss these positions from my own point of view. I shall confine myself to some general remarks concerning them.

This will constitute the first part of my talk and will prepare the second part, where I'll try to sketch an original perspective.

1.1

According to Shapiro's *ante rem* structuralism [cf. [8] and [9], ch. 10, 257-289]:

- Pure mathematics pertains to structures, these being defined as being the “abstract form of a system”, or better as “the deductive study of structures as such” [cf. [9], 259];
- “Structures exist whether they are exemplified in a nonstructural realm or not” [cf. [8], 89].
- “Each mathematical object is a place in a particular structure.” [cf. [8], 78]

To afford a response to Benacerraf's Dilemma starting from such a premise, one has at least to account for the way we can be acquainted with mathematical structures.

Shapiro suggests we manage to do it in three ways:

- By pattern recognition or sensory abstraction, a process through which a human subject “comes to recognize a pattern by observing patterned systems” [cf. [8], 111]. This way is only available for finite small structures.
- By “linguistic abstraction” [cf. [8], 112] or projection [cf. [8], 118]:

“The small finite structures, once abstracted, are seen to display a pattern themselves. [...] We then *project* this pattern of patterns beyond the structures obtained by simple abstraction.”

The structure of natural numbers may, for example, be apprehended in such a way starting from the recognition of the patterns of small cardinal numbers. This way is only available for small structures (even if infinite).

- By implicit definition, that is, by a description of the structure itself by means of “a simultaneous characterization of a number of items in terms of their relation *to each other*” [cf. [9], 283-284].

1.2

According to neo-Fregean Platonism, the key to afford a response to Benacerraf’s Dilemma resides in Frege’s Context Principle.

This is because, according to the neo-Fregean reading of such a principle, a consequence of it is that [cf. [5], 115]:

in order to establish an intelligible use for singular terms purporting reference to numbers, or other abstract objects [...] it suffices merely to explain the truth-conditions of statements incorporating such terms [...] [; and that] if under a suitable explanation of the truth-conditions of an appropriate range of such statements, suitable such statements are—or may warrantably be claimed to be—true, then those of their ingredient terms which purport reference to numbers or other abstract objects will in fact refer—or may warrantably be claimed to succeed in referring—to such objects; and the intelligent contemplation of such a statement will constitute thought directed upon the objects concerned.

If I understand correctly this claim in the context of the neo-Platonist program, it consists in maintaining that:

- In order to respond to the access problem for a certain domain of abstract objects it suffices to afford suitable truth-conditions for a suitable range of statements involving the terms used to denote these objects;
- Provided that these conditions fit a genuine Tarskian semantics, this responds *ipso facto* to a generalization of Benacerraf’s Dilemma concerning this domain of abstract objects;

- When (natural) numbers—and thus Benacerraf’s Dilemma in its original form—are concerned, the suitable range of statements is constituted by identity statements of the form:

the number of F = the number of G ,

where F and G vary on concepts; and suitable truth-conditions for these statements are afforded by Hume’s Principle:

[the number of F = the number of G] \Leftrightarrow [$F \approx G$],

where the relation \approx is the equivalence relation ‘to be equinumerous’, or ‘to be such that the objects which fall under F are in a one-one correspondence with the objects which fall under G ’.

- This is because “the adjunction of Hume’s Principle to a suitable underlying second-order logic enables us to prove the five Dedekind-Peano axioms for arithmetic” [cf. [5], 123].

1.3

I begin my remarks by observing that the (conservative) responses to Benacerraf’s Dilemma, or to any generalization of it concerned with a certain domain of (supposed) mathematical objects afforded both by Shapiro’s *ante rem* structuralism and by neo-Fregean Platonism depend more on the recognition that a certain definition, or more generally a certain system of principles, enables us to make warranted statements involving singular or sortal terms denoting these objects, than on an explanation of the modalities of access to the (abstract) objects that, according to an already accepted semantics, constitute the reference of these terms.

1.3.1

This seems to me quite clear for neo-Fregean Platonism, but I maintain that it is also the case for *ante rem* structuralism.

There are two reasons for that:

- If, as Shapiro argues, the surface grammar of any mathematical statement “give[s] the underlying logical form of mathematical language”

[cf. [9], 269], the singular terms of such a statement refer to *bona fide* objects, and these objects are positions in structures, then any suitable semantics for mathematics has to entail that a statement that claims that a certain object has a certain property or that certain objects are each other in a certain relation is true—if true—only because of the internal properties of the structure.

- Since there is no way to study deductively a structure that has been recognized but has not been yet linguistically described, or, better, characterized by means of a certain system of principles, if mathematics is “the deductive study of structures as such”, then there is no room in mathematics for structures whose internal properties do not coincide with the consequences of such a characterization.

1.3.2

This means that, for *ante rem* structuralism, as well as for neo-Fregean Platonism, the accepted system of principles that enables us to make warranted statements involving certain (singular or sortal) terms also provides us *ipso facto*:

- With a semantics for these terms (that is in general relatively easy to construe as a genuine Tarskian semantics);
- With a justification for the content of these statements and thus with knowledge of such a content and of the objects these terms refers to, according to such a semantics.

1.4

But, to say or to admit that it is so, it is not of course sufficient to provide a plausible philosophy of mathematics that affords a response to Benacerraf’s Dilemma or to a generalization of it.

One should add at least that the warranted statements that we are enabled to utter because of the accepted system of principles coincide with, or at least correspond (according to an acceptable translation) to the usual theorems of the branch of mathematics which is taken to be about the current structure.

There are two ways to meet this obvious requirement.

1.4.1

The first one consists in adopting a sober attitude and admitting, as Shapiro does, that [cf. [8], 32]:

[...] the primary task of philosophy of mathematics is to *interpret* mathematics and thus illuminate its place in the world view. [...] it is *mathematics* that is to be interpreted, and not what the philosopher hopes mathematics can be or should be, and not what a prior (or a priori) philosophical theory says mathematics should be. In general, interpretation can and should involve criticism, but here at least, criticism does not come from outside—from preconceived first principles.

From such a perspective, the system of principles which enables us to make warranted statements are those that govern real mathematical theories. The mathematical structures philosophers are speaking about—if they adopt a structuralist perspective—are nothing but already accepted mathematical theories.

