

# On the rate of convergence in the central limit theorem for martingale difference sequences

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**ABSTRACT:** We established the rate of convergence in the central limit theorem for stopped sums of a class of martingale difference sequences.

**Sur la vitesse de convergence dans le théorème limite central pour les différences de martingale**

**RÉSUMÉ:** On établit la vitesse de convergence dans le théorème limite central pour les sommes arrêtées issues d'une classe de suites de différences de martingale.

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## 1 Introduction

Let  $(X_i)_{i \in \mathbb{N}}$  be a sequence of random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We shall say that  $(X_k)_{k \in \mathbb{N}}$  is a martingale difference sequence if, for any  $k \geq 0$

1.  $\mathbb{E}\{|X_k|\} < +\infty$ .
2.  $\mathbb{E}\{X_{k+1} | \mathcal{F}_k\} = 0$ , where  $\mathcal{F}_k$  is the  $\sigma$ -algebra generated by  $X_i, i \leq k$ .

For each integer  $n \geq 1$  and  $x$  real number, we denote

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i, \quad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-\frac{t^2}{2}) dt, \quad \sigma_{n-1}^2 = \mathbb{E}\{X_n^2 | \mathcal{F}_{n-1}\},$$

$$\nu(n) = \inf\{k \in \mathbb{N}^* / \sum_{i=0}^k \sigma_i^2 \geq n\}, \quad S_{\nu(n)}^2 = \sum_{k=1}^{+\infty} S_k^2 I_{\nu(n)=k}, \quad \sigma_{\nu(n)}^2 = \sum_{k=1}^{+\infty} \sigma_k^2 I_{\nu(n)=k},$$

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$F_n(x) = \mathbb{P}(S_{\nu(n)} \leq x\sqrt{n})$ ,  $S'_{\nu(n)} = S_{\nu(n)} + \sqrt{\gamma(n)}X_{\nu(n)+1}$ ,  $H_n(x) = \mathbb{P}(S'_{\nu(n)} \leq x\sqrt{n})$ , and  $\gamma(n)$  is a random variable such that

$$\sum_{i=0}^{\nu(n)-1} \sigma_i^2 + \gamma(n)\sigma_{\nu(n)}^2 = n \quad p.s. \quad (1)$$

If the random variables  $X_i$  are independent and identically distributed with  $\mathbb{E}X_i = 0$  and  $\mathbb{E}X_i^2 = 1$ , we have by the central limit theorem (CLT)

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(S_n \leq x\sqrt{n}) - \phi(x)| = 0.$$

By the theorem of Berry ([1], 1941) and Esseen ([3], 1942), if moreover,  $\mathbb{E}|X_i^3| < +\infty$ , the rate of convergence in the limit is of order  $n^{-\frac{1}{2}}$ . If  $(X_i)_{i \in \mathbb{N}}$  is an ergodic martingale difference sequence with  $\mathbb{E}X_i^2 = 1$ , by the theorem of Billingsley ([9], 1968) and Ibragimov ([6], 1963), see also ([10], 1980)) we have the CLT. The rate of convergence can, however, be arbitrarily slow even if  $X_i$  are bounded and  $\alpha$ -mixing (cf [7]). There are several results showing that with certain assumption on the conditional variance  $\mathbb{E}(X_i^2 | \mathcal{F}_{i-1})$ , the rate of convergence becomes polynomial (Kato ([13], 1979), Grams ([12], 1972), Nakata ([11], 1976), Bolthausen ([4], 1982), Haeusler ([5], 1988), ...).

In 1963, Ibragimov [6] has shown that for  $X_i$  uniformly bounded, if instead of usual sums  $S_n$ , the stopped sums  $S_{\nu(n)}$  or  $S'_{\nu(n)}$  are considered, one gets the rate of convergence of order  $n^{-\frac{1}{4}}$ ; the only assumption beside boundedness is that  $\sum_{i=0}^{+\infty} \sigma_i^2$  diverge to infinity a.s.

In the present paper we give a rate of convergence for a larger class of martingale difference sequences, the Ibragimov's case will be a particular one.

## 2 Main result

We consider a sequence  $(X_i)_{i \in \mathbb{N}}$  of square integrable martingale differences.

