Semi classical limit for a NLS with potential

Keraani Sahbi*

Abstract

This paper is dedicated to the semiclassical limit of t the nonlinear focusing Schrödinger equation (NLS) with a potential ,

$$i\epsilon\partial_t u^\epsilon + \frac{\epsilon^2}{2}\Delta u^\epsilon - V(x)u^\epsilon + |u^\epsilon|^{2\sigma}u^\epsilon = 0$$

with initial data in the form $Q\left(\frac{x-x_0}{\epsilon}\right)e^{i\frac{x.v_0}{\epsilon}}$, where Q is the ground state of the associated unscaled elliptic problem. Using a refined version of the method introduced in [2] by J. C. Bronski, R.L. Jerrard, we prove that, up to a time-dependent phase shift, the initial shape is conserved with parameters that are transported by the classical flow of the classical Hamiltonian $H(t,x) = \frac{|\xi|^2}{2} + V(x)$. This gives, in particular, a complete description of the dynamics of the time-dependent Wigner measure associated to the family of solutions.

keywords. Schrödinger equation, ground state, stability, semiclassical limit, Wigner measure, WKB method.
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1 Introduction

This paper is a sequel to [?]. We continue to study the semi classical limit of the nonlinear focusing Schrödinger equation with a potential:

$$i\epsilon\partial_t u^\epsilon + \frac{\epsilon^2}{2}\Delta u^\epsilon - V(x)u^\epsilon + |u^\epsilon|^{2\sigma}u^\epsilon = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N. \tag{1}$$

*IRMAR, Université de Rennes1, Campus de Beaulieu, F-35042 Rennes Cedex, France.

Here, $\Delta = \sum_{j=1}^{j=N} \partial_{x_j}^2$ is the Laplace operator on $\mathbb{R}^N$, $u^\epsilon : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{C}$ is a complex-valued family of functions, ϵ is a small parameter (referring to Planck's constant) and V a real-valued potential. This equation arises in many fields of physics such that the propagation of light in some nonlinear optical materials. Roughly speaking the potential V is due to the inhomogeneities of the medium (see [15] for more details). The case $V = |x|^2$ describe the Bose-Einstein condensate.
The semi classical analysis of equation (1) aims to describe the asymptotic behavior of the family of solutions when $\epsilon \to 0$. The common situation is to associate to (1) a family of initial data which oscillate or concentrate with scale ϵ (or both) and then study the evolution of these properties in times. There are many methods to deal with this problem. The main usual one is the geometrical optic-or WKB method. It consists in representing the solution in the form $u^\epsilon = U^\epsilon e^{\frac{i}{\epsilon}\varphi(x,t)}$ where U^ϵ has the formal expansion $U^\epsilon = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \cdots$. The phase φ is a solution of a Hamilton-Jacobi type equation called eikonal equation and the amplitudes U_j are solutions of a recurrent infinite system of nonlinear equations (called transport equations). The justification of this formal solution is the main difficulty of this method. In general we have to linearize the equation around the approximative solution and use the *a priori* estimates (energy estimate, strichartz estimate..) to prove that error term goes to 0 when $\epsilon \longrightarrow 0$.
An other related topic, which is well developed in the last few years, is to concentrate on the existence and the stability of the associated standing waves (see [1],[?],[?]). A perturbed elliptic equation are then studied and some different behaviors (related to the properties of the potential V) are found.
In [2] Bronski and Jerrard have considered the equation (1) with the particular family of initial data

$$u^\epsilon(0, x) = Q\left(\frac{x - x_0}{\epsilon}\right) e^{i\frac{x.\xi_0}{\epsilon}}, \tag{2}$$

where $(x_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^N$ and Q is the ground state of the associated unscaled elliptic problem (see preliminary section below). The very particular form of the initial data allows them to use an alternative approach to prove that the solution of Eq.(1)-(2) has an asymptotic soliton dynamics. Their method does not use a linearizion argument as usually done, but the conservation laws (quantum and classical) and the stability of the ground state Q. In [?], we have used the same method combined with a WKB intuitions to improve

the results of [2]. The main objective of the present paper is to improve our result in [?] and to give a sharper description of the asymptotic behavior of the family of solutions to Eq.(1)-(2). It will be clear to the reader that this work relies strongly to the arguments developed by J. C. Bronski and R. L.Jerrard in [2]. Let us now give the precise assumptions of this paper.
(A0) $\sigma < \frac{2}{N}$: we are interesting in the sub-critical nonlinearity[1] .
For potential $V(x)$, the following assumptions is required.
(A1) $V(x) = V_1(x) + V_2(x)$ where V_1 and V_2 are real functions.
(A2) $V_1(x)$ belongs to the $\mathcal{C}^3$ class, bounded as well as its derivatives,
(A3) $\partial^\alpha V_2$ belongs to C^2 for every $|\alpha| = 2$ and V_2 is bounded from below.

Remark 1 An example of potential satisfying assumptions below is $V = \frac{1}{2}|x|^2$ the harmonic potential.

If $V \equiv 0$ then pure Galilee transformation and the definition of the ground state Q yield an explicit solution to Eq.(1)-(2)

$$u^\epsilon = Q(\frac{x - (\xi_0 t + x_0)}{\epsilon})e^{i\frac{t - \frac{|\xi_0|^2}{2}t + x\xi_0}{\epsilon}}.$$

However,$(x,t) \mapsto t - \frac{|\xi_0|^2}{2}t + x\xi_0$ and $t \mapsto (\xi_0 t + x_0)$ are respectively the solution of Hamilton-Jacobi equation (7) and the classical Hamiltonian system

$$\dot{X}(t) = \boldsymbol{\xi}(t), \quad \dot{\boldsymbol{\xi}}(t) = -\nabla V(\mathbf{x}(t)), \quad (\mathbf{x}, \boldsymbol{\xi})_{|t=0} = (x_0, \xi_0). \tag{3}$$

Observe that, in view of the properties of V, the system (3) is globally solvable. Furthermore, the classical Hamiltonian

$$H(t) = \frac{|\boldsymbol{\xi}(t)|^2}{2} + V(\mathbf{x}(t)) \tag{4}$$

is conserved along the evolution in time. Keeping this in mind, we seek a solution of Eq.(1) in the form

$$u^\epsilon(t,x) = u(t,x,\epsilon)e^{i\frac{\varphi(t,x)}{\epsilon}}, \tag{5}$$

with

$$u(t,x,\epsilon) = \sum_{j=0}^{\infty} \epsilon^j U^j(\frac{t}{\epsilon}, \frac{x - \mathbf{x}(t)}{\epsilon}). \tag{6}$$

[1]Some remarks on the critical case are given in section 4.

Substitution of (5) and (6) into Eq.(1) implies that the phase φ is the unique solution of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t\varphi(t,x) + \frac{1}{2}|\nabla\varphi(t,x)|^2 + V(x) - 1 = 0, \\ \varphi(0,x) = x\xi_0, \end{cases} \tag{7}$$

to which we refer as the eikonal equation. Also we obtain that $U^0 = Q$ and $U^1(t,x) = <D^2\varphi(t,\mathbf{x}(t))x, x>Q(x)$, where $D^2\varphi$ denotes the Hessian matrix of φ.

By Taylor expansion, one obtains

$$\varphi(t,x) = \varphi(t,\mathbf{x}(t)) + \nabla\varphi(t,\mathbf{x}(t))(x-\mathbf{x}(t)) + \mathcal{O}(|x-\mathbf{x}(t)|^2).$$

It is not hard to check that $\nabla_x\varphi(t,\mathbf{x}(t)) = \boldsymbol{\xi}(t)$ and that

$$Q(\frac{x-\mathbf{x}(t)}{\epsilon})e^{i\frac{\varphi(t,x)}{\epsilon}} \overset{H^1_\epsilon}{\simeq} Q(\frac{x-\mathbf{x}(t)}{\epsilon})e^{i\frac{\theta(t)+x\boldsymbol{\xi}(t)}{\epsilon}} \qquad \text{as} \quad \epsilon \downarrow 0,$$

where $\theta(t) := \varphi(t,\mathbf{x}(t)) - \boldsymbol{\xi}(t).\mathbf{x}(t)$ and H^1_ϵ stands for the H^1 space equipped with the rescaled norm:

$$\|f\|^2_{H^1_\epsilon} := \frac{1}{\epsilon^N}\|f\|^2_{L^2} + \frac{1}{\epsilon^{N-2}}\|\nabla f\|^2_{L^2}.$$

An easy computation yields

$$\theta(t) = t(1-H(0)) + \int_0^t \nabla V(\mathbf{x}(s)).\mathbf{x}(s)ds$$

a quantity which is defined for all $t \in \mathbb{R}$.

We expect that, up an error term of size ϵ, $u^\epsilon(t)$ is equal to $e^{i\frac{\cdot\boldsymbol{\xi}(t)+\theta(t)}{\epsilon}}Q(\frac{\cdot-\mathbf{x}(t)}{\epsilon})$. In the main theorem of this paper we give a partial justification of this predicted behavior. More precisely, we prove the following

Theorem 1 *Assume* (A1)-(A3) *and* $\sigma < \frac{2}{N}$. *Let* (u^ϵ) *be the family of solutions to (1)-(2), then*

$$u^\epsilon(t,x) = e^{i\frac{x\boldsymbol{\xi}(t)+\epsilon\theta^\epsilon(t)}{\epsilon}}Q(\frac{x-\mathbf{x}(t)}{\epsilon}) + \mathcal{O}(\epsilon), \quad \text{in} \quad H^1_\epsilon \quad \text{as} \quad \epsilon \downarrow 0,$$

locally uniformly in $t \in \mathbb{R}$, *where* $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ *is the solution of the classical Hamiltonian system* (3) *and* θ^ϵ *is a t-dependent shift term.*

Some remarks are in order.

Remark 2 The novelty of this result is that the concentration center is showed to be exactly the one predicted by the WKB method. The x-dependant part of the phase function $\frac{x\boldsymbol{\xi}(t)}{\epsilon}$ is also obtained. Also, the rate of convergence is the optimal one given by the WKB formal calculus, since $U^1 \neq 0$.