It is of course possible—either because of the informal nature of many mathematical theories, or because of the adoption, on behalf of the philosophers, of a general description of some formal feature that such a system of principles is supposed to satisfy—that these mathematical theories are submitted to an interpretation that unveils, or aims to unveil, an underlying structure that is far from the surface [cf. [8], 34].

But, in any case, from this perspective, philosophy of mathematics is a reflection on something that is given before it, and is, because of this, literally, a philosophy of something.

1.4.2

The second way to meet the previous requirement consists in adopting a genuine foundationalist perspective and showing (or trying to show) that it is possible to select a suitable range of statements whose truth-conditions can be fixed both:

- according to a certain predetermined system of criteria selected because of their agreement with a certain philosophical perspective,

- and in such a way that the well known theorems of an already accepted mathematical theory follow from this stipulation.

This is the attitude of neo-Fregean Platonism.

As a matter of fact, the neo-Fregean strategy to respond to Benacerraf's Dilemma is a multifarious one:

- The Context Principle—or better its consequences according to the neo-Fregean Platonist reading—are firstly evoked in order to argue that “what is needed in order to respond to the access problem [for (natural) numbers] is to ‘explain the sense of a proposition in which a number word [...] occurs’” [cf. [5], 115; and Frege, *Grundlagen*, § 62];
- Then, Frege's interpretation of statements involving numerical terms is evoked in order to argue that “what, more specifically, must be explained [...] is *the sense of identity statements* connecting terms for numbers” and that “such identity statements will, in the fundamental case, be of the form: The number of *Fs* = the number of *Gs*” [cf. [5], 115];
- Finally, Hume's Principle is employed in order to “explain the sense” of these statements, so that this is done:
 - by affording truth-conditions for them;
 - and in such a way that nothing else (except “a suitable underlying second-order logic”) is needed in order to get Peano's axioms.

1.4.3

Now, it seems to me quite clear that it is this very last step that justifies *a posteriori* the whole neo-Fregean argument.

Suppose that Hume's Principle had failed to be the starting point of a proof of Peano's axioms, or more dully, that someone had some (philosophical) reasons to reject this proof, for example because of its use of two axiom schemas of comprehension, one for unary predicates and the other for binary relations (note that the supposition here is not that these reasons are good reasons, but only that they could be evoked by someone) [cf. for example [10]].

It would then be hard to admit—either in general, or at list for such an opponent of Peano’s axioms proof—that it suffices “to explain the sense” of statements of the form \lceil The number of F s = the number of G s \rceil , in order to respond to the access problem for (natural) numbers.

And, without a clear identification of an “appropriate range of [...] statements” whose “sense” could be “explained” by supplying *ipso facto* a premise starting from which the usual theorems of arithmetic can be deduced, it would be hard to admit that in order to respond to the access problem for (natural) numbers there is nothing to do but explain the sense of certain propositions.

It suffices to evoke these possibilities to understand that in neo-Fregean Platonism, Peano’s arithmetic—together with the system of second-order logic in which this arithmetic is deduced from Hume’s Principle—legitimizes Hume’s Principle at least as well as this principle provides Peano’s arithmetic with a suitable foundation.

But, if this is so, why neo-Fregean Platonism should be preferable to any other philosophical strategy that:

- Recognizes that in order to respond to Benacerraf’s Dilemma it suffices to show that a certain system of principles enables us to utter warranted statements involving numeral terms which coincide with, or correspond to the well known theorems of arithmetic;

and

- Specifies these principles and makes manifest that their consequences coincide with or correspond to these theorems?

In other terms, why neo-Fregean Platonism should be preferable, as a (conservative intellectual) response to Benacerraf’s Dilemma, to the mere presentation of any accepted version of arithmetic, together with a suitable argument arguing that what enables us, within this version of arithmetic, to utter warranted statements involving numeral terms also provides us *ipso facto* with: a semantics for these terms (that it is easy to construe as a Tarskian one); a justification for the content of these statements; and thus knowledge of such a content and of the objects these terms refer to, according to such a semantics?

1.4.4

I know two different answers to such a question.

- The first one is a very general one and consists in arguing that the neo-Fregean Platonist response is preferable because of the logical or *a priori* nature of Hume's Principle and of the deduction of Peano's arithmetic from this principle.
- The second one is more specific, since it is concerned with the particular logical form of what, according to the neo-logicist perspective, enables us to utter warranted statements involving numeral terms, that is, an abstraction principle in Frege's sense (like Hume's Principle), grafted on a system of second order logic.

As I understand it, this latter answer is unacceptable. But since it concerns more specific issues than those that we are now discussing here, I shall come back to it later. Let me simply here discard it provisionally.

The reason why the first answer is also untenable is easier to appreciate.

This is not because neither this principle nor this deduction are logical or *a priori*. Personally I do not know whether they are. I only know that, at least the first of these qualifications has been seriously questioned both for Hume's Principle—for example by Boolos—and for this latter deduction—for example by Shapiro and Weir [cf.: [1], [2], and [10]].

But, though this principle and this deduction are taken as purely logical, analytic and *a priori*, in order to argue that neo-Fregean Platonism is the good response of Benacerraf's Dilemma since it responds to such a dilemma in showing at the same way that arithmetic is nothing but logic, and is thus analytic and *a priori*, one should also add as a premise either that: logicism is the best philosophy of mathematics, or that arithmetic is just as this deduction shows that it is.

Since the first premise should seem too strong even to a logicist—who should rather be able to conclude that this is so starting from other premises—, only the second one is actually available.

This was also the premise of Frege's original argument of the *Grundlagen*. One could rephrase it as follows:

Frege's Premise: If it is possible to deduce the accepted theorems of arithmetic (or more particularly, in the case of the neo-logicist argument,

Peano's arithmetic) from some purely logical premises, then we have showed that arithmetics is nothing but logic.

This is the same as discarding any alternative version of arithmetic *vis-à-vis* of the logicist one, to suppose that such a latter version unveils the intrinsic nature of arithmetic.

I do not deny that it is so. I simply maintain that:

- Once mathematics is understood as as being a given reality, the only way to construe the notion of the intrinsic nature of a mathematical theory in a plausible way is to look for some common aspects of all the versions of this theory (that are often very different from each other from a logical point of view);
- The possibility to deduce Peano's arithmetic within a certain system cannot be taken as being an argument for a certain philosophical perspective unless Peano's arithmetic is not understood as such as being a given reality and is thus accepted as such prior to any alternative version of it.