**Theorem 1.** *If the series  $\sum_{i=0}^{+\infty} \sigma_i^2$  diverges a.s. and if there exists a nondecreasing sequence  $(Y_i)_{i \in \mathbb{N}}$  adapted to the filtration  $(\mathcal{F}_i, i \in \mathbb{N})$  such that, for all  $i \in \mathbb{N}^*$*

$$\mathbb{E}(Y_i^4) < +\infty, \quad 1 \leq Y_i \quad \text{and} \quad \mathbb{E}(|X_i|^3 | \mathcal{F}_{i-1}) \leq Y_{i-1} \sigma_{i-1}^2 \quad a.s.$$

then for all  $n$  sufficiently large

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left( 11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right), \quad (2)$$

$$\sup_{x \in \mathbb{R}} \left| H_n(x) - \phi(x) \right| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left( 11 + \frac{9}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) \quad (3)$$

where  $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$ .

If we put  $Y_i = M$  a.s. where  $M > 0$  is a constant, one obtains the following corollaries:

**Corollary 1.** *If the series  $\sum_{i=0}^{+\infty} \sigma_i^2$  diverges a.s. and there exists  $M > 0$  such that, for all  $i \in \mathbb{N}^*$ ,  $\mathbb{E}(|X_i|^3 | \mathcal{F}_{i-1}) \leq M \mathbb{E}(X_i^2 | \mathcal{F}_{i-1})$  a.s. then there is a constant  $0 < c_M < +\infty$*

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{c_M}{n^{\frac{1}{4}}}, \quad (4)$$

$$\sup_{x \in \mathbb{R}} \left| H_n(x) - \phi(x) \right| \leq \frac{c_M}{n^{\frac{1}{4}}}. \quad (5)$$

**Corollary 2.** *If there exists  $0 < \alpha \leq M < +\infty$  satisfying  $\sigma_{i-1}^2 \geq \alpha$  and  $\mathbb{E}(|X_i|^3 | \mathcal{F}_{i-1}) \leq M$  a.s. for all  $i \in \mathbb{N}^*$ , then there is a constant  $0 < c_{(\alpha, M)} < +\infty$  such that (4) and (5) hold.*

Moreover, if we suppose that  $(X_i)_{i \in \mathbb{N}}$  is uniformly bounded, we obtain the result of Ibragimov [6].

**Corollary 3.** *If the series  $\sum_{i=0}^{+\infty} \sigma_i^2$  diverges a.s. and  $|X_i| \leq M < +\infty$  a.s. for all  $i \geq 0$ , then (4) and (5) hold.*

**Example.** Let  $A = (A_k)_{k \in \mathbb{N}}$  be a sequence of real valued random variables such that  $\sup_{k \in \mathbb{N}} \mathbb{E}(A_k^4)^{1/4} = \beta < \infty$  and consider an arbitrary sequence of variables  $\zeta = (\zeta_k)_{k \in \mathbb{N}^*}$  with zero means, unit variances, bounded third moments and which are also independent of  $A$ . We define  $X = (A_{k-1} \zeta_k)_{k \in \mathbb{N}^*}$  and  $\mathcal{F}_k$  the  $\sigma$ -algebra generated by  $A_0, A_1, \dots, A_k$ .

Clearly  $(X_k, \mathcal{F}_k, k \in \mathbb{N}^*)$  is a martingale difference sequence, and for all  $k \in \mathbb{N}^*$ ,

$$\mathbb{E}(A_{k-1}^2 \zeta_k^2 | \mathcal{F}_{k-1}) = A_{k-1}^2 \quad \text{a.s.},$$

$$\mathbb{E}(|A_{k-1} \zeta_k|^3 | \mathcal{F}_{k-1}) \leq |A_{k-1}| \sup_{i \in \mathbb{N}^*} \mathbb{E}(|\zeta_i|^3) A_{k-1}^2 \quad \text{a.s.}$$

If  $(|A_k|)_{k \in \mathbb{N}}$  is nondecreasing, then using Theorem 1, one obtains

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq c \beta \frac{\sup_{k \in \mathbb{N}^*} \mathbb{E}(|\zeta_k|^3)^{\frac{1}{4}}}{n^{\frac{1}{4}}}$$

where  $c$  is a positive constant.