Remark 3 We could deal with an initial data in the form $Q\left(\frac{x-x_0}{\epsilon}\right) e^{i\frac{\varphi(x)}{\epsilon}}$. This adds no difficulty since it can be approximated by $Q\left(\frac{x-x_0}{\epsilon}\right) e^{i\frac{x.\xi_0}{\epsilon}} e^{i\frac{\varphi(X_0)}{\epsilon}}$ where $\xi_0 = \nabla\varphi(x_0)$.

Remark 4 An interesting discussion on the semiclassical nonlinear Schrödingfer equations with potential and focusing initial data can be found in [4].

In the context of semi classical analysis, some positive measure in the phase space was developed independently by P. Gérard [6] and P-L. Lions & T. Paul [12]. The pertinence of this measure, which is called *Wigner measure*, lies on the informations which gives on both spatial and frequency behaviors of the bounded sequences of $\mathbf{L}^2$ and on the fact that the t-dependent measure associated to the solutions of an evolution equation

$$\epsilon D_t u^\epsilon + p(x, \epsilon D)u^\epsilon = 0, \qquad u^\epsilon(0, x) = u_I^\epsilon(x),$$

where $p(x, \xi)$ a smooth real-valued function, is obtained by solving the transport equation

$$\partial_t \mu = H_p \mu, \qquad \mu_{|t=0} = \mu_I,$$

where H_p is the vector field associated to p and μ_I is the Wigner measure associated to the family of initial data u_I^ϵ.
The Wigner measure of a bounded family (ψ^ϵ) in $\mathbf{L}^2$ is the weak limit, up to subsequence, of its Wigner transform

$$W^\epsilon(\psi^\epsilon)(x, \xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \psi^\epsilon(x - \frac{\epsilon v}{2})\bar{\psi}^\epsilon(x + \frac{\epsilon v}{2}) e^{i\xi v} dv.$$

This limit is a positive Radon measure ν on $\mathbb{R}^N \times \mathbb{R}^N$ satisfying

$$\|\nu\|_{\mathcal{M}} \leq \limsup_{\epsilon \to 0} \|\psi^\epsilon\|^2_{\mathbf{L}^2(\mathbb{R}^N_x)},$$

where $\|\cdot\|_{\mathcal{M}}$ *denotes* the norm in the space of bounded Radon measures. For instance, it is not hard to check that

$$\nu\left(\frac{1}{\epsilon^{N/2}}F(\frac{\cdot - x_0}{\epsilon})e^{i\frac{\cdot\xi_0}{\epsilon}}\right) = \delta_{(x-x_0)} \otimes |\hat{F}(\xi - \xi_0)|^2 d\xi/(2\pi)^N, \quad \forall F \in L^2(\mathbb{R}^N).$$

For more details and precise statements about Wigner measures the reader is referred to [6], [7], [8], [12] and the references quoted therein.
Theorem 1 allows us to describe the dynamics of the t-dependent Wigner measure associated to the family $\left(\frac{1}{\epsilon^{N/2}}u^\epsilon(t)\right)$.

Theorem 2 *Under the same notations used in Theorem 1, we have*

$$W^\epsilon\left(\frac{1}{\epsilon^{N/2}}u^\epsilon(t)\right) \rightharpoonup \delta_{(x-\mathbf{x}(t))} \otimes |\hat{Q}(\xi - \boldsymbol{\xi}(t))|^2 \frac{d\xi}{(2\pi)^N}, \qquad \text{as} \quad \epsilon \downarrow 0,$$

locally uniformly in $t \in \mathbb{R}$.

This theorem follows from Theorem 1 via straightforward calculus. The main point is that the unknown shift term of Theorem 1 disappears and the dynamics of t-dependent Wigner measure associated to the family $\left(\frac{1}{\epsilon^{N/2}}u^\epsilon(t)\right)$ is rigorously described.

The rest of this paper is structured as follows. In section 2 we present some results about Eq.(1) and the ground state Q needed for the proofs of our results which are given in section 3. Section 4 is devoted to the harmonic potential.

2 Preliminaries

In this preliminary section, we are going to recall some definitions and basic properties of the objects that will be used in our analysis.

2.1 Properties of Eq. (1)

It is well-known (see for example T. Cazenave [3]) that Eq.(1) is locally well posed in $\mathbf{H}^1$. Furthermore, the solutions of Eq.(1) have the following conservation laws as t varies

$$\begin{aligned}\mathcal{N}^\epsilon(t) &= \frac{1}{\epsilon^N}\int |u^\epsilon|^2 dx,\\ E^\epsilon(t) &= \frac{1}{2\epsilon^{N-2}}\int |\nabla u^\epsilon|^2 dx - \frac{1}{\epsilon^N(\sigma+1)}\int |u^\epsilon|^{2\sigma+2}dx + \frac{1}{\epsilon^N}\int V(x)|u^\epsilon|^2 dx.\end{aligned}$$

Notice that , in view of the assumptions of V, we have

$$E^\epsilon \leq C, \tag{8}$$

where C is a constant depending only on N, Q, (x_0, ξ_0) and V, but not on ϵ. If the sub-critical case ($\sigma < \frac{2}{N}$) the a priori bound (8) leads the the global well-posedness of the Schrödinger equation (1). The heart of the globalization is the existence of an *a priori* bound of the $\mathbf{H}^1$ norm of the solution u^ϵ of Eq.(1). More precisely, it has been proved (see e.g [3]) that the length of the interval of existence can be taken to depend only on the $\mathbf{H}^1$ norm of the solution. Thus, if one has an *a priori* estimate in the following type

$$\|\nabla u^\epsilon\|_{\mathbf{L}^2} \leq C_\epsilon \tag{9}$$

the global existence follows. An estimate in the type (9) can be derived as follows. From the Galiardo-Nirenberg inequalities, it ensures that

$$\|u^\epsilon\|_{\mathbf{L}^{2\sigma+2}} \leq C\|u^\epsilon\|_{\mathbf{L}^2}^{1-\theta}\|\nabla u^\epsilon\|_{\mathbf{L}^2}^{\theta}, \quad \text{where} \quad \theta = \frac{N\sigma}{2\sigma+2}. \tag{10}$$

By using the conservation laws below (10), one obtains

$$\frac{1}{\epsilon^{N-2}}\|\nabla u^\epsilon\|_{\mathbf{L}^2}^2 \leq C\left(1 + (\frac{1}{\epsilon^{N-2}}\|\nabla u^\epsilon\|_{\mathbf{L}^2})^{N\sigma}\right). \tag{11}$$

When $N\sigma < 2$ the $\mathbf{L}^2$ norm of the gradient of the solution

$$\frac{1}{\epsilon^{N-2}}\|\nabla u^\epsilon\|_{\mathbf{L}^2}^2 \leq C \tag{12}$$

for all $t \in \mathbb{R}$.

2.2 Ground states of NLS

The nonlinear focusing Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + |u|^{2\sigma}u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, \\ u(0,x) = \varphi \in \mathbf{H}^1 \end{cases} \tag{13}$$

has a family of localized, finite energy solutions which result from a competition between the dispersion and the focusing nonlinearity. Such solutions can be found in the form

$$u(t,x) = e^{it}Q(x). \tag{14}$$

Substitution of (14) into (13) yields

$$\frac{1}{2}\Delta R - R + |R|^{2\sigma}R = 0. \tag{15}$$

Equation (15) has an infinite number of $\mathbf{H}^1$ solutions (see [S]). Among them is a real, positive and radial solution Q which is called *ground state*. In [**?**] it has been proved that such solution is unique : the elliptic problem (15) has a unique real, positive and radial solution.
The ground state has the following properties
i) Q is positive and radially symmetric,
ii)$Q \in \mathbf{H}^1 \cap \mathcal{C}^\infty(\mathbb{R}^N)$,
iii) there exists a constant α, such that $Qe^{\alpha|x|} \in \mathbf{L}^\infty$.
iv) Q is the unique solution , up to translation and rotation, of the minimization problem

$$\begin{cases} v \in \mathcal{F} = \{v \in \mathbf{H}^1, \|v\|_{\mathbf{L}^2} = M\}, \qquad M = \|Q\|_{\mathbf{L}^2}; \\ E(u) = I_M = \inf\{E(v), v \in \mathcal{F}\}, \end{cases} \tag{16}$$

where

$$E(v) := \frac{1}{2}\int |\nabla v|^2 dx - \frac{1}{\sigma+1}\int |v|^{2\sigma+2}dx. \tag{17}$$

The nonlinear stability theory of the ground state yields the following

Proposition 1 *For every $\alpha_0 > 0$ there exists a constant $h(\alpha_0)$, such that*

$$\inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0, 2\pi[}} \|\phi - e^{i\theta}Q(\cdot - y)\|_{\mathbf{H}^1} \le \alpha_0 \tag{18}$$

for all $\phi \in \mathbf{H}^1$, such that $\|\phi\|^2_{\mathbf{L}^2} = M$ and $E(\phi) - E(Q) < h(\alpha_0)$.

Proof. For the convenience of the reader we outline the proof of Proposition 1.
We proceed by contradiction. If the statement of Proposition 1 does not hold then there exist a sequence (ϕ_n) in $\mathbf{H}^1$ and $\alpha_0 > 0$, such that

$$\begin{cases} \|\phi_n\|_{\mathbf{L}^2} = \|Q\|_{\mathbf{L}^2}, \quad \text{for every } n, \\ E(\phi_n) \underset{n\to\infty}{\longrightarrow} E(Q), \\ \inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0, 2\pi[}} \|\phi_n - e^{i\theta} Q(\cdot - y)\|_{\mathbf{H}^1} > \alpha_0. \end{cases} \tag{19}$$

Therefore, (ϕ_n) is a minimizing sequence of the problem (16). The contradiction will follow from the following

Lemma 1 *The minimization problem* (16) *has a solution u. In addition, for every minimizing sequence (u_n), there exist a subsequence (u_{n_k}) and a family $(y_k) \subset \mathbb{R}^N$, such that $(u_{n_k}(\cdot - y_k))$ has a strong limit u in $\mathbf{H}^1$.*

According to Lemma 1, there exist $(y_n) \subset \mathbb{R}^N$ and a solution u of (16), such that $\|\phi_n - u(\cdot - y_n)\|_{\mathbf{H}^1} \to 0$. The uniqueness of the solution to (16), up to translation and rotation, yields that $u = e^{i\theta_0} Q(\cdot - y_0)$. Hence,$\|\phi_n - e^{i\theta_0} Q(\cdot - y_n - y_0)\|_{\mathbf{H}^1} \to 0$, which contradicts (19).
Proof of Lemma 1. The proof is divided into three steps.
Step 1. Firstly, a classical argument of homogeneity shows that

$$_M < 0.$$

Secondly, the use of Galiardo-Nirenberg inequalities as in (10) implies the existence of $\delta > 0$ and $K < \infty$, such that

$$E(u) \geq \delta \|u\|_{\mathbf{H}^1} - K, \quad \text{for all } u \in \mathcal{F}. \tag{20}$$

Step 2. A direct consequence of Step 1 is that every minimizing sequence of the problem (16) is bounded in $\mathbf{H}^1$ and bounded from below in $\mathbf{L}^{2\sigma+2}$.
Step 3. Let (u_n) be a minimizing sequence of the problem (16). By step 2, (u_n) is bounded in $\mathbf{H}^1$ and bounded from below in $\mathbf{L}^{2\sigma+2}$. At this stage we need the following two results.
• Sobolev's inequality.

$$\int |u|^{2\sigma+2} dx \leq C(\sup_{y \in \mathbb{R}^N} (\int_{\{|x-y|\leq 1\}} |u|^2 dx)^\sigma \|u\|^2_{\mathbf{H}^1}, \qquad \sigma < \frac{2}{N} \tag{21}$$

for all $u \in \mathbf{H}^1$.