I conclude that:

- Either neo-logicists renounce to appeal to the deduction of Peano's arithmetic from the Hume' Principle as part of an argument aiming to show that this principle affords the good response to Benacerraf's Dilemma;
- Or they have no room to argue that their answer is actually the good one, rather than merely a possible one; namely a particular version of a more general possible answer.

1.4.5

If I'm right it suffices to observe that:

- neo-logicists actually appeal to the deduction of Peano's arithmetic from the Hume' Principle as part of their argument,

- Hume's principle is in fact an implicit definition and the system of second order logic used in order to deduce from it Peano's arithmetic is nothing but the framework of a logical deduction (which is in fact an acceptable one),

in order to conclude that neo-Fregean Platonism should be understood as a particular, though very sophisticate, form of *ante rem* structuralism in disguise.

Conversely, one could argue that *ante rem* structuralism is a sort of generalization of neo-Fregean Platonism founded on the idea that there are several other systems of stipulations which are able to create the conditions under which one can claim that there are mathematical objects.

1.5

The conclusions of the first part of my talk are thus the following:

- The two examples of conservative intellectual responses to Benacerraf's Dilemma I have considered amount in fact both to argue that there is room to construe the crucial notions of mathematical object, mathematical truth and mathematical knowledge in such a way that (pure) mathematics has in itself the power to respond to such a dilemma, and to do it by dissolving this dilemma.
- Moreover, they succeed in this by adopting a similar strategy: to argue that there are abstract objects (together with genuine truth and genuine knowledge) as soon as there are systems of principles that enable us to make warranted statements involving singular or sortal terms denoting these objects.

Since it would be too artless to argue that these systems of principles are part of a reality that is independent of humans acts, and it should thus be unquestionable that they are nothing but human stipulations (though one could add that they are not purely unjustified conventions), it follows from here that the previous philosophical perspectives show that:

A way to argue in favor of an appropriate and plausible form of Platonism is to construe the notion of mathematical object (together with other crucial connected notions) in such a way that it is possible to maintain that

mathematical objects are constituted when certain systems of stipulations are adopted.

But the adoption of a system of stipulations is, in its very essence, a human act taking place in space and time, or better in history.

Thus:

- An appropriate and plausible form of Platonism is a philosophical perspective on mathematical activity understood as an activity taking place in history;
- The crucial component of this perspective is given by the elaboration of an appropriate notion of mathematical objects fixing the conditions under which one can claim that a system of mathematical objects has been constituted.

The constitution of a system of mathematical objects is what I term “mathematical objectivation”.

The second part of my talk will be precisely concerned with mathematical objectivation.

2

The notion of object can be, and actually has been construed in several ways. Two natural perspectives have been explored: the perceptual perspective, and the linguistic perspective.

In the first perspective, an object is the content of a perceptual act or better of a class of equivalence of perceptual acts, and the problem consists in defining the conditions under which this content is perceived as being a unity and this unity has to be taken as invariant under the passage from an act to another one.

In the second perspective, an object is the referent of a linguistic expression, and the problem consists in defining the conditions under which a linguistic expression refers, it refers properly to an object (rather than to a property, or a relation, for example), and this object has to be taken as invariant under different occurrences of this expression or of other related expressions.

I do not think that the first perspective is as such stranger to philosophy of mathematics, since I’m far to exclude that mathematics has an essential

empirical component (to have an obvious example of this fact, consider the necessity to recognize graphical patterns in order to deal with symbol a long a formal argument).

Nevertheless, I shall not consider this question here. I suppose rather that at a first approximation one can suppose that the philosophical problems concerned with the perception of the empirical objects entering mathematical activity can be considered as being solved, *i.e.* that this perception can be taken as philosophically unproblematic.

I thus address here only the linguistic problem, or better some aspects of it.

2.1

I also suppose the following necessary condition, suggested by the previous considerations about *ante rem* structuralism and neo-Fregean Platonism:

C.i: A mathematical object is not constituted unless a linguistic term, which is understood—within a certain community of human subjects—as being a singular or a sortal term, is introduced or selected from already given terms, in such a way that the members of such a community consider themselves as being warranted in using this term in statements that they consider as being warranted in their turn.

This is a quite general condition. According to it, the constitution of a mathematical object or of a domain of mathematical objects is relative to a certain community of human subjects and to the appraisal of the members of this community.

But, once it is used in an historical context, *i.e.* it is referred to a certain historical situation in which the constitution of the relevant objects is supposed to have occurred, extra-logical variables can be determined by means of an historical reconstruction. And, once some relevant categories to be used in such a reconstruction have been fixed, this reconstruction can be taken as philosophically unproblematic.

Thus, we can reduce such a condition to the following one:

C.ii: A mathematical object is not constituted unless a linguistic singular or sortal term is introduced or selected from already given terms, in such a way that such an introduction or selection enables the utterance of warranted statements involving this term.

There are several problems concerned with this condition. Some of them are the following:

- What has to be understood as being a singular or a sortal term?
- What counts as an introduction or a selection of such a term?
- What does it mean that such an introduction or selection enables the utterance of certain statements?
- What counts as a warranted statement?

All these problems are genuine philosophical ones and need of course a response in any philosophy of mathematics that takes **C.i** or **C.ii** as being accepted relevant conditions. But I'll not consider them here.

I confine myself to suppose that the first problem has been solved (relatively to any language which is relevant for the constitution of mathematical objects), while the three other ones can be solved by assuming the appropriateness and the complete intelligibility of the following answers:

- The introduction or selection of a singular or sortal term is an explicit or implicit definition (where the expression "implicit definition" has to be understood both in a technical Hilbertian sense and in a non technical sense).
- Such a definition enables the utterance of certain statements when it takes place in a context that makes possible, once this definition is given, to formulate some arguments counting as possible justifications for some statements where the defined term occurs (notice that part of this context can be constituted by a simultaneous definition of other terms, as is the case for implicit definitions in Hilbertian sense).
- A statement is warranted when these arguments are taken as being correct ones.

2.2

I'm far to argue that these answers are satisfactory. I simply suppose that they are possible. And I do that, since I maintain that:

- It would be possible to detail them appropriately making appeal both to other philosophical considerations or to some historical reconstruction;
- The adoption of the condition $C.ii$ addresses a problem that is independent both of these answers and of any any other possible answer for the previous questions.

It is precisely this problem which I'm interested in.