### 3 Proof of Theorem

According to Esseen's theorem ( see, e.g., ([2], 1954) p. 210 and ([8], 1955) p. 285), for all  $y > 0$ ,

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{1}{\pi} \int_{-y}^y \left| \mathbb{E} \left\{ \exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right) \right\} - \exp\left(-\frac{t^2}{2}\right) \right| \frac{dt}{|t|} + \frac{24}{\pi \sqrt{2\pi y}}. \quad (6)$$

Below we shall prove the following inequalities

$$\left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \leq a_n e^{\frac{t^2}{2}} \left( \frac{|t|}{3\sqrt{n}} + \frac{t^2}{4n} + \frac{a_n |t|^3}{3n^{\frac{3}{2}}} + \frac{a_n t^4}{4n^2} \right), \quad (7)$$

$$\left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2} \right) \right\} \right| \leq \frac{a_n t^2}{2n} \exp\left(\frac{t^2}{2}\right), \quad (8)$$

$$\left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} \right) \right\} - \mathbb{E} \left\{ \exp \left( \frac{itS'_{\nu(n)}}{\sqrt{n}} \right) \right\} \right| \leq \frac{3a_n t^2}{2n} \quad (9)$$

where  $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$ .

### 3.1 Proof of the Inequality (7)

We have

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \\ &= \sum_{k=1}^{+\infty} \mathbb{E} \left\{ \left( \exp \left( \frac{itS_k}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{k-1} \sigma_p^2 \right) - 1 \right) I_{\nu(n)=k} \right\} \\ &= \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ \exp \left( \frac{itS_{j-1}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \left( e^{\frac{itX_j}{\sqrt{n}}} - e^{-\frac{t^2\sigma_{j-1}^2}{2n}} \right) I_{\nu(n)=k} \right\}. \end{aligned}$$

For real  $x$ , put

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + u(x), \quad e^{-x} = 1 - x + \beta(x) \frac{x^2}{2} \quad (*)$$

It is easily seen that, for all  $x \in \mathbb{R}$

$$|u(x)| \leq \frac{|x|^3}{6}, \quad |u(x)| \leq \frac{x^2}{2}, \quad \text{and} \quad |\beta(|x|)| \leq 1.$$

Observing that the random variable  $W_{j-1}^n = \exp \left( \frac{itS_{j-1}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right)$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_{j-1}$  and using the identities (\*), we obtain

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \\ &= \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ W_{j-1}^n \mathbb{E} \left\{ \left( \frac{itX_j}{\sqrt{n}} - \frac{t^2 X_j^2}{2n} + u\left(\frac{tX_j}{\sqrt{n}}\right) + \frac{t^2 \sigma_{j-1}^2}{2n} + \beta\left(\frac{t^2 \sigma_{j-1}^2}{2n}\right) \frac{t^4 \sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} \middle| \mathcal{F}_{j-1} \right\} \right\} \end{aligned} \quad (10)$$

Since  $\{\nu(n) = k\}$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_k$ , for all  $j \geq 2$ , we have

$$\sum_{k=1}^{j-1} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k=1}^{j-1} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

On the other hand, for all  $j \geq 1$  we have

$$\sum_{k=1}^{+\infty} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k=1}^{+\infty} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

It follows that, for all  $j \geq 1$

$$\sum_{k \geq j} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k \geq j} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

So, from (10) we derive

$$\begin{aligned} & \left| \mathbb{E}\left\{ \exp\left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\ &= \left| \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E}\left\{ W_{j-1}^n \mathbb{E}\left\{ \left( u\left(\frac{tX_j}{\sqrt{n}}\right) + \beta\left(\frac{t^2\sigma_{j-1}^2}{2n}\right) \frac{t^4\sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \right| \\ &\leq \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E}\left\{ \exp\left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \left( \frac{|t|^3 |X_j|^3}{6n^{\frac{3}{2}}} + \frac{t^4\sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\}. \end{aligned} \quad (11)$$

For any  $j \geq 2$  and any real function  $\psi$  such that  $\mathbb{E}(\psi(X_k)) < \infty$  for any positive  $k$ , we have

$$\begin{aligned} & \sum_{k=1}^{j-1} \mathbb{E}\left\{ \exp\left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\ &= \sum_{k=1}^{j-1} \mathbb{E}\left\{ \exp\left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \end{aligned} \quad (12)$$