• Concentration compactness Lemma (cf. P-L. Lions [11]). If (u_n) is a bounded sequence in $\mathbf{H}^1$, such that

$$\int |u_n|^2 dx = M > 0,$$

then, up to a subsequence, one of the following properties holds.

(i) There exists a sequence $(y_n) \subset \mathbb{R}^N$, such that for every $\varepsilon > 0$, there exists $A < \infty$, such that $\lim_{n\to\infty} \int_{\{|x-y_n|\leq A\}} |u_n|^2 dx \geq M - \epsilon$.

(ii)

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\{|x-y|\leq 1\}} |u_n|^2 dx = 0.$$

(iii) There exists $\gamma \in]0, M[$, such that for every $\epsilon > 0$, there exists two sequences $(v_n), (w_n) \subset \mathbf{H}^1$, with compact disjoint supports, such that

$$\|v_n\|_{\mathbf{H}^1} + \|w_n\|_{\mathbf{H}^1} \leq 4 \sup_n \|u_n\|_{\mathbf{H}^1}; \tag{22}$$

$$\|u_n - v_n - w_n\|_{\mathbf{L}^2} \leq \epsilon; \tag{23}$$

$$|\textstyle\int |v_n|^2 - \gamma| \leq \epsilon; \tag{24}$$

$$|\textstyle\int |w_n|^2 + \gamma - M| \leq \epsilon; \tag{25}$$

$$\textstyle\int |\nabla u_n|^2 - |\nabla v_n|^2 - |\nabla w_n|^2 \geq -\epsilon. \tag{26}$$

Let us continue the proof of Lemma 1. Applying (21) to (u_n) it follows that $\sup_{y\in\mathbb{R}^N}(\int_{\{|x-y|\leq 1\}} |u_n|^2 dx$ is bounded from below. Hence, (ii) of Concentration compactness lemma cannot occurs. Furthermore, (iii) does not hold. Otherwise, it follows easily from (22), (23), (26) and the disjointness of the supports of (v_n) and (w_n) that

$$E(u_n) - E(v_n) - E(w_n) \geq -\delta(\epsilon) \longrightarrow 0, \quad \text{as}\, \epsilon \downarrow 0. \tag{27}$$

Let $a_n = \sqrt{M}/\|v_n\|_{\mathbf{L}^2}$. Since $a_n v_n \in \mathcal{F}$, it follows that

$$E(v_n) \geq \frac{I_M}{a_n^2} + \frac{a_m^{2\sigma} - 1}{2\sigma + 2} \int |v_n|^{2\sigma+2}.$$

Since $a_n - 1 > c$ for some c which is independent of n, then

$$E(v_n) \geq \frac{I_M}{a_n^2} + c \int |v_n|^{2\sigma+2}. \tag{28}$$

In the same manner we get

$$E(w_n) \geq \frac{I_M}{b_n^2} + c \int |w_n|^{2\sigma+2}, \tag{29}$$

where $b_n = \sqrt{M}/\|w_n\|_{\mathbf{L}^2}$. Putting (27), (28) and (29) together, it follows that

$$E(u_n) \geq \frac{I_M}{M} \int |w_n + v_n|^2 + c \int |v_n + w_n|^{2\sigma+2} - \delta(\epsilon). \tag{30}$$

In the last line we have used the fact that the supports of (v_n) and (w_n) are disjoint. Applying (22), (23) and Hölder's inequality, one obtains

$$E(u_n) \geq \frac{I_M(M + \delta(\epsilon))}{M} + c(1 - \delta(\epsilon)) \int |u_n|^{2\sigma+2} - \delta(\epsilon). \tag{31}$$

Letting $\epsilon \downarrow 0$, one gets

$$E(u_n) \geq I_M + c \int |u_n|^{2\sigma+2}. \tag{32}$$

Since $E(u_n) \to I_M$ as $n \to \infty$, (32) implies that $\int |u_n|^{2\sigma+2} \to 0$ as $n \to \infty$. This contradicts the fact that (u_n) is bounded from below in $\mathbf{L}^{2\sigma+2}(\mathbb{R}^N)$. Therefore, (i) occurs. Set $\tilde{u}_n = u_n(\cdot - y_n)$. Since the sequence $\tilde{u}_n$ is bounded in $\mathbf{H}^1$, then there exists $u \in \mathbf{H}^1$, such that

$$\tilde{u}_n \rightharpoonup u \quad \text{in} \quad \mathbf{H}^1, \quad \text{as}\, n \to \infty.$$

The compactness of the embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^2(\{|x| \leq R\})$ and (i) imply that

$$\int |u|^2 \geq M - \epsilon,$$

for every $\epsilon > 0$, so that

$$\int |u|^2 = M.$$

Thus $\tilde{u}_n \to u$ in $\mathbf{L}^2$, and in particular $u \in \mathcal{F}$. Hölder's inequality, the weak lower semicontinuity of the $\mathbf{H}^1$ norm and the definition on I_M imply that $E(u) = I_M$. Hence, $\nabla \tilde{u}_n \to \nabla u$ in $\mathbf{L}^2$, which means that $w_n \to u$ strongly in in $\mathbf{H}^1$. This closes the proof of Lemma 1. □

In fact we have more than Proposition 1.

Proposition 2 *There exist constants C, h, such that*

$$\inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0,2\pi[}} \|\phi - e^{i\theta} Q(\cdot - y)\|^2_{\mathbf{H}^1} \leq C\,(E(\phi) - E(Q)) \tag{33}$$

for all $\phi \in \mathbf{H}^1$, such that $\|\phi\|^2_{\mathbf{L}^2} = M$ and $E(\phi) - E(Q) < h$.

Proof of Proposition 2. For the convenience of the reader we sketch from [16] the proof of Proposition 2. M. Weinstein have introduced the Lyapunov method in the study of the stability theory of ground states. The lyapunov functional constructed in our situation is given by

$$\mathcal{E}(\phi) = \frac{1}{2}\int |\nabla\phi|^2 dx - \frac{1}{\sigma+1}\int |\phi|^{2\sigma+2} dx + \int |\phi|^2 dx. \tag{34}$$

Let $\phi \in \mathbf{H}^1$ and $(y,\theta) \in \mathbb{R}^N \times [0, 2\pi[$. One writes

$$\phi(x+y)e^{i\theta} = Q(x) + w \qquad \text{and} \quad w = u + iv. \tag{35}$$

One has

$$\begin{aligned} \Delta\mathcal{E} &\equiv \mathcal{E}(\phi) - \mathcal{E}(Q) && (36)\\ &= \mathcal{E}(\phi(\cdot + y)e^{i\theta}) - \mathcal{E}(Q) && \text{by scale invariance,}\\ &= \mathcal{E}(R + w) - \mathcal{E}(Q) && \text{by (35).} \end{aligned}$$

By Taylor expansion, one obtains

$$\mathcal{E}(Q+w) - \mathcal{E}(Q) \;=\; w\frac{d\mathcal{E}}{d\phi}(Q) + \frac{w^2}{2}\frac{d^2\mathcal{E}}{d^2\phi}(Q) + \mathcal{O}(|w|^3). \tag{37}$$

However,

$$\frac{d\mathcal{E}}{d\phi}(Q) = \frac{\Delta}{2}Q - Q + Q^{2\sigma+1} = 0 \qquad \text{by (15).} \tag{38}$$

Then,

$$\mathcal{E}(Q+w) - \mathcal{E}(Q) \;=\; \frac{w^2}{2}\frac{d^2\mathcal{E}}{d\phi^2}(Q) + \mathcal{O}(|w|^3). \tag{39}$$

On one hand, by a direct computation we obtain

$$\frac{w^2}{2}\frac{d^2\mathcal{E}}{d^2\phi}(Q) = (L_+u, u) + (L_-v, v), \tag{40}$$

where

$$L_+ = -\Delta + 1 - (2\sigma + 1)Q^{2\sigma} \quad \text{as} \quad L_- = -\Delta + 1 - Q^{2\sigma} \tag{41}$$

are, respectively, the real and imaginary part of the linearized NLS operator about the ground state Q. On the other hand, the remaining $\mathcal{O}(|w|^3)$ terms can be estimated from below by an interpolation estimate of Galiardo-Nirenberg as

$$\mathcal{O}(|w|^3) \geq -C_1\|w\|_{H^1}^{2+\alpha} - C_2\|w\|_{H^1}^6 \quad \text{with} \quad \alpha > 0. \tag{42}$$

Thus, we infer

$$\mathcal{E}(Q + w) - \mathcal{E}(Q) \ \geq \ (L_+u, u) + (L_-v, v) - C_1\|w\|_{H^1}^{2+\alpha} - C_2\|w\|_{H^1}^6, \tag{43}$$

with $\alpha > 0$. The crucial step is the following

Lemma 2 (cf. [16], (2.8), p 56) *If* y_0 *and* θ_0 *minimize*[2] $\|\phi(\cdot + y)e^{i\theta} - R\|_{H^1}$ *then*

$$(L_+u, u) + (L_-v, v) \geq C_3\|w\|_{\mathbf{H}^1}^2 - C_4\|w\|_{H^1}^3 - C_5\|w\|_{H^1}^4, \tag{44}$$

where $u + iv = w = \phi(x + y_0)e^{i\theta_0} - Q(x)$.