Though $C.ii$ is admitted as being a necessary condition, it is quite obvious that it is not also a sufficient one. Thus, it is natural to look for other necessary or even sufficient conditions.

I see two different attitudes it would be possible to take to approach this problem. As a matter of fact, they correspond to two different insights in the problem itself.

- One could firstly maintain that $C.ii$ is not a sufficient condition because it is satisfied by different sorts of abstract objects among which there are some that could very hardly be considered as mathematical. A way to get from it a sufficient condition could then be to refer it to a certain domain of objects, whose mathematical nature is taken as being unproblematic, as (integer positive) numbers, and to look for a suitable way to specify the different parts of it with respect to such a domain.
- One could, on the other hand, maintain that $C.ii$ is not a sufficient condition because mathematical objects are, so to say, objects in a deeper sense, that is, not only because they satisfy this necessary condition but also, and above all, because they enjoy other relevant features of empirical objects (or of other entities we take unquestionably as being objects).

The two attitudes are not independent, since it is possible to specify $C.ii$ with respect to a certain domain of mathematical objects and then argue or suggest that at least some of the features that are ascribed to these objects *via* such a specification are shared, or should be shared, by any mathematical object.

It is thus possible to adopt both these attitudes together. This is what I shall do.

2.2.1

The first attitude seems to me to be adopted by the neo-logicist program.

The question addressed within this program is not “What is a mathematical object?”, but rather: “What is a number?” And, since it is taken for granted that numbers are mathematical entities, no particular effort is devoted (as long as I know) to look for a suitable characterization for mathematical objects in general (notice however that though the heart of the neo-logicist program is given by a theory about natural numbers, an extension of the program to real numbers is envisaged).

In the neo-logicist perspective, **C.ii** is transformed in a sufficient condition for (natural) numbers by supposing that:

- The introduction of the sortal terms “(integer) number” and “finite (integer) numbers”, of the singular terms “0”, “1”, “2”, etc., and of the coextensional relevant definite descriptions, like “the number less or equal to n ” or “the successor of 0” takes place within a well established system of second-order logic including Hume’s Principle (or any other equivalent proper axiom), assumes the form of an ordered series of licensed definitions and cooperates to a formal deduction of Peano’s arithmetic within this system.
- The warranted statements involving these terms are nothing but the theorems proved throughout such a deduction.

The implicit suggestion contained in such a specification is that:

C.iii: A domain of mathematical objects has been constituted if and only if an abstraction principle like Hume’s Principle and an ordered series of licensed definitions are added to the other components of a well established system of second order logic within which it is possible to deduce theorems counting as warranted statements involving the terms referring to these objects.

It is quite obvious to argue that this would be too strong a requirement, at least if one wanted to count as mathematical objects the entities forming the subject matter of a large number of theories, both formal and informal (in the sense of which a logical system is formal), which have unquestionably populated the history of mathematics.

Once an historical perspective is adopted it is certainly natural to amend and extend such a definition by substituting the well established logical system with an accepted deductive context and the abstraction principle with a suitable system of (explicit or implicit) axioms and definitions.

In doing that, one would move away from the neo-logicist perspective not only because one would adopt a very weak conception of what a deduction is, renouncing in fact to the reassuring protection of formal logic from any sort of relativism, but also in a more specific and technical sense.

Moreover, one would not provide any plausible solution for our problem, since **C.iii** is not only too strong, but it is also too weak.

2.2.2

Before considering the reasons why this condition is too weak, and thus adopt the second one of the two attitudes distinguished at the beginning of the present section 2.2, let me consider the technical sense in which, by adopting such a possible amendment, one would move far away from the neo-logicist perspective.

Formerly, I have provisionally discarded a possible answer to the question concerning the reasons for which neo-Fregean Platonism should be preferable, as a (conservative intellectual) response to Benacerraf's Dilemma, to other philosophical perspectives. It is now time to consider this answer and justify its discard.

I have said already that this answer is concerned with the fact that, according to the neo-logicist perspective, what enables us to utter warranted statements involving numeral terms is an abstraction principle in Frege's sense, grafted on a system of second order logic.

This abstraction principle is of course Hume's Principle, but what is important here is not its specific content, but its logical form, the fact that it is precisely an abstraction principle in Frege's sense, *i. e.* an example of the general schema:

$$\forall\alpha\forall\beta[\Phi(\alpha) = \Phi(\beta)] \Leftrightarrow \Theta(\alpha, \beta)$$

where $\Phi(-)$ is an operator acting on entities falling under the range of the universal quantifications, $\Theta(-, -)$ is an equivalence relation defined on this range, and $- = -$ is the identity relation which is defined, just because of this principle, on the entities produced by the application of $\Phi(-)$ to the entities of this range.

Here, the relation of identity is taken for granted. In defining it on the entities produced by the application of the operator $\Phi(-)$ to the entities falling under the range of the universal quantifications, one does not define it, but makes appeal to an equivalence relation, which is supposed to have been already defined on the latter entities, to introduce a criterion of identity for the former entities, and thus defines these former entities as being individuals falling under a new sortal concept, the concept of Φs .

Said in other terms: supposing that any equivalence relation $\Theta(-, -)$ has been defined on the elements of a domain X of certain entities, *i. e.* on the Xs , one introduces—under the form of the images $\Phi(x)$ of the elements of X given by a supposed operator $\Phi(-)$ acting upon the Xs —new singular terms to denote the classes of equivalence defined on the Xs by the relation $\Theta(-, -)$.

The logical properties of any relation of equivalence make possible to introduce these terms simply by stating that $\Phi(x) = \Phi(x')$ if and only if $\Theta(x, x')$.

This move is not only very common in the history of mathematics, but it is also so logically trivial that it is often left implicit and simply appears because of the adoption of a language which depends on it.

Frege's classical example of an abstraction principle other than Hume's is that of parallel lines and their common direction. Another one, that, as a matter of fact, governs Euler's version of the Cartesian theory of curves, is the following:

The algebraic curve of Cartesian equation $F(x, y) = 0$ is the same as the algebraic curve of Cartesian equation $G(x, y) = 0$ if and only if it is possible to convert $F(x, y) = 0$ in $G(x, y) = 0$ and *viceversa* (that is, the polynomial $F(x, y)$ in the polynomial $G(x, y)$ and *viceversa*) by applying to x and y a linear transformation corresponding to a change of Cartesian co-ordinates.