On the other hand, for all  $j \geq 1$ , we have

$$\begin{aligned} & \sum_{k=1}^{+\infty} \mathbb{E}\left\{ \exp\left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\ &= \mathbb{E}\left\{ \exp\left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \psi(X_j) \right\} \\ &= \sum_{k=1}^{+\infty} \mathbb{E}\left\{ \exp\left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \end{aligned} \quad (13)$$

It follows from (12) and (13) that

$$\begin{aligned} & \sum_{j=1}^{+\infty} \sum_{k \geq j} \mathbb{E} \left\{ \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E} \left\{ \psi(X_j) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\ &= \sum_{j=1}^{+\infty} \sum_{k \geq j} \mathbb{E} \left\{ \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E} \left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \end{aligned} \quad (14)$$

Applying (11) and (14) for  $\psi(x) = |x|^3$  we deduce that

$$\begin{aligned} & \left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\ & \leq \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \left( \mathbb{E} \left\{ \frac{|t|^3 |X_j|^3}{6n^{\frac{3}{2}}} | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} + \mathbb{E} \left\{ \frac{t^4 \sigma_{j-1}^4}{8n^2} I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right) \right\} \\ & \leq \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \left( \frac{|t|^3 Y_{j-1} \sigma_{j-1}^2}{6n^{\frac{3}{2}}} I_{\nu(n)=k} + \frac{t^4 \sigma_{j-1}^4}{8n^2} I_{\nu(n)=k} \right) \right\} \end{aligned} \quad (15)$$

By the Hölder inequality, for all  $j \in \mathbb{N}^*$

$$\sigma_{j-1}^2 = \mathbb{E}(X_j^2 | \mathcal{F}_{j-1}) \leq \mathbb{E}(|X_j|^3 | \mathcal{F}_{j-1})^{\frac{2}{3}} \leq Y_{j-1}^{\frac{2}{3}} \sigma_{j-1}^{\frac{4}{3}} \quad a.s.,$$

whence

$$\sigma_{j-1}^2 \leq Y_{j-1}^2 \quad a.s. \quad (16)$$

From (15), (16) and using the fact that  $Y_k \geq Y_{j-1} \geq 1$  for all  $j \leq k$ , we deduce that

$$\begin{aligned} & \left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\ & \leq \left( \frac{|t|^3}{6n^{\frac{3}{2}}} + \frac{t^4}{8n^2} \right) \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ Y_{j-1}^2 \sigma_{j-1}^2 \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) I_{\nu(n)=k} \right\} \\ & \leq \left( \frac{|t|^3}{6n^{\frac{3}{2}}} + \frac{t^4}{8n^2} \right) \sum_{k=1}^{+\infty} \mathbb{E} \left\{ Y_k^2 \sum_{j=1}^k \sigma_{j-1}^2 \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) I_{\nu(n)=k} \right\}. \end{aligned} \quad (17)$$

To bound up the terms appearing in (17), we will use the following elementary lemma.

**Lemma 1.** *Let  $k \geq 1$ , then on the event  $\{\nu(n) = k\}$  we have*

$$\sum_{j=1}^k \exp \left( \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \frac{t^2}{2n} \sigma_{j-1}^2 \leq \exp \left( \frac{t^2}{2} \right) \left( 1 + \frac{Y_k^2 t^2}{n} \right).$$

**Proof of Lemma.** On the event  $\{\nu(n) = k\}$ , we have

$$\begin{aligned} \exp\left(\frac{t^2}{2}\right) &\geq \exp\left(\frac{t^2}{2n} \sum_{p=0}^{k-1} \sigma_p^2\right) - \exp\left(\frac{t^2}{2n} \sigma_0^2\right) \\ &\geq \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \left(\exp\left(\frac{t^2 \sigma_j^2}{2n}\right) - 1\right) \end{aligned}$$

Using the inequality,  $\exp(x) - 1 \geq x$  for all  $x \geq 0$ , one obtains

$$\exp\left(\frac{t^2}{2}\right) \geq \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \frac{t^2}{2n} \sigma_j^2.$$