Putting together (43) and (44), it follows that

$$\Delta\mathcal{E} = \mathcal{E}(\phi) - \mathcal{E}(Q) \geq G(\inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0, 2\pi[}} \|\phi - e^{i\theta}Q(\cdot - y)\|_{H^1}^2), \tag{45}$$

where

$$G(t) = ct^2(1 - at^\alpha - bt^4) \quad \text{with} \quad a, b, c, \alpha > 0. \tag{46}$$

Proposition 2 can be derived as follows. Let $\delta_0 > 0$, such that $G(t) \geq \frac{c}{2}t$, for every $t \in [0, \delta_0[$. According to Proposition 1 if $\Delta\mathcal{E} < h(\delta_0)$ then $\inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0, 2\pi[}} \|\phi - e^{i\theta}Q(\cdot - y)\|_{H^1}^2 \leq \delta_0$. Thus, (45) reads

$$\mathcal{E}(\phi) - \mathcal{E}(Q) = E(\phi) - E(Q) \geq \frac{c}{2} \inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0, 2\pi[}} \|\phi - e^{i\theta}Q(\cdot - y)\|_{H^1}^2. \tag{47}$$

In the last line we have used the fact that $\|\phi\|_{L^2}^2 = \|Q\|_{L^2}^2$ to pass from $\mathcal{E}$ to E. Thus we may take $C = \frac{c}{2}$ and $h = h(\delta_0)$.

[2] The infimum is attained (see [Bo]).

The proof of Lemma 2 is contained in [16] in spatial dimensions $N = 1, 3$. The paper of M.K. Kwong [**?**] allows the extension of these results to all spatial dimensions.

3 Proof of Theorem 1

3.1 Preparation of the Proof

Our test functions will be taken in the $C^2(\mathbb{R}^N)$ Banach space equipped with following norm:

$$\|\phi\|_{C^2} = \sum_{|\alpha|\le 2} \|\partial_x^\alpha \phi\|_\infty.$$

We let C^{2*} denote the dual space of C^2, equipped with the dual norm

$$\|\mu\|_{C^{2*}} := \sup\{\int \phi(x)\mu(dx) : \phi \in C^2(\mathbb{R}^N), \|\phi\|_{C^2} \le 1\}.$$

It is clear that C^{2*} contains the space of bounded Radon measures. One can check the following result.

Lemma 3 *For every $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$, we have*

$$\|\delta_\xi - \delta_\eta\|_{C^{2*}} \simeq \frac{2|\xi - \eta|}{2 + |\xi - \eta|}. \tag{48}$$

In our proofs later we shall use the following trivial consequence of (48).

Lemma 4 *There exits two constants $C > 0$ and $K_0 > 0$ such that if $\|\delta_\xi - \delta_\eta\|_{C^{2*}} \le K$ then*

$$|\xi - \eta| \le C\frac{2K}{2 - K}, \tag{49}$$

for every $K < K_0$.

Proof of Lemma 3. Set $\alpha := |\eta - \xi|$. If $\alpha = 0$ the result is trivial. Let us prove (48) for $\alpha \neq 0$.
On one hand, for every $\theta \in [0, 1]$ and every $f \in C^2$, one has

$$\begin{aligned} f(\xi) - f(\eta) &= \theta\,(f(\xi) - f(\eta)) + (1 - \theta)\,(f(\xi) - f(\eta)) \\ &\le 2\theta\|f\|_{L^\infty} + (1 - \theta)\alpha\|Df\|_{L^\infty}. \end{aligned}$$

If we take $\theta = \frac{\alpha}{2+\alpha}$ we get that $\|\delta_\xi - \delta_\eta\|_{C^{2*}} \leq \frac{2\alpha}{2+\alpha}$.
On the other hand, let f_α and g_α be the family of C^2 functions given by

$$f_\alpha(x) = \alpha^2 \left(1 - \sin(\frac{\pi}{2}\frac{(\xi - x)(\xi - \eta)}{\alpha^2})\right)$$

and

$$g_\alpha(x) = 1 - \sin(\frac{(\xi - x)(\xi - \eta)}{\alpha}),$$

where α is as above. By a straightforward computation we obtain that there exits some constant $C > 0$ such that

$$\frac{\alpha^2}{2\alpha^2 + \alpha + C} \leq \frac{|f_\alpha(\xi) - f_\alpha(\eta)|}{\|f_\alpha\|_{C^2}}.$$

and,

$$\frac{\sin(\alpha)}{C} \leq \frac{|g_\alpha(\xi) - g_\alpha(\eta)|}{\|g_\alpha\|_{C^2}}.$$

Thus, we get

$$\frac{\alpha^2}{2\alpha^2 + \alpha + C} \leq \|\delta_\xi - \delta_\eta\|_{C^{2*}}$$
$$\frac{\sin(\alpha)}{C} \leq \|\delta_{(x-\xi)} - \delta_{(x-\eta)}\|_{C^{2*}}.$$

An interpolation between these two estimates completes the proof of Lemma 3. □

The total energy can be rewritten as

$$E^\epsilon(t) = \underbrace{\frac{1}{2\epsilon^{N-2}}\int |\nabla|u^\epsilon||^2 dx - \frac{1}{\sigma+1}\int \frac{1}{\epsilon^N}|u^\epsilon|^{2\sigma+2}dx}_{E_b^\epsilon(u^\epsilon):\text{ binding energy}} + \\ + \underbrace{\frac{1}{2}\int \frac{|\xi^\epsilon|^2}{m^\epsilon}dx}_{\mathrm{E_k^\epsilon(u^\epsilon): kinetic\ energy}} + \underbrace{\int V(x)m^\epsilon(t,x)dx}_{\mathrm{E_p^\epsilon(u^\epsilon): potential\ energy}}, \quad (50)$$

where

$$m^\epsilon(t) := \frac{1}{\epsilon^N}|u^\epsilon(t,x)|^2, \quad \xi^\epsilon := \frac{i}{2\epsilon^{N-1}}(u^\epsilon\nabla\bar{u}^\epsilon - \bar{u}^\epsilon\nabla u^\epsilon), \quad (51)$$

are the position and momentum densities. Also, we set

$$\boldsymbol{\xi}^\epsilon(t) = \frac{1}{M}\int \xi^\epsilon(t,x)dx. \tag{52}$$

For our future convenience we state the following identities:

$$\frac{dm^\epsilon}{dt}(t) = -\mathrm{div}\xi^\epsilon(t,x) \tag{53}$$

and

$$\int \frac{d\xi^\epsilon}{dt}(t,x)dx = -\int \nabla V(x) m^\epsilon dx. \tag{54}$$

The following lemma will be useful.

Lemma 5 *Assume V satisfying* (A1)-(A3). *Then, there exists $C >$ such that*

$$|\int V(\epsilon x + y)|Q(x)|^2 dx - MV(y)| \leq C\epsilon^2$$

for every $y \in \mathbb{R}^N$.

Proof. Taylor expansion and the fact that $\partial^\alpha V \in L^\infty$, for every $|\alpha| = 2$, yield

$$|V(\epsilon x + y) - V(y) - \epsilon x \nabla V(y)| \leq C\epsilon^2 |x|^2$$

uniformly in $y \in \mathbb{R}$. The result follows from the fact that Q is radial (this cancels the term $\int x\nabla V(y)|Q(x)|^2 dx$) and the integrability[3] of $|x|^2 Q$.
Let us finally give the following adapted version of Proposition 2.

Proposition 3 *There exist constants C, h, such that*

$$\inf_{y \,\in\, \mathbb{R}^N} \|\phi - Q^\epsilon(\cdot - y)\|^2_{H^1_\epsilon} \leq C(E^\epsilon_b(\phi) - E^\epsilon_b(Q^\epsilon)),$$

for every nonnegative function $\phi \in H^1$, such that $\frac{1}{\epsilon^N}\|\phi\|^2_{L^2} = M$ and $E^\epsilon_b(\phi) - E^\epsilon_b(Q^\epsilon) < h$. Here, we have used the notation $Q^\epsilon := Q(\frac{\cdot}{\epsilon})$.

[3]Recall that there exists a constant α, such that $Qe^{\alpha|x|} \in \mathbf{L}^\infty$.

3.2 Proof of Theorem 1

Let us remark firstly that without loss of generality we may assume that $V(x) \geq 0$. In fact if u^ϵ is a solution to (1) with a potential V then $e^{-i\frac{Lt}{\epsilon}} u^\epsilon$ is a solution to the same equation with a potential $V(x) + L$. Since V is bounded from below we choose L such that $V(x) + L \geq 0$.
We set

$$v^\epsilon(t, x) = e^{-i\frac{(\epsilon x + \mathbf{x}(t))\boldsymbol{\xi}(t)}{\epsilon}} u^\epsilon(\epsilon x + \mathbf{x}(t)). \tag{55}$$

It is clear that

$$\|v^\epsilon(t, \cdot)\|_{L^2}^2 = \frac{1}{\epsilon^N} \|u^\epsilon(t, \cdot)\|_{L^2}^2 = M,$$

for every $t \in \mathbb{R}$. The idea is simple. It consists in applying Proposition 2 to the family $v^\epsilon(t, \cdot)$. By a direct computation and under the notations (51) and (17) we get

$$\begin{aligned}
\mathcal{E}(v^\epsilon(t)) &= \frac{1}{2} \int |\nabla v^\epsilon|^2 dx - \frac{1}{\sigma + 1} \int |v^\epsilon|^{2\sigma+2} dx \\
&= \frac{M|\boldsymbol{\xi}(t)|^2}{2} + \frac{1}{2\epsilon^{N-2}} \int |\nabla u^\epsilon|^2 dx - \boldsymbol{\xi}(t) \int \xi^\epsilon dx \\
&\quad - \frac{1}{(\sigma + 1)\epsilon^N} \int |u^\epsilon|^{2\sigma+2} dx
\end{aligned}$$

which, in term of the total energy, gives

$$\mathcal{E}(v^\epsilon(t)) = E^\epsilon(u^\epsilon(t)) + \frac{M|\boldsymbol{\xi}(t)|^2}{2} - \boldsymbol{\xi}(t) \int \xi^\epsilon dx - \int V(x) m^\epsilon dx. \tag{56}$$