By adopting this principle, Euler certainly did not introduce the single algebraic curves in mathematics. A lot of single algebraic curves were well known and studied from the antiquity, being characterized for their specific properties. And, extensionally speaking, the domain of all the algebraic curves and the sortal concept '[(to be an) algebraic curve]', were at the mathematicians' disposal from Descartes, which had defined these curves as being geometric, *i. e.* geometrically constructible according to a certain particular criterion of geometric construction.

Euler simply understood that these curves could be treated as classes of equivalence of polynomial equations and characterized them in this way by changing the intensional characterization of Descartes' domain of geometric curves.

This was certainly a crucial advance in mathematics, but one could hardly maintain that the previous abstraction principle shows that geometric Cartesian curves can be understood as being mathematical objects. At most one could argue that this principle shows that they can be understood as being analytical rather than geometric objects (and in fact the crucial advance actually consists in this change of understanding).

At the same way, by adopting Hume's Principle, a neo-logicist certainly does not introduce the natural numbers in mathematics. Extensionally speaking, the domain of natural numbers was at mathematicians' disposal at least from Euclide.

Once this principle is immersed in a suitable system of second order logic and is associated to a suitable definition of the successor relation on numbers of concepts (which is in fact a transposition of a definition of a corresponding relation between concepts), this principle simply allows to define the natural numbers as a particular sort of numbers of concepts, that is as the elements of a suitable proper subset of the set of the classes of equivalence defined on the domain of concepts by the equivalence relation \ulcorner (to be) equinumerous with \urcorner .

The deduction of Peano's arithmetic from Hume's Principle amounts precisely to such a definition together with the proof of some crucial theorems on natural numbers so defined (which allow to prove, outside the system of second order logic within which Hume's Principle is immersed, that the set of these numbers satisfies Peano's axioms).

Thus the question arises:

Q: How can neo-logicists maintain that it is just by defining natural numbers in the virtue of Hume's Principle that one shows that they are (genuine) objects?

If I understand well their answer, it is twofold.

2.2.2.1

Its first part relies on a technical remark that can be put under the following form (notice that from this remark also follows that by speaking of classes

of equivalence defined on the domain of concepts, as I have just done, we do not open the door to the classical paradoxes about sets).

Once the general abstraction schema

$$\forall\alpha\forall\beta[\Phi(\alpha) = \Phi(\beta)] \Leftrightarrow \Theta(\alpha, \beta)$$

is immersed in a system of second order logic by:

- taking the variables α and β as being unary predicates of this system;
- taking the equivalence relation $\Theta(-, -)$ as being explicitly defined within this system by introducing the schema of formulas $\Theta(P, Q)$ as nothing but an abbreviation of a (well formed) licensed and suitable formula where the unary predicates P and Q occur free;
- admitting in this system one and only one proper axiom as the following [cf. Boolos (1987)]

$$\forall P\exists!z\forall Q[\eta(Q, z) \Leftrightarrow \Theta(P, Q)],$$

where $\eta(-, -)$ is a non logical binary predicative constant implicitly defined by this axiom, whose two places are supposed to be occupied respectively by an unary predicate and an individual variable or constant;

- and finally defining the image $\Phi(P)$ of any unary predicate P stating that this image is just the individual z which satisfies the open formula

$$\forall Q[\eta(Q, z) \Leftrightarrow \Theta(P, Q)],$$

which is evidently licensed by the previous proper axiom;

then,

- the principle obtained from the general abstraction schema according to the previous replacements can be proved within this system¹, and

¹The (easy) proof is the following. Suppose first that $\Phi(F) = x$ and $\Phi(G) = x$. For the definition of $\Phi(-)$, it follows: $\forall Q[\eta(Q, x) \Leftrightarrow \Theta(F, Q)]$ and $\forall Q[\eta(Q, x) \Leftrightarrow \Theta(G, Q)]$. Take in both these formulas Q to be G . It follows: $\eta(G, x) \Leftrightarrow \Theta(F, G)$ and $\eta(G, x) \Leftrightarrow \Theta(G, G)$. But $\Theta(-, -)$ is an equivalence relation, thus $\Theta(G, G)$, and then $\eta(G, x)$, and so, for $\eta(G, x) \Leftrightarrow \Theta(F, G)$: $\Theta(F, G)$. Suppose now $\Phi(F) = x$, $\Phi(G) = y$ and $\Theta(F, G)$. For the definition of $\Phi(-)$ and $\Phi(F) = x$, it follows: $\forall Q[\eta(Q, x) \Leftrightarrow \Theta(F, Q)]$. Take Q to be G . For $\Theta(F, G)$, it follows: $\eta(G, x)$. For the definition of $\Phi(-)$ and $\Phi(G) = y$, it follows: $\forall Q[\eta(Q, y) \Leftrightarrow \Theta(G, Q)]$. Take Q to be G . For $\Theta(G, G)$, it follows $\eta(G, y)$. Take now in the proper axiom P to be F and Q to be G . For $\Theta(F, G)$, it follows: $\exists!z(\eta(G, z))$. Thus $x = y$, that is, $\Phi(F) = \Phi(G)$. ■

thus the classes of equivalence defined on its unary predicates by any equivalence relation $\Theta(-, -)$ can be appropriately treated in this system as individual variables.

This treatment is moreover authorized by the introduction of two comprehension axioms schemas for unary and binary predicates which assure respectively that:

- For any (well formed) formula of the system $A(z)$, where one and only one individual variable occurs free, it exists a unary predicate $[x : A(x)]$ which applies to all the individuals that satisfy $A(x)$ and only to them;
- For any (well formed) formula of the system $A(w, z)$, where two and only two individual variables occurs free, it exists a binary predicate $[x, y : A(x, y)]$ which applies to all the pair of individuals that satisfy $A(x, y)$ and only to them.

It is then possible to take the equivalence relation Θ as being the relation which applies to a pair of equinumerous unary predicates, and then define :

- An individual constant 0 , as being the image $\Phi([x : x \neq x])$ of the unary predicate $[x : x \neq x]$;
- A binary relation S which applies to a pair of individuals if and only if they are respectively the images $\Phi(P)$ and $\Phi(Q)$ of two unary predicates P et Q which are such that it exists an individual z such that Pz and $Q = [x : Px \wedge x \neq z]$, and works as the successor relation on the Φ -images of unary predicates;
- An unary predicate FN which applies to all the images $\Phi(P)$ of an unary predicate P to which the constant 0 is related according to the ancestral of the relation S , and which works as the sortal concept \ulcorner (to be) a natural number \urcorner (since the ancestral of S works as a relation of strict order on the Φ -images of unary predicates).