Therefore

$$\begin{aligned} &\sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \frac{t^2}{2n} \sigma_{j-1}^2 \\ &\leq \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \frac{t^2}{2n} (\sigma_{j-1}^2 - \sigma_j^2) + \exp\left(\frac{t^2}{2}\right) \\ &= \sum_{j=1}^{k-2} \left( \exp\left(\frac{t^2}{2n} \sum_{p=0}^j \sigma_p^2\right) - \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \right) \frac{t^2}{2n} \sigma_j^2 - \frac{t^2}{2n} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{k-2} \sigma_p^2\right) \sigma_{k-1}^2 \\ &\quad + \frac{t^2}{2n} \exp\left(\frac{t^2}{2n} \sigma_0^2\right) \sigma_0^2 + \exp\left(\frac{t^2}{2}\right) \\ &\leq \frac{t^2}{2n} Y_k^2 \sum_{j=1}^{k-2} \left( \exp\left(\frac{t^2}{2n} \sum_{p=0}^j \sigma_p^2\right) - \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \right) + \frac{t^2}{2n} Y_k^2 \exp\left(\frac{t^2}{2n} \sigma_0^2\right) + \exp\left(\frac{t^2}{2}\right) \\ &\leq \left(1 + \frac{t^2}{2n} Y_k^2\right) \exp\left(\frac{t^2}{2}\right). \end{aligned}$$

We conclude the proof of the lemma by noting that  $\sigma_{k-1}^2 \leq Y_k^2$  and  $\sum_{p=0}^{k-1} \sigma_p^2 \leq n$  a.s..

Finally, according to Lemma 1 and the (17) we get

$$\left| \mathbb{E} \left\{ \exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2\right) \right\} - 1 \right| \leq a_n \exp\left(\frac{t^2}{2}\right) \left( \frac{|t|}{3\sqrt{n}} + \frac{t^2}{4n} + \frac{a_n |t|^3}{3n^{\frac{3}{2}}} + \frac{a_n t^4}{4n^2} \right),$$

where  $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$ .

### 3.2 Proof of the Inequality (8)

Using (1) and the inequality  $|1 - \exp(-x)| \leq x$ , for all  $x \geq 0$  we see that

$$\begin{aligned}
& \left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2} \right) \right\} \right| \\
&= \left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2} \right) \left( \exp \left( -\frac{t^2}{2n} \gamma(n) \sigma_{\nu(n)}^2 \right) - 1 \right) \right\} \right| \\
&\leq \mathbb{E} \left\{ \left| 1 - \exp \left( -\frac{t^2}{2n} \gamma(n) \sigma_{\nu(n)}^2 \right) \right| \right\} \exp \left( \frac{t^2}{2} \right) \\
&\leq \mathbb{E} \left\{ \frac{t^2}{2n} |\gamma(n)| \sigma_{\nu(n)}^2 \right\} \exp \left( \frac{t^2}{2} \right) \\
&\leq (\mathbb{E} Y_{\nu(n)}^4)^{\frac{1}{2}} \frac{t^2}{2n} \exp \left( \frac{t^2}{2} \right).
\end{aligned}$$

Therefore (8) holds true.

From (7) and (8) we conclude that

$$\left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} \right) \right\} - \exp \left( -\frac{t^2}{2} \right) \right| \leq a_n \left( \frac{|t|}{3\sqrt{n}} + \frac{3t^2}{4n} + \frac{|t|^3}{3n^{\frac{3}{2}}} a_n + \frac{t^4}{4n^2} a_n \right).$$

Using Esseen's theorem, we derive

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n}{\pi} \int_{-y}^y \left( \frac{1}{3\sqrt{n}} + \frac{3|t|}{4n} + \frac{t^2}{3n^{\frac{3}{2}}} a_n + \frac{|t|^3}{4n^2} a_n \right) dt + \frac{24}{\pi \sqrt{2\pi} y}.$$

Hence

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n}{\pi} \left( \frac{2y}{3\sqrt{n}} + \frac{3y^2}{4n} + \frac{2y^3}{9n^{\frac{3}{2}}} a_n + \frac{y^4}{8n^2} a_n \right) + \frac{24}{\pi \sqrt{2\pi} y}.$$

Choosing  $y$  in such a way that  $y/\sqrt{n} = 1/(ya_n)$ , i.e.  $y = (n/a_n^2)^{\frac{1}{4}}$ , we infer that

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left( 11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right).$$

The proof of the inequality (2) in theorem is complete.