However, the conservation law of the total energy yields

$$\begin{aligned}
E^\epsilon(u^\epsilon(t)) &= E^\epsilon(u^\epsilon(0)) \\
&= E^\epsilon(Q(\frac{\cdot - x_0}{\epsilon}) e^{i\frac{\cdot \xi_0}{\epsilon}}) \\
&= \frac{M|\xi_0|^2}{2} + \mathcal{E}(Q) + \int V(\epsilon x + x_0) |Q|^2 dx.
\end{aligned} \tag{57}$$

Putting together (56) and (57), it follows that

$$\begin{aligned}
\mathcal{E}(v^\epsilon(t)) - \mathcal{E}(Q) = M\frac{|\xi_0|^2}{2} + \int V(\epsilon x + x_0)|Q|^2 dx + M\frac{|\boldsymbol{\xi}(t)|^2}{2} - \\
- \boldsymbol{\xi}(t) \int \xi^\epsilon dx - \int V(x) m^\epsilon dx.
\end{aligned}$$

However, by Lemma 5, it holds that

$$\int V(\epsilon x + x_0)|Q|^2 dx = MV(x_0) + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \downarrow 0.$$

Thus, and under notation (4), we infer

$$\mathcal{E}(v^\epsilon(t)) - \mathcal{E}(Q) = MH(0) + M\frac{|\boldsymbol{\xi}(t)|^2}{2} - \boldsymbol{\xi}(t)\int \xi^\epsilon dx - \int V(x)m^\epsilon dx + \mathcal{O}(\epsilon^2),$$

as $\epsilon \downarrow 0$ uniformly in $t \in \mathbb{R}$.
The proofs of our results will follow from the following

Proposition 4 *Under the notations* (51) *and* (3)*, we have*

$$\|m^\epsilon dx - M\delta_{\mathbf{x}(t)}\|_{C^{2*}} + \|\xi^\epsilon dx - M\boldsymbol{\xi}(t)\delta_{\mathbf{x}(t)}\|_{C^{2*}} = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0, \quad (58)$$

locally uniformly in $t \in \mathbb{R}$.

Let us postpone the proof of the proposition and finish the proof of Theorem 1.
Let $T > 0$ to be an arbitrary fixed time. Let $\alpha = \sup_{0 \le t \le T} |\mathbf{x}(t)|$. One takes a bump function $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$\zeta(x) = 1 \quad \text{if} \quad |x| < \alpha, \quad \zeta(x) = 0 \quad \text{if} \quad |x| > 2\alpha.$$

Since $V \geq 0$ (recall that V is not necessary in C^2)

$$\int V(x)m^\epsilon dx \geq \int \zeta(x)V(x)m^\epsilon dx.$$

Proposition 4 implies that

$$\begin{aligned} \boldsymbol{\xi}(t)\int \xi^\epsilon dx + \int \zeta(x)V(x)m^\epsilon dx &= M\frac{|\boldsymbol{\xi}(t)|^2}{2} + M(\zeta V)(\mathbf{x}(t)) + \mathcal{O}(\epsilon^2) \\ &= MH(t) + \mathcal{O}(\epsilon^2) \end{aligned}$$

uniformly in $t \in [0, T]$, since $\zeta(\mathbf{x}(t)) = 1$ for every $t \in [0, T]$.
This gives finally,

$$\mathcal{E}(v^\epsilon) - \mathcal{E}(Q) \leq -MH(t) + MH(0) + \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0,$$

locally uniformly in $t \in [0,T]$. Since $H(t) = H(0)$ then

$$\mathcal{E}(v^\epsilon) - \mathcal{E}(Q) = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0,$$

uniformly in $t \in [0,T]$. In view of Proposition 2, we obtain

$$\inf_{\substack{y \in \mathbb{R}^N \\ \theta \in [0,2\pi[}} \|v^\epsilon - e^{i\theta} Q(\cdot + y)\|_{H^1}^2 \le C\left(\mathcal{E}(v^\epsilon(t)) - \mathcal{E}(Q)\right) \le \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0,$$

uniformly in $t \in [0,T]$. Hence, there exist two families of functions y^ϵ and θ^ϵ, such that

$$\|v^\epsilon - e^{i\theta^\epsilon(t)} Q(\cdot + y^\epsilon(t))\|_{H^1}^2 = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0, \tag{59}$$

uniformly in $t \in [0,T]$. In term of u^ϵ, (59) can be rewritten as

$$\|u^\epsilon - \tilde{Q}^\epsilon\|_{H_\epsilon^1}^2 = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0, \tag{60}$$

uniformly in $t \in [0,T]$, where

$$\tilde{Q}^\epsilon(x) := e^{i\frac{x\boldsymbol{\xi}(t)+\epsilon\theta^\epsilon(t)}{\epsilon}} Q\left(\frac{x - \mathbf{x}(t) + \epsilon y^\epsilon(t)}{\epsilon}\right).$$

From (60), we have

$$\|m^\epsilon dx - M\delta_{(\mathbf{x}(t)-\epsilon y^\epsilon)}\|_{C^{2*}} = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0, \tag{61}$$

uniformly in $t \in [0,T]$. Combined with (58), (61) gives

$$\|M\delta_{\mathbf{x}(t)} - M\delta_{(\mathbf{x}(t)-\epsilon y^\epsilon)}\|_{C^{2*}} = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0,$$

uniformly in $t \in [0,T]$. Thus, in view of (49), we infer

$$|\epsilon y^\epsilon| = |\mathbf{x}(t) - (\mathbf{x}(t) - \epsilon y^\epsilon)| \le C\frac{2\mathcal{O}(\epsilon^2)}{2 - \mathcal{O}(\epsilon^2)} = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0,$$

uniformly in $t \in [0,T]$, which means that

$$|y^\epsilon| = \mathcal{O}(\epsilon), \quad \text{as} \quad \epsilon \downarrow 0,$$

uniformly in $t \in [0,T]$. Finally, since

$$\|Q - Q(\cdot - y^\epsilon(t)\|_{H^1}^2 \le |y^\epsilon|^2 \|\nabla Q\|_{H^1}^2,$$

we can take $y^\epsilon = 0$. In the last line we have used the fact that $Q \in H^2$ (in fact $Q \in W^{2,p}(\mathbb{R}^N)$, for every $2 \le p < \infty$.)
This completes the proof of Theorem 1 .

$\square$

3.3 Proof of Proposition 4

The essential part of our proof is taken from [2]. We shall proceed in two steps. The first and main one consists in proving the proposition on some interval $[0, T_0]$. In the second one we shall use an argument of iteration to extend the results of step 1 to $[0, T]$, for every $T > 0$.

Step 1. Let $T_0 > 0$ be a certain positive number which will be explicited later. Let $A = A(T_0)$ be a large number to be chosen later too (among its properties is that $|\mathbf{x}(t)| \leq A$, for every $t \in [0, T_0]$). One takes $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, such that

$$\zeta(x) = 1 \quad \text{if} \quad |x| < A, \quad \zeta(x) = 0 \quad \text{if} \quad |x| > 2A. \tag{62}$$

One defines

$$\begin{aligned} X^\epsilon(t) &:= \frac{1}{M} \int x\zeta(x) m^\epsilon dx \\ \tilde{X}^\epsilon(t) &:= \frac{1}{M} \int \nabla V_2(x) m^\epsilon dx, \end{aligned}$$

and, under notations 3), (51) et (52)

$$\begin{aligned} \eta_1^\epsilon(t) &= |X^\epsilon(t) - \mathbf{x}(t)|, \quad \eta_2^\epsilon(t) = |\tilde{X}^\epsilon(t) - \nabla V_2(\mathbf{x}(t))|, \\ \eta_3^\epsilon(t) &= |\boldsymbol{\xi}^\epsilon(t) - \boldsymbol{\xi}(t)|, \quad \eta_4^\epsilon(t) = |\int V_1(x) m^\epsilon(t) dx - M V_1(\mathbf{x}(t))|, \\ \eta_5^\epsilon(t) &= |\int \zeta(x) V_2(x) m^\epsilon(t) dx - M V_2(\mathbf{x}(t))|. \end{aligned}$$

We let η^ϵ denote the following quantity

$$\eta^\epsilon(t) = \eta_1^\epsilon(t) + \eta_2^\epsilon(t) + \eta_3^\epsilon(t) + \eta_4^\epsilon(t) + \eta_5^\epsilon(t).$$

Observe that, in view of Lemma 5, we can check easily

$$\eta^\epsilon(0) = \mathcal{O}(\epsilon^2). \tag{63}$$

In the sequel we let C denote every constant which depends on the problem (dimension, V, x_0, ξ_0, Q, N,...) and on T_0 , but not on ϵ. Mutatis mutandis for $\mathcal{O}$.
The main ingredient of the proof of Proposition 4 is

Proposition 5 *There exist $C > 0$, $h_0 > 0$ and $\epsilon_0 > 0$, such that if*

$$T_\epsilon^* := \sup\{t \in [0, T^0] : \eta^\epsilon(s) \le h_0 \quad \forall s \in (0, t)\}$$

then

$$\|m^\epsilon(t)dx - M\delta_{\mathbf{x}(t)}\|_{C^{2*}} + \|\xi^\epsilon(t)dx - M\boldsymbol{\xi}(t)\delta_{\mathbf{x}(t)}\|_{C^{2*}} \le C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2) \quad (64)$$

whenever $t \le T_\epsilon^*$ and $0 \le \epsilon < \epsilon_0$.

Let us postpone the proof of Proposition 5 for a while and conclude the proof of Proposition 4.
The conclusion of the proof of Proposition 4 has the following key tool.

Lemma 6 *There exists $C > 0$, such that*

$$\eta^\epsilon(t) \le \mathcal{O}(\epsilon^2) + C\int_0^t \eta^\epsilon(s)ds \quad (65)$$

for all $t \le T_\epsilon^*$.