In the light of these logical facts, the neo-logicist thesis according to which numbers are objects as long as they are defined by means of Hume's Principle admits of a very austere reading:

Natural numbers are objects as long as they can be defined as individual constants within a system of second order logic, where traditionally unary

predicates are identified with concepts, binary predicates are identified with relations and individuals are identified with objects.

But is this thesis admissible?

I do not deny that natural numbers so defined can be considered as objects. But are they so just because they are reduced to individuals of a second order logical system?

Should one suppose that in order to introduce a domain of objects, it is sufficient and necessary to immerse the abstraction schema in a second order logical system in virtue of a proper axiom that associates any class of equivalence defined by the relation Θ to one and only one individual of the system and allows thus to treat these classes of equivalence as individuals of the system?

It seems to me that the answer should be in the negative. The reason is that a positive answer would entail implausible consequences. The following are some of them:

- If natural numbers are places in Peano's axioms structure, these axioms being given as such, then they are not objects; if they are individuals of a second order logical system which satisfies Peano's axioms, then they are objects.
- Segments, angles and polygons, as they are defined both in Euclide's or Hilbert's geometries, are not objects.
- Algebraic Eulerian curves as defined by the previous example of the abstraction schema are not objects just because, under the interpretation that produces this example, this schema is not immersed in a second order logical system; and if one succeeded in making such an immersion by expressing the equivalence relation Θ by a suitable open formula of this system and introducing a correspondent proper axiom associating any one of the classes of equivalence that this relation defines to one and only one individual of the system, then these curves would be converted to objects.
- Linguistic singular or sortal terms do not denote objects unless predicates can be quantified.

2.2.2.2

A possible defense from this objection comes from the second part of the neo-logicist answer to the question **Q**:

In defining natural numbers through Hume's Principle immersed in a suitable system of second order logic, one does more than "secur[ing] the existence of numbers by stipulation", and it is just in virtue of this something more that numbers so defined are objects [cf. [5], 121]:

[...] what is stipulated when an abstraction principle is advanced as an implicit definition, is not the existence of certain objects—referent for the terms featured on its left hand side—but the truth of (indefinitely many) biconditionals co-ordinating identity-statements linking such terms with statements involving the relevant equivalence relation over the underlying domain. The truth of any given one of those identity statements, and hence the existence of objects to which its ingredient terms refer is not stipulated, but follows only given the truth of the co-ordinate statements to the effect that the abstraction's equivalence relation holds among the relevant objects (or concepts, in case of a higher-order abstraction such as Hume's Principle). And the truth of that latter statement will be always a matter of independently constituted facts (about parallelism of certain lines, or one-one correspondence between certain concepts, etc.). What is brought into existence by stipulation—if anything is—is not objects, but a certain *sortal* concept. What objects, if any, fall under it is—as we've said—entirely dependent upon the truth of instances of the abstractive biconditional's right hand side. [cf. Hale and Wright (2002), 121].

If this argument were admissible, it would be certainly a very strong one. Said in other terms it states that:

What makes that the referents of a singular or a sortal term are genuine objects is that, once suitable definitions are given, the truth of statements that ascribe some properties to these referents is "a matter of independently constituted facts".

But, what are these facts in the case of an abstraction principle?

If I understand well, they should be facts that verify the right hand side of this principle when it is interpreted as referring to them.

This right hand side is the statement $\Theta(\alpha, \beta)$, that becomes, in the case of natural numbers, $P \approx Q$, or better, once this equivalent is written in the

system of second order logic that has been chosen:

$$\exists R \left[\begin{array}{l} \forall x (P \Rightarrow \exists! y (R(x, y) \wedge Qy)) \wedge \\ \forall y (Qy \Rightarrow \exists! x (R(x, y) \wedge Px)) \end{array} \right]$$

where P and Q are “relevant concepts”, *i. e.* suitable unary predicative constants.

Now, once:

- 0 is defined as the number of the concept $[x : x \neq x]$ (whose existence is, if not stipulated, at least licensed by a suitable comprehension axiom);
- a binary relation S is defined on numbers of concepts (taken as individuals in the virtue of a suitable proper axiom) by assuming that $S(m, n)$ if and only if there exist F and y such that Fy , $\Phi(F) = n$ and $\Phi([x : Fx \wedge x \neq y]) = m$;
- an unary predicate FN is defined on individuals by assuming that FNn if and only if $S^*(0, n)$ or $n = 0$, where S^* is the ancestral of S ;

what “facts” verifying an equivalence of the form $P \approx Q$ are needed to deduce that the set of individuals z which satisfy the formula FNz forms a progression and is thus a model for Peano’s axioms?

If the answer is that these facts are logical facts about the considered system of second order logic—*i. e.* the facts that certain theorems of the form $P \approx Q$ like

$$\begin{array}{ll} [x : x \neq x] \approx [x : x \leq 0 \wedge x \neq 0] & \text{or} \\ [x : x \leq 0] \approx [x : x \leq \Phi([x : x \leq 0]) \wedge x \neq 0] & \text{or} \\ [x : x \leq n] \approx [x : x \leq \Phi([x : x \leq n]) \wedge x \neq 0] & \text{or} \\ [x : Px \wedge x \neq a] \approx [x : Px \wedge x \neq b] & \text{(with } Pa \text{ and } Pb) \end{array}$$

are proved in this system—, then the reply is that either it is very hard to admit that these facts are “independently constituted”, or they are not relevant.

Consider the first three of the previous theorems. As for any individuals x and z “ $x \leq z$ ” is nothing but an abbreviation of

$$S^*(x, z) \vee x = z$$

and the relation S^* is defined as being the ancestral of S that is defined as it has been said previously, these theorems, as well as any other similar theorem involving the relation \leq , only follow if the individuals $\Phi(P)$ are introduced in virtue of Hume's Principle (notice moreover that, strictly speaking, no one of these theorems is actually needed as such to define the natural numbers).

The case of the fourth theorem is different. This theorem is independent of Hume's Principle (though certainly not of the "underlying logic" used to define natural numbers starting from the introduction of this principle), but it is also perfectly irrelevant for the neo-logicist definition of natural numbers, as well as any other theorem of the same form involving neither the relation \leq nor an individual $\Phi(P)$.