### 3.3 Proof of the Inequality (9)

Observing that the random events  $\{\gamma(n) \leq x\} \cap \{\nu(n) = k\}$  and consequently the random variables  $\sqrt{\gamma(n)} I_{\nu(n)=k}$  are measurable with respect to  $\mathcal{F}_k$ , we find that

$$\begin{aligned}
& \left| \mathbb{E} \left\{ \exp \left( \frac{itS_{\nu(n)}}{\sqrt{n}} \right) \right\} - \mathbb{E} \left\{ \exp \left( \frac{itS'_{\nu(n)}}{\sqrt{n}} \right) \right\} \right| \\
&= \left| \sum_{k=0}^{+\infty} \mathbb{E} \left\{ \left( \exp \left( \frac{itS_k}{\sqrt{n}} \right) - \exp \left( \frac{itS_k}{\sqrt{n}} + \frac{it\sqrt{\gamma(n)}}{\sqrt{n}} x_{\nu(n)+1} \right) \right) I_{\nu(n)=k} \right\} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left( \frac{itS_k}{\sqrt{n}} \right) \left( 1 - \exp \left( \frac{it\sqrt{\gamma(n)}}{\sqrt{n}} X_{\nu(n)+1} \right) \right) I_{\nu(n)=k} \right\} \right| \\
&= \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left( \frac{itS_k}{\sqrt{n}} \right) \left( -\frac{it\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} + \frac{t^2}{2n} \gamma(n) X_{k+1}^2 - u \left( \frac{t\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} \right) \right) I_{\nu(n)=k} \right\} \right| \\
&= \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left( \frac{itS_k}{\sqrt{n}} \right) \mathbb{E} \left\{ -\frac{it\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} + \frac{t^2}{2n} \gamma(n) X_{k+1}^2 - u \left( \frac{t\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} \right) \middle| \mathcal{F}_k \right\} I_{\nu(n)=k} \right\} \right| \\
&= \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left( \frac{itS_k}{\sqrt{n}} \right) \left( \frac{t^2}{2n} \gamma(n) X_{k+1}^2 - u \left( \frac{t}{\sqrt{n}} \sqrt{\gamma(n)} X_{k+1} \right) \right) I_{\nu(n)=k} \right\} \right| \\
&\leq \sum_{k=0}^{+\infty} \mathbb{E} \left\{ I_{\nu(n)=k} \frac{3t^2}{2n} \gamma(n) X_{k+1}^2 \right\} \\
&\leq \frac{3t^2}{2n} \sum_{k=0}^{+\infty} \mathbb{E} \left\{ I_{\nu(n)=k} \mathbb{E} \{ X_{k+1}^2 | \mathcal{F}_k \} \right\} \\
&\leq \frac{3t^2}{2n} \mathbb{E} (Y_{\nu(n)}^4)^{\frac{1}{2}}.
\end{aligned}$$

The proof of the inequality (9) is complete.

### 3.4 Proof of the inequality (3).

According to Esseen's theorem where  $y = (n/a_n^2)^{\frac{1}{4}}$  and the inequality (9), one obtains

$$\begin{aligned}
\sup_{x \in \mathbb{R}} \left| H_n(x) - \phi(x) \right| &\leq \frac{1}{\pi} \int_{-y}^y \left| \mathbb{E} \left\{ \exp \left( \frac{itS'_{\nu(n)}}{\sqrt{n}} \right) \right\} - \exp \left( -\frac{t^2}{2} \right) \right| \frac{dt}{|t|} + \frac{24}{\pi \sqrt{2\pi} y} \\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left( 11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) + \frac{1}{\pi} \int_{-y}^y \frac{3|t|}{2n} E(Y_{\nu(n)}^4)^{\frac{1}{2}} dt \\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left( 11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) + \frac{3}{2\pi \sqrt{n}} \\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left( 11 + \frac{9}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right).
\end{aligned}$$

The proof of theorem is complete.  $\square$

Proofs of corollaries 1, 2 and 3 are easy so, it is left to the reader.

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