¿From (65) and the well-known Gronwall inequality, we get

$$\eta^\epsilon(t) \le \mathcal{O}(\epsilon^2)e^{Ct} \le \mathcal{O}(\epsilon^2)e^{CT^0} \le \mathcal{O}(\epsilon^2) \quad (66)$$

for all $t \le T_\epsilon^*$. By the definition of T_ϵ^* and the continuity of η^ϵ it follows that $T_\epsilon^* = T^0$ if ϵ is small enough. Hence, Proposition 5 yields

$$\|m^\epsilon(t)dx - M\delta_{\mathbf{x}(t)}\|_{C^{2*}} + \|\xi^\epsilon(t)dx - MP\delta_{\mathbf{x}(t)}\|_{C^{2*}} \le \mathcal{O}(\epsilon^2)$$

whenever $t \le T^0$ and $0 \le \epsilon < \epsilon_0$. This concludes the proof of Proposition 4 on $[0, T^0]$.
To extend the result to every $T > 0$, we shall use an argument of iteration. This argument shall be developed further in Step 2 of this proof.

Proof of Lemma 6. For every $t \le T_\epsilon^*$, we have

$$\eta^\epsilon(t) \le \eta^\epsilon(0) + \int_0^t |\dot\eta_1^\epsilon| + |\dot\eta_2^\epsilon| + |\dot\eta_3^\epsilon| + |\dot\eta_4^\epsilon| + |\dot\eta_5^\epsilon| ds.$$

Firstly,

$$\begin{aligned}
\dot{\eta}_1^\epsilon &= \frac{1}{M}\int x\zeta(x)\frac{dm^\epsilon}{dt}(t,x)dx - \boldsymbol{\xi}(t) \\
&= -\frac{1}{M}\int x\zeta(x)\mathrm{div}\xi^\epsilon(t,x)dx - \boldsymbol{\xi}(t) \qquad \text{by (53)} \\
&= \frac{1}{M}\int \nabla(x\zeta)\xi^\epsilon(t,x)dx - \boldsymbol{\xi}(t).
\end{aligned}$$

By the definition of ζ, we have

$$\nabla(x\zeta)(\boldsymbol{x}(t)) = (1,1,..,1).$$

Thus, in view of (64) , it ensues that

$$|\dot{\eta}_1^\epsilon| \leq \frac{1}{M^2}\|\nabla(x\zeta)\|_{C^2}\|\xi^\epsilon(t)dx - M\boldsymbol{\xi}(t)\delta_{\boldsymbol{x}(t)}\|_{C^{2*}} \leq C\eta^\epsilon(t).$$

Secondly, by (53), we have

$$\begin{aligned}
|\dot{\eta}_2^\epsilon| &= \left|\int \nabla V_2(x)\mathrm{div}\xi^\epsilon dx + \boldsymbol{\xi}(t)\nabla^2 V_2(\boldsymbol{x}(t))\right| \\
&= \left|\int \nabla^2 V_2(x)\xi^\epsilon dx - \boldsymbol{\xi}(t)\nabla^2 V_2(\boldsymbol{x}(t))\right| \\
&\leq \frac{1}{M^2}\|\nabla^2 V_2\|_{C^2}\|\xi^\epsilon(t)dx - M\boldsymbol{\xi}(t)\delta_{\boldsymbol{x}(t)}\|_{C^{2*}} \\
&\leq C\eta^\epsilon(t).
\end{aligned}$$

In the last line we have used the assumption (A3) (i.e $\nabla^2 V_2 \in C^2$).
Thirdly, in view of (3) and (54), we get

$$\begin{aligned}
|\dot{\eta}_3^\epsilon| &= \left|-\frac{1}{M}\int \nabla V m^\epsilon(t)dx + \nabla V(\mathbf{x}(t))\right| \\
&\leq \left|\frac{1}{M}\int \nabla V_1 m^\epsilon(t)dx - \nabla V_1(\mathbf{x}(t))\right| + |\eta_2^\epsilon| \\
&\leq \|\nabla V_1\|_{C^2}\|m^\epsilon(t)dx - M\delta_{\mathbf{x}(t)}\|_{C^{2*}} + |\eta_2^\epsilon| \\
&\leq C\eta^\epsilon + \mathcal{O}(\epsilon^2).
\end{aligned}$$

In the last line we have used the assumption (A1) and Proposition 5.

Finally,

$$
\begin{aligned}
|\dot{\eta}_4^\epsilon| + |\dot{\eta}_5^\epsilon| &\leq \left| \int V_1(x) \mathrm{div} \xi^\epsilon dx + M \nabla V_1(\mathbf{x}(t)) \boldsymbol{\xi}(t) \right| \\
&+ \left| \int \zeta(x) V_2(x) \mathrm{div} \xi^\epsilon dx + M \nabla V_2(\mathbf{x}(t)) \boldsymbol{\xi}(t) \right| \\
&\leq \left| \int \nabla V_1(x) \xi^\epsilon dx - M \nabla V_1(\mathbf{x}(t)) \boldsymbol{\xi}(t) \right| \\
&+ \left| \int \nabla (\zeta V_2)(x) \xi^\epsilon dx - M \nabla V_2(\mathbf{x}(t)) \boldsymbol{\xi}(t) \right| \\
&\leq (\|\nabla V_1\|_{C^2} + \|\nabla (\zeta V_2)\|_{C^2}) \, \|\xi^\epsilon(t) dx - M \boldsymbol{\xi}(t) \delta_{\mathbf{x}(t)}\|_{C^{2*}} \quad (67) \\
&\leq \eta^\epsilon(t) + \mathcal{O}(\epsilon^2).
\end{aligned}
$$

In the last line we have used that $\nabla(\zeta V_2)(\mathbf{x}(t)) = \nabla V_2(\mathbf{x}(t))$ for every $t \in [0, T_0]$ and Proposition 5. This achieves the proof of Lemma 6. ☐

Proof of Proposition 5 . Let us observe first that, under notation (50), we have

$$
E_k^\epsilon(t) - \frac{M}{2} |\boldsymbol{\xi}^\epsilon(t)|^2 = \frac{1}{2} \int |\frac{\xi^\epsilon}{\sqrt{m^\epsilon}} - \boldsymbol{\xi}^\epsilon \sqrt{m^\epsilon}|^2 dx \geq 0. \tag{68}
$$

Also, by Cauchy-Schwartz inequality and (12), we obtain

$$
|\boldsymbol{\xi}^\epsilon(t)| \leq M \int_{\mathbb{R}^N} |\xi^\epsilon| dx \leq \frac{C}{\epsilon^{N-1}} \|\nabla u^\epsilon\|_{L^2} \|u^\epsilon\|_{L^2} \leq C, \tag{69}
$$

for every $t \in \mathbb{R}$.
The quantum and classical conservation laws and the assumption (A3) give the following

Lemma 7 *There exists $C > 0$, such that*

$$
E_b^\epsilon(t) - E_b^\epsilon(Q^\epsilon) \leq C \eta^\epsilon(t) + \mathcal{O}(\epsilon^2), \tag{70}
$$

$$
E_k^\epsilon(t) - \frac{M}{2} |\boldsymbol{\xi}^\epsilon(t)|^2 \leq C \eta^\epsilon(t) + \mathcal{O}(\epsilon^2), \tag{71}
$$

where $Q^\epsilon := Q(\frac{\cdot}{\epsilon})$.

Proof. On the one hand, one has

$$E_b^\epsilon(t) + E_k^\epsilon(t) \quad = \quad E^\epsilon(t) - E_p^\epsilon(t).$$

On the other hand, we have the conservation laws of total energy yields

$$E^\epsilon(0) = E_b^\epsilon(Q^\epsilon) + \frac{M}{2}|\xi_0|^2 + MV(x_0) + \mathcal{O}(\epsilon^2).$$

Since V is nonnegative, we have

$$E_p^\epsilon(t) \geq \int \zeta(x)V(x)m^\epsilon(t)dx.$$

This yields, in particular,

$$E_b^\epsilon(t) + E_k^\epsilon(t) \quad \leq \quad E_b^\epsilon(Q^\epsilon) + MH(0) - \int \zeta(x)V(x)m^\epsilon(t)dx + \mathcal{O}(\epsilon^2).$$

Which yields, by definition of η^ϵ and the fact that $\zeta(\mathbf{x}(t)) = 1$ for every $t \in [0, T]$,

$$E_b^\epsilon(t) + E_k^\epsilon(t) \quad \leq \quad E_b^\epsilon(Q^\epsilon) + MH(0) - MV(\mathbf{x}(t)) + \eta^\epsilon(t) + \mathcal{O}(\epsilon^2).$$

By the conservation of the classical energy, we obtain

$$E_b^\epsilon(t) + E_k^\epsilon(t) \quad \leq \quad E_b^\epsilon(Q^\epsilon) + \frac{M}{2}|\boldsymbol{\xi}(t)|^2 + \eta^\epsilon(t) + \mathcal{O}(\epsilon^2).$$

However, from (69) we get

$$|\frac{|\boldsymbol{\xi}(t)|^2}{2} - \frac{|\boldsymbol{\xi}^\epsilon(t)|^2}{2}| \leq C(|\boldsymbol{\xi}(t)| + |\boldsymbol{\xi}^\epsilon(t)|)\eta^\epsilon \leq C\eta^\epsilon, \tag{72}$$

This gives,

$$E_b^\epsilon(t) + E_k^\epsilon(t) \leq E_b^\epsilon(Q^\epsilon) + M\frac{|\boldsymbol{\xi}^\epsilon(t)|^2}{2} + \eta^\epsilon(t) + \mathcal{O}(\epsilon^2).$$

The latter inequality and the definition of η^ϵ therefore yield

$$\underbrace{E_b^\epsilon(t) - E_b^\epsilon(Q^\epsilon)}_{\geq 0 \text{ by } (16)} + \underbrace{E_k^\epsilon(t) - \frac{M}{2}|\boldsymbol{\xi}^\epsilon(t)|^2}_{\geq 0 \text{ by } (68)} \leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2). \tag{73}$$

The required conclusion then follows. □

Remark 5 In the last lemma, and the whole of the proof, we have used only the fact that the family of initial data (φ^ϵ) satisfies the following properties :

$$\begin{cases} \frac{1}{\epsilon^N}\|\varphi^\epsilon\|_{L^2}^2 = M, \\ \int < \varphi^\epsilon, \nabla\varphi^\epsilon > dx - M\xi_0 = \mathcal{O}(\epsilon^2), \\ \|\frac{1}{\epsilon^N}|\varphi^\epsilon|^2 dx - M\delta_{x_0}\|_{C^{2*}} = \mathcal{O}(\epsilon^2), \\ E^\epsilon(\varphi^\epsilon) = I_M + \frac{M}{2}|\xi_0|^2 + MV(x_0) + \mathcal{O}(\epsilon^2). \end{cases} \tag{74}$$

(Notice that $E_b^\epsilon(Q^\epsilon) = I_M$). It is easy to see that all the results of the paper hold for every family of initial data (φ^ϵ) satisfying the properties above.