If the answer is that the "independently constituted facts" are not logical facts, but just the facts that two concepts like

$$\begin{array}{llll}
 [x : x \neq x] & \text{and} & [x : x \leq 0 \wedge x \neq 0] & \text{or} \\
 [x : x \leq 0] & \text{and} & [x : x \leq \Phi([x : x \leq 0]) \wedge x \neq 0] & \text{or} \\
 [x : x \leq n] & \text{and} & [x : x \leq \Phi([x : x \leq n]) \wedge x \neq 0] & \text{or} \\
 [x : Px \wedge x \neq a] & \text{and} & [x : Px \wedge x \neq b] & \text{(with } Pa \text{ and } Pb)
 \end{array}$$

are equinumerous, then the reply is that: either these facts can only be established by expressing them through suitable formulas of the system and proving these formulas as theorems; or they are strictly irrelevant for the definition of natural numbers.

The conclusion of this long *excursus* is thus that both the neo-logicist twofold answer to the question **Q** and the neo-logicists possible answer to the question concerning the reasons for which neo-Fregean Platonism should be preferable, as a (conservative intellectual) response to Benacerraf's Dilemma, to other philosophical perspectives that I had provisionally discarded, are implausible.

2.3

It is thus time to come back at **C.iii** and say why it is too weak.

My point is not that the Platonism that such a condition would promote is too frail, because, according to such a condition, the existence of mathematical objects would depend on a stipulation [cf. [5], 12; [7]; and [4]].

To advance this reason, one should dispose both of a criterion and a definition of existence as a condition that does not depend and cannot depend

on a stipulation, and of course I neither dispose of them nor I think that it be possible to introduce them without being forced to deny that there are mathematical objects.

If I maintain that **C.iii** is too weak, it is not because I maintain that neo-logicist numbers are not genuine (mathematical) objects, but because I hold that they are genuine (mathematical) objects because of a feature of them that neither the condition **C.iii**, nor the usual neo-logicist arguments, put forward.

I make the same point with respect to *ante rem* structuralism: I maintain that places in a structure which is known by linguistic abstraction or projection or by an implicit definition are objects, but I deny that they are genuine objects just because they are places in a structure and it is then possible to make warranted statements about them.

2.3.1

Platonism is an attractive philosophy of mathematics, since it explains—though in a way that is often implausible—why mathematics appears, both to mathematicians and to learners, not only as reliable, but also as reliable because of a constraint that does not depend on them, or at least not directly. As people often say, mathematics resists to any change of opinion.

One could think that an easy way to make this feeling compatible with an historical perspective justifying a condition like **C.ii** consists in:

- transforming this condition by clarifying that the introduction or selection of a singular or sortal term should amount to a definition taking place in a deductive system, and the warranted statements should amount to theorems proved within this system;
- and remarking that once the deductive system has been fixed and the definition has been introduced, there is no way to disclaim a well established proof within this system.

If such an amended condition were taken as being not only necessary but also sufficient, then one would have recovered the amended version of **C.iii** I have suggested in section 2.2.1:

C.iv. A mathematical object is constituted if and only if a linguistic singular or sortal term is introduced by means of a definition taking place in a

deductive system, in such a way that such an introduction or selection enables to prove theorems where these terms occur.

After all, though both the existence and the properties of mathematical objects would then depend on human stipulations, these stipulations could be considered, in many situations, like already given constraints that cannot be disregarded.

I do not think, however, that this solution is good, since I think that, like C.iii, C.iv is too weak a condition: mathematical objects are certainly the referents of singular or sortal terms that are introduced by means of a definition taking place in a deductive system, in such a way that such an introduction or selection enables to prove theorems where these terms occur, but they are genuine objects because of that.

2.3.2

In a quite stimulating book devoted to the notion of mathematical object, E. Giusti—one of the most influential historians of mathematics of the present generation—has suggested that mathematical objects are created by a “process of objectivation of procedures” concerned with previous objects, and that this process necessarily involves three stages: what is becoming a mathematical object has to appear as being a “tool” of mathematical researches (concerned with other objects), an “object matter” of other researches, and a “solution of problems” [cf. [3]].

The main idea of this account is that the definition of a mathematical object is generally the last act of a long process. It is the act that, so to say, crystallizes a previous practice on other objects, or fixes the conditions under which an already given reality (that one could imagine as a given patterned system) is formally characterized under the form of a structure that such a reality is supposed to exemplify.

A definition for Fregean abstraction could and should be seen as an act of this sort, as long as it supposes that a previous domain of objects is given and a partition of it in classes of equivalence is admitted. This definition would then correspond to the act of taking these classes of equivalence as the object matter of a new theory, and thus as objects *per se*, and this act would be motivated by an aim that is as such external to this new theory.

Cavaillès’ thematisation is also an act of this sort, as long as it supposes that a certain formalism is already operative on already given objects. It corresponds to the act of isolating an aspect of this formalism and transforming

it in the correlate of a new object defined either as being independent of this formalism or as making appeal to it.

In both cases the introduction of new objects depends on a re-organization of a previous domain and aims to a re-orientation of an established practice.

History of mathematics presents other acts of this sort. One of the most frequent consists for example in the introduction of objects as solutions of problems that cannot be solved within a previous domain, though some evident analogy suggests that a solution should be admitted (what seems to me to be close to Cavallès' idealization). The introduction of imaginary roots of an algebraic equation, or of irrational numbers, just defined (and even denoted) as real solutions of equations whose geometric solution was obvious, are easy examples (notices that in both cases these objects have been successively re-defined as classes of equivalence respectively of rational and real numbers).

In a sense, the problem of characterizing mathematical objects could then be solved by pointing out the main "patterns of mathematical changes" exemplified in history of mathematics. This seems to have been the suggestion of P. Kitcher [cf. [6]].

But in this way, one could only obtain an extensional characterization. Though I think that no intensional characterization is possible without a support of an historical inquiry, it seems also to me that this inquiry should be conducted under the guidance of an hypothesis. It is just this hypothesis that I would like to advance.

2.4

A crucial character of empirical objects is that they can be presented in two different ways: either they can be shown or they can be described.

To show an empirical object means to characterize it ostensively, by pinpointing its place in space and time. To describe it means to characterize it linguistically, by listing some of its properties. Notice that a changing property—for example a changing color on a surface that is perceived otherwise as being uniform—can mark the spatio-temporal bounds of an object and thus make possible to place it ostensively in space and time. But even in this case, to show an object means to characterize it ostensively because of the fact that it occupies the spatio-temporal region determined by these bounds and not because of the fact that these bounds are perceived or even determined in the virtue of some other changing property.

When an object is shown, it does not make sense to ask for a better identification of it. If a singular term “ x ” (for example a demonstrative) denotes an empirical object *qua* showed, it does not make sense to ask: “which is the object that is x ?”. The appropriate question is rather: “What is the object that is x like?”; and any property P that can be ascribed to an empirical object—though different from the properties ‘to be “ x ”’, or ‘to be denoted by “ x ”’, or similar—is a good candidate for entering the answer: “ x is P ”.

On the contrary, if a singular term “ x ” denotes an empirical object *qua* described, it makes perfectly good sense to ask: “which is the object that is x ?”. And though one can also ask appropriately “What is the object that is x like?”, there are certain possible answers to such a question that simply state that the object that is x is x .

This difference is connected with a difference about beliefs. If a belief concerns an object x *qua* shown, then it is a *de re* belief; the statement “one believes that x is P ” means: of x one believes that it is P . If a belief concerns an object *qua* described, then it is a *de dicto* belief; the statement “one believes that x is P ” means: one believes of x that it is P .

2.4.1

Though in case of empirical objects, the latter difference is owed to the nature of an ostensive act, it is as such independent of the possibility to fulfill such an act. One could thus wonder whether it is possible to have *de re* beliefs concerned with non empirical objects. In my view, this is equivalent to wonder whether a non empirical object can be presented in such a way that when a singular term “ x ” denotes this object *qua* presented in this way, it does not make sense to ask: “which is the object that is x ?”.

My guess is that for mathematical objects *de re* belief is possible. Moreover, I argue that such a possibility is a necessary condition for having mathematical objects.

More precisely, I argue that

C.v: A mathematical object is not constituted unless a linguistic singular term is introduced or selected from already given terms in such a way that such an introduction or selection enables the utterance of statements that, if considered as expressions of certain beliefs, have to be considered as expressions of *de re* beliefs.

2.4.2

This does not mean of course that if a singular term “ x ” denotes a mathematical object, then the statement “one believes that x is P ” certainly means: of x one believes that it is P . Rather it means that if a singular term “ x ” denotes a mathematical object, then there is a singular term “ y ” such that it is possible to state warrantably that “ y ” denotes the same object as “ x ” and the statement “one believes that y is P ” certainly means: of y one believes that it is P .

Suppose that “ x ” is “the eleventh prime number” and “ y ” is “31”. Then “ x ” and “ y ” denote the same object and the belief that y [or 31] is greater than 30 is, I guess, a *de re* belief, though the belief that x [or the eleventh prime number] is greater than 30 is a *de dicto* belief.

My argument is that the question “which is the (natural) number that is 31?” does not make sense within arithmetic, while the question “which is the eleventh prime number?” is a perfectly legitimate question, and has even the form of some crucial question in number theory.

Once one accepts that **C.iv.** is at least a necessary condition, *i. e.* one maintains that mathematical objects are the referents of singular or sortal terms that are introduced by means of a definition taking place in a deductive system, in such a way that such an introduction or selection enables to prove theorems where these terms occur, my point is equivalent to the following one:

An act of mathematical objectivation has been completed only when this definition and this deductive system are such that it is then possible to denote in this system the same object in different ways, and that one of these ways is, so to say, a privileged mode of presentation: it enters statements that have to be considered as expressions of *de re* beliefs.

2.4.3

At this point the question arises:

What’s the privileged mode of presentation of a mathematical object?

I think that such a question cannot have a general answer, since the criterion used to select such a mode of presentation cannot be but an intra-theoretic one.

2.4.3.1

I have just implicitly argued that to present a natural number as being the number denoted by the singular term “31” is to present it in a privileged mode.

I argue that this is due to the fact that, once the Indo-Arabic numerical system of denotation for natural numbers has been adopted, the singular term “31” immediately indicates the position of the number it denotes in the succession of natural numbers, and that the determination of such a position counts for a natural number as an act of showing it, rather as a description of it.

In general, once the Indo-Arabic numerical system of denotation for natural numbers has been adopted, the elementary symbols “0”, “1”, . . . , “9”, taken in their established order, and the symbols that are composed by any juxtaposition of them count, I guess, as denotations of natural numbers *qua* presented in a privileged mode.

This does not depend only on the formal nature of the deductive systems which constitute accepted versions of arithmetic, like Peano’s or the neo-logicist systems. It depends on the way this system is understood and, so to say, used in order to make arithmetic.

As long as arithmetic is a mathematical theory, it is more than a formal system. It is also, for example, a context where mathematical problems can be advanced and solved.

In it, the question “is the eleventh prime number greater than 30 or not?” counts as a genuine and appropriate problem. And the answer: “the eleventh prime number is 31, thus it is greater than 30” counts a genuine and appropriate answer.

But it is not so because “31” is a privileged mode of presentation of a natural number and “the eleventh prime number” is not. Rather, it is the other way around: “31” is a privileged mode of presentation of a natural number and “the eleventh prime number” is not so because this question and this answer count as a genuine and appropriate question and as a genuine and appropriate answer.

And if this so, it is, in its turn, because of the nature of the previous practice or the already given reality that any version of arithmetic is aiming to crystallize or formally characterize. Thus, it is so because of an intrinsically historical fact.

2.4.3.2

This means that the criterion used to select a privileged mode of presentation of a mathematical object is a transforming criterion.

As long as it cannot be but an intra-theoric criterion, its transformation is then a pattern of mathematical change, or better the first stage of a process of constitution of new mathematical objects.

The constitution of analytical geometry is an example of that: the display of a polynomial equation in two real variables that counted in Cartesian geometry as a non privileged mode of presentation of a curve becomes a privileged mode of presentation of a mathematical object, and the formalization of analytical geometry, that is the constitution of it as a well presented mathematical theory, is in a sense nothing but the process of legitimation of this change.

2.5

Provided that I'm right in stating the necessary condition **C.v** and in accepting **C.iv** as another necessary condition, one should wonder whether the conjunction of these two necessary condition gives a necessary and sufficient condition.

Let me conclude my talk today leaving this question open.

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