The following lemma relies mainly on Proposition 3.

Lemma 8 *There exists $h_1 > 0$ and $\epsilon_1 > 0$, such that if $\eta^\epsilon(t) < h_1$ and $\epsilon < \epsilon_1$ then there exists some point $z^\epsilon(t) \in \mathbb{R}^N$, such that*

$$\|m^\epsilon(t)dx - M\delta_{z^\epsilon(t)}\|_{C^{2*}} + \|\xi^\epsilon(t)dx - M\boldsymbol{\xi}^\epsilon\delta_{z^\epsilon(t)}\|_{C^{2*}} \leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2). \tag{75}$$

Proof. Let $h_1 = \frac{h}{2C}$ (h is the constant in Proposition 3 and C is the constant of Lemma 7). According to Lemma 7 if $\eta^\epsilon(t) < h_1$ and $\epsilon < \epsilon_1$ sufficiently small then $E_b^\epsilon(u^\epsilon(t)) - E_b^\epsilon(Q^\epsilon) \leq C(h_1 + \mathcal{O}(\epsilon_1^2)) < h$. Thus, in view of Proposition 3, there exists $z^\epsilon(t) \in \mathbb{R}^N$, such that

$$\frac{1}{\epsilon^N}\||u^\epsilon| - \tau_{z^\epsilon(t)}Q^\epsilon\|_{L^2}^2 \leq CE_b^\epsilon(t) - E_b^\epsilon(Q^\epsilon)(t) \leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2),$$

where the notation $\tau_{x_0}f := f(\cdot + x_0)$ is used.
Let $\psi \in C^2(\mathbb{R}^N)$, such that $\|\psi\|_{C^2} \leq 1$. Set

$$\Gamma^\epsilon(t) = \left|\int \psi(x)m^\epsilon dx - M\psi(z^\epsilon)\right| + \left|\int \psi(x)\xi^\epsilon dx - M\boldsymbol{\xi}^\epsilon\psi(z^\epsilon)\right|. \tag{76}$$

The triangle inequality yields

$$\Gamma^\epsilon(t) \leq (1 + |\boldsymbol{\xi}^\epsilon|)\left|\int \psi(x)m^\epsilon dx - M\psi(z^\epsilon)\right| + \left|\int \psi(x)(\xi^\epsilon - \boldsymbol{\xi}^\epsilon m^\epsilon)dx\right|.$$

Combined with the fact that $\boldsymbol{\xi}^\epsilon$ is bounded and $\int(\xi^\epsilon - \boldsymbol{\xi}^\epsilon m^\epsilon)dx = 0$, the latter inequality gives

$$\Gamma^\epsilon(t) \leq C\left|\int \tilde{\psi}(x)m^\epsilon dx\right| + \left|\int \tilde{\psi}(x)(\xi^\epsilon - \boldsymbol{\xi}^\epsilon m^\epsilon)dx\right|, \tag{77}$$

where $\tilde{\psi}(x) = \psi(x) - \psi(z^\epsilon)$. Using (77) and the trivial inequality $ab \leq a^2 + b^2$, one obtains

$$\Gamma^\epsilon(t) \leq C \int (|\tilde{\psi}(x)| + |\tilde{\psi}(x)|^2) m^\epsilon dx + \int |\frac{\xi^\epsilon}{\sqrt{m^\epsilon}} - \boldsymbol{\xi}^\epsilon \sqrt{m^\epsilon}|^2 dx. \quad (78)$$

On one hand, (68) and (71) give

$$\int |\frac{\xi^\epsilon}{\sqrt{m^\epsilon}} - \boldsymbol{\xi}^\epsilon \sqrt{m^\epsilon}|^2 dx \leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2). \quad (79)$$

On the other hand, the triangle inequality yields

$$\int (|\tilde{\psi}(x)| + |\tilde{\psi}(x)|^2) m^\epsilon dx \leq 2 \int (|\tilde{\psi}(x)| + |\tilde{\psi}(x)|^2) \frac{1}{\epsilon^N} \left| |u^\epsilon| - \tau_{z^\epsilon(t)} Q^\epsilon \right|^2 dx + \\ + 2 \int (|\tilde{\psi}(x)| + |\tilde{\psi}(x)|^2) \frac{1}{\epsilon^N} |\tau_{z^\epsilon(t)} Q^\epsilon|^2 dx. \quad (80)$$

Thus,

$$\begin{aligned} \int (|\tilde{\psi}(x)| + |\tilde{\psi}(x)|^2) m^\epsilon dx &\leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2) + \epsilon^2 \int |x|^2 |Q(x)|^2 dx \quad (81) \\ &\leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2). \end{aligned}$$

In the last line we have used $\| |\tilde{\psi}(x)| + |\tilde{\psi}(x)|^2 \|_{C^2} \leq C$ and $|\tilde{\psi}(z^\epsilon)| + |\tilde{\psi}(z^\epsilon)| = 0$.
Putting together (76), (78), (79) and (81), it follows that

$$|\int \psi(x) m^\epsilon dx - M\psi(z^\epsilon)| + |\int \psi(x) \xi^\epsilon dx - M\boldsymbol{\xi}^\epsilon \psi(z^\epsilon)| \leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2),$$

for every $\psi \in C^2(\mathbb{R}^N)$, such that $\|\psi\|_{C^2} \leq 1$. Thus (75) follows. □
Our next task is to prove that the family $z^\epsilon(t)$ introduced in Lemma 8 is close to $\mathbf{x}(t)$.

Lemma 9 *There exist $h_2 > 0$ and $\epsilon_2 > 0$, such that if $\eta^\epsilon(t) \leq h_2$ and $\epsilon < \epsilon_2$ then*

$$\|\delta_{\mathbf{x}(t)} - \delta_{z^\epsilon(t)}\|_{C^{2*}} \leq |\mathbf{x}(t) - z^\epsilon(t)| \leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2). \quad (82)$$

Proof. The first inequality is trivial. Let us prove the second one. Recall that T^0 and A are not yet chosen. We shall choose them at this stage of the proof. If $|z^\epsilon(t)| \leq A$, for every $t \in [0, T_\epsilon^*[$, then the definition of η^ϵ and the properties of ζ imply

$$\begin{aligned}
|\mathbf{x}(t) - z^\epsilon(t)| &\leq |X^\epsilon(t) - z^\epsilon(t)| + \eta^\epsilon(t) \\
&\leq C\|x\zeta\|_{C^2}\|m^\epsilon(t)dx - M\delta_{z^\epsilon(t)}\|_{C^{2*}} + \eta^\epsilon(t) \\
&\leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

In the last line we have used (75). Hence, it suffices to choose T^0 and A, such that $|z^\epsilon(t)| \leq A$, for every $t \leq T_\epsilon^* \leq T^0$.
Let $\psi \in C^2(\mathbb{R}^N)$ and $t_1, t_2 \in [0, T_\epsilon^*]$, such that $t_2 > t_1$. One has

$$\begin{aligned}
\int \psi(m^\epsilon(t_2) - m^\epsilon(t_1))dx &= \int_{t_1}^{t_2}\int \psi \frac{dm^\epsilon}{dt}(t)dxdt \\
&= -\int_{t_1}^{t_2}\int \psi div\xi^\epsilon dxdt \qquad \text{by (53)} \\
&= \int_{t_1}^{t_2}\int \nabla\psi\xi^\epsilon dxdt \\
&\leq \|\nabla\psi\|_\infty \int_{t_1}^{t_2}\int |\xi^\epsilon|dxdt. \\
&\leq C_0|t_2 - t_1|\|\psi\|_{C^2} \qquad \text{by} \quad (69). \qquad (83)
\end{aligned}$$

The constant C_0 in (83) depends only upon the problem (V, M, N,...). The triangle inequality and (75) yield

$$\|M\delta_{z^\epsilon(t_2)} - M\delta_{z^\epsilon(t_1)}\|_{C^{2*}} \leq C(\eta^\epsilon(t_2) + \eta^\epsilon(t_1)) + C_0|t_2 - t_1| + \mathcal{O}(\epsilon^2). \quad (84)$$

Since $t_1, t_2 \in [0, T_\epsilon^*] \subset [0, T^0]$, it follows that

$$C\left(\eta^\epsilon(t_2) + \eta^\epsilon(t_1)\right) + C_0|t_2 - t_1| + \mathcal{O}(\epsilon^2) \leq 2h_2C + C_0T^0 + \mathcal{O}(\epsilon^2). \quad (85)$$

We choose $T^0 = \frac{M}{2C_0}$, where C_0 is the constant in (85), then ϵ_2 and h_2, such that $2h_2C + C_0T^0 + \mathcal{O}(\epsilon_2) < MK_0$ (where K_0 is the constant in Lemma 4). With this choice we get

$$\|\delta_{z^\epsilon(t_2)} - \delta_{z^\epsilon(t_1)}\|_{C^{2*}} < K_0.$$

The inequality (49) gives

$$|z^\epsilon(t_2) - z^\epsilon(t_1)| \leq K_0. \quad (86)$$

Since $z^\epsilon(0) = x_0$ then

$$|z^\epsilon(t)| \leq K_0 + |x_0|,$$

for every $t \in [0, T_\epsilon^*]$. Thus, we take $A = K_0 + |x_0|$.
Let us notice that the term $C_0|t_2 - t_1|$ depends only upon the problem (V, M, N,...) and the size of the interval $[0, T^0]$. This fact shall be crucial in the extension of the result after T^0.
This concludes the proof of Lemma 9.

□

Let us now conclude the the proof of Proposition 5. We take $h_0 = \min\{h_1, h_2\}$ and $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ (where (h_1,ϵ_1) and (h_2, ϵ_2) are successively defined in Lemma 8 and Lemma 9). According to Lemma 8 and Lemma 9, it follows that

$$\begin{aligned}
\|m^\epsilon(t)dx - M\delta_{\mathbf{x}(t)}\|_{C^{2*}} &\leq \|m^\epsilon(t)dx - M\delta_{z^\epsilon(t)}\|_{C^{2*}} \\
&\quad + \|M\delta_{\mathbf{x}(t)} - M\delta_{z^\epsilon(t)}\|_{C^{2*}} \\
&\leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

whenever $t \leq T_\epsilon^*$ and $0 \leq \epsilon < \epsilon_0$. Also, the triangle inequality and some elementary properties of C^{2*} norm yield

$$\begin{aligned}
\|\xi^\epsilon dx - M\boldsymbol{\xi}(t)\delta_{\mathbf{x}(t)}\|_{C^{2*}} &\leq \|\xi^\epsilon dx - M\boldsymbol{\xi}^\epsilon\delta_{\mathbf{x}(t)}\|_{C^{2*}} + 2M|\boldsymbol{\xi}^\epsilon(t) - \boldsymbol{\xi}(t)| \\
&\quad + M|\boldsymbol{\xi}^\epsilon||\mathbf{x}(t) - z^\epsilon(t)| \\
&\leq C\eta^\epsilon(t) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

whenever $t \leq T_\epsilon^*$ and $0 \leq \epsilon < \epsilon_0$. In the latter inequality we have used (75), (82), the definition of η^ϵ and (69). This achieves the proof of Proposition 5.
□

Step 2. Our purpose in this step is to extend the result of Step 1 to every $T > 0$. By Step 1, we have

$$\|m^\epsilon dx - M\delta_{\mathbf{x}(t)}\|_{C^{2*}} + \|\xi^\epsilon dx - M\boldsymbol{\xi}(t)\delta_{\mathbf{x}(t)}\|_{C^{2*}} = \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \downarrow 0, \tag{87}$$

uniformly in $t \in [0, T^0]$. The trace $\psi^\epsilon = u^\epsilon(T^0, \cdot)$ of u^ϵ on $t = T^0$ then satisfies

$$\frac{1}{\epsilon^N}\|\psi^\epsilon\|_{L^2}^2 = M, \tag{88}$$

$$\boldsymbol{\xi}^\epsilon \underset{\epsilon\to 0}{\longrightarrow} \boldsymbol{\xi}(T^0), \tag{89}$$

$$\|\frac{1}{\epsilon^N}|\psi^\epsilon|^2 dx - M\delta_{\mathbf{x}(T^0)}\|_{C^{2*}} \underset{\epsilon\to 0}{\longrightarrow} 0, \tag{90}$$

and

$$E^\epsilon(\psi^\epsilon) \underset{\epsilon\to 0}{\longrightarrow} I_M + \frac{M}{2}|\boldsymbol{\xi}(T^0)|^2 + MV(\mathbf{x}(T^0)). \tag{91}$$

Besides, the convergence rates are of order ϵ^2.
As explained in Remark 5, this allows us to repeat the same argument on $[T^0, 2T^0]$. Since T^0 depends only upon the problem (V, M, N,...) one can reach any $T > 0$ after a finite number of iterations. This concludes the proof of Proposition 4.

4 The critical case

In this section we give some remarks on the case of critical nonlinearity. The argument of the proof of Theorem 1 does not work in the case of critical nonlinearity. More precisely, the orbital stability of the ground state, which is the main tool in the proof of Theorem 1, holds only for subcrtitical nonlinearities ($\sigma < \frac{2}{N}$). The first remark is that the asymptotic behavior proved in Theorem 1 breaks down when the nonlinearity is critical. This follows from the following (see [] for a complete discussion about harmonic potential)

Proposition 6 *Take $V = \frac{1}{2}|x|^2$. Then the solution of (1)-(2) is given by*

$$\frac{1}{\epsilon^{N/2}} u^\epsilon(t,x) = \frac{1}{(\epsilon\cos t)^{N/2}} e^{i\frac{\varphi(t,x)}{\epsilon}} Q\left(\frac{x - \mathbf{x}(t)}{\epsilon\cos(t)}\right),$$

where $\mathbf{x}(t) = \cos(t)x_0 + \sin(t)\xi_0$ *and* $\varphi(t,x) = (1 - \frac{|\xi_0|^2}{2} - \frac{|x|^2}{2})\tan t + \frac{x\xi_0}{\cos t}$.

In the critical case, the profile is the modulated ground state $\frac{1}{(\cos t)^{N/2}} Q(\frac{\cdot}{\cos t})$. This modulation term is caused by the harmonic potential. In fact when we consider a less "strong" potential (the stark potential, for example) then Theorem 1 holds for $\sigma \leq \frac{2^*-2}{2}$ (here we put $2^* = \infty$ if $N = 1, 2$, and $2^* = \frac{2N}{N-2}$ if $N \geq 3$).

Proposition 7 *Let $V(x) = b.x + a$ and (u^ϵ) be the family of solutions to (1)-(2) with $\sigma = \frac{2}{N}$. Then*

$$u^\epsilon(t,x) = e^{i\frac{tb(x-x_0) - \frac{t^3}{6}|b|^2 - t - at - \frac{t|\xi_0|^2}{2} + x\xi_0}{\epsilon}} Q\left(\frac{x + \frac{t^2}{2}b - \xi_0 t - x_0}{\epsilon}\right),$$

for every $t \in \mathbb{R}^n$.

In the case of harmonic potential we have some informations even when the initial profile is not Q. More precisely , we have the following

Proposition 8 *Take* $V = \frac{1}{2}|x|^2$, $\sigma \leq \frac{2^*-2}{2}$ *and* (u^ϵ) *be the family of solutions to (1) with initial data* $u^\epsilon(0,x) = \psi\left(\frac{x-x_0}{\epsilon}\right) e^{i\frac{x.\xi_0}{\epsilon}}$, *where* $(x_0,\xi_0) \in \mathbb{R}^N \times \mathbb{R}^N$ *and* $\psi \in \Sigma := \{f \in H^1; xf \in L^2\}$. *Then*

$$\begin{aligned} \frac{1}{\epsilon^N}\int x|u^\epsilon(t,x)|^2dx &= \|\psi\|_{L^2}^2\mathbf{x}(t) + \mathcal{O}(\epsilon), \\ \int \xi^\epsilon(t,x)dx &= \|\psi\|_{L^2}^2\boldsymbol{\xi}(t) \ + \mathcal{O}(\epsilon), \end{aligned}$$

uniformly on I_ϵ *the interval of definition*[4] *of* u^ϵ. *Moreover, if* ψ *is real and radial then* $\mathcal{O}(\epsilon) = 0$.

Remark 6 This proposition tells us roughly that the average position and momentum of the quantum particles are approximated by the the punctual classical trajectories under harmonic potential.

Proof. It is a consequence of the easy fact that

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{\epsilon^N}\int x|u^\epsilon(t,x)|^2dx\right) &= \int \xi^\epsilon(t,x)dx, \\ \frac{d}{dt}\left(\int \xi^\epsilon(t,x)dx\right) &= -\frac{1}{\epsilon^N}\int x|u^\epsilon(t,x)|^2dx. \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\epsilon^N}\int x|u^\epsilon(0,x)|^2dx &= \|\psi\|_{L^2}^2x_0 + \epsilon\int x|\psi(x)|^2dx, \\ \int \xi^\epsilon(0,x)dx &= \|\psi\|_{L^2}^2\xi_0 - \epsilon\Im\left(\int \psi(x)\nabla\bar{\psi}(x)dx\right). \end{aligned}$$

■

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[4] If $\sigma < \frac{2}{N}$ then $I_\epsilon = \mathbb{R}$

References

[1] A. AMBROSETTI, M. BADIALE, S. CINGOLANI :*Semiclassical states on nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal **140** (1997), n. 3, 285-300.

[2] J. C. BRONSKI, R.L. JERRARD :*Soliton Dynamics in a Potential*, Mathematical Research Letters **7** (2000), no. 2-3, 329–342.

[3] R. CARLES :*Nonlinear Schrödingfer equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal. **35**, No. 4, 823-843 (2003).

[4] R. CARLES, L. MILLER :*Semiclassical nonlinear Schrödingfer equations with potential and focusing initial data* to appear in Usaka J. Math.

[5] T. CAZENAVE :*An introduction to Nonlinear Schrödinger Equations*, Text. Metod. Mat. **26**, Univ. Fed. Rio de Janeiro, 1993.

[6] P. GÉRARD :*Mesures semi-classiques et ondes de Bloch* Séminaire Equations aux Dérivées Partielles, exp 6 (1990-1991), Ecole Polytechnique, Palaiseau.

[7] P. GÉRARD, E. Leitchnam :*Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J. **71** , 559-607, (1993).

[8] P. GÉRARD, A. MARKOWICH, N.J MAUSER, F. POUPAUD:*Homogenization limits and Wigner transforms*, Comm. in Pure and Applied Math. Vol. L, 0323-0379, (1997).

[9] S. KERAANI :*Semiclassical limit of a class of Schrödinger equations with potential.*, Comm. Partial Differential Equations **27** (2002), no. 3-4, 693–704.

[10] M.K. KWONG :*uniqueness of positive solutions to* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^n$, Arch. Rat. Mech. Anal **105** (1989), n. 3 243-266.

[11] P-L. LIONS :*The concentration-compactness principle in the calculus of variations. The compact case. Part 1*, Ann. Inst. Henri Poincaré, Analyse non linéaire **1** (1984), 109-145.

[12] P-L. LIONS, T. PAUL :*Sur les mesures de Wigner*, Revista Mat.Iberoamericana, **9**, 553-618, (1993).

[13] Y.G. OH :*Existence of semiclassical bound states of nonlinear Schrödinger equations with potential of class* $(v)_\alpha$, Comm. in PDE **13** (1988), 1499-1519.

[14] ——— :*on positive multi-lump bound states of nonlinear Schrödinger equations under multiple-well potentials*, Comm. Math. Phys. **131** (1990), 223-253.

[15] Sulem, Catherine; Sulem, Pierre-Louis The nonlinear Schrödinger equation. Self-focusing and wave collapse. Applied Mathematical Sciences, **139.** Springer-Verlag, New York, 1999.

[16] M. WEINSTEIN :*Lyapunov stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **16** (1985), 472-491.