Weyl chamber flow on irreducible quotients of products of $\mathrm{PSL}(2,\mathbb{R})$

Damien FERTE

Abstract: We study the topological dynamics of the action of the diagonal subgroup on quotients $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, where Γ is an irreducible lattice. Closed orbits are described and a set of points of dense orbit is explicitly given. Such properties are expressed using the Furstenberg boundary of the symmetric space $\mathbb{H} \times \mathbb{H}$.

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1 Introduction

Let G be a real connected semi-simple Lie group, K a maximal compact subgroup, A a Cartan subgroup and M the centralizer of A in K. If Γ is a lattice of G, the action (by right-translation) of A on the coset $\Gamma\backslash\mathrm{G}/\mathrm{M}$ is the *Weyl chamber flow* over the locally symmetric space $\Gamma\backslash\mathrm{G}/\mathrm{K}$. When the group G is $\mathrm{PSL}(2,\mathbb{R})$, this action is conjugated to the geodesic flow on the unit tangent bundle of the finite-volume hyperbolic surface $\Gamma\backslash\mathbb{H}$. In this case, there are a lot of different kinds of closed A-invariant subsets. By contrast, when the dimension of the Cartan subgroup A (the rank of G) is greater than 2, the situation is expected to be very rigid. In [Ma], G. A. Margulis conjectured that, excluding a situation of factorisation by a rank one action, each A-orbit closure is *algebraic* i.e. is the orbit of a closed connected subgroup of G containing A. Points of $\Gamma\backslash\mathrm{G}$ whose orbit is closed or dense satisfy this conjecture. Closed and compact orbits come from unipotent or semi-simple elements of the lattice Γ. (See [TW] for a description of closed orbits and [PR] for a condition of compactness.) Moore's ergodicity theorem (see [Z]) implies that, for the finite Haar measure, almost all point in $\Gamma\backslash\mathrm{G}$ have a dense orbit. But there is no explicit way to find or construct such dense orbits. For the groups $\mathrm{SL}(n, \mathbb{R})$ and their products, E. Lindenstrauss and B. Weiss proved that each point whose A-orbit closure contains a compact orbit also satisfies Margulis' conjecture. See also [Mo] for the study of the group $\mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_l)$.

This article deals with the special case of the group $\mathrm{G} = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$. The subgroup A is the maximal diagonal subgroup and we will also consider the semi-subgroup

$$\mathrm{A}^+ = \left\{ \left(\pm \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \pm \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \right) : \lambda_1, \lambda_2 \geq 1 \right\}.$$

If the lattice Γ in G is *reducible* (up to finite index, it is a product of two lattices of $\mathrm{PSL}(2, \mathbb{R})$), this is the simplest case of factorisation: every closed invariant subset for the action of A is the product of closed invariant subsets for the geodesic flows. Therefore we shall assume that Γ is an *irreducible* lattice of G. For instance, if $\mathbb{K}$ is a real quadratic field (of Galois automorphism σ) and $\mathcal{O}$ is its integers ring, the injection of the group $\mathrm{PSL}(2, \mathcal{O})$ in G given by

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} \right)$$

is an irreducible lattice called *Hilbert modular lattice* associated to $\mathbb{K}$. For irreducible lattices of G, Margulis' conjecture can be strentghtened in the following way.

Conjecture 1.1 (Margulis). *Let Γ be an irreducible lattice of $\mathrm{G} = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ and A be the maximal diagonal subgroup of G. Let x be a point in $\Gamma \backslash \mathrm{G}$. Thus the orbit $x\mathrm{A}$ is either closed or dense in $\Gamma \backslash \mathrm{G}$.*

In [L], the question of classification of A-ergodic finite measures on $\Gamma \backslash \mathrm{G}$ is studied and related to a "quantum unique ergodicity" conjecture.

In this article, we give an explicit set of points whose orbit (by the semi-group A^+) is dense. The condition is expressed on the factors $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$ of the Furstenberg boundary $\mathcal{F} = \partial \mathbb{H} \times \partial \mathbb{H}$ of the symmetric space $\mathbb{H} \times \mathbb{H}$ associated to the group G.

We use here the terminology of [Sh]. An element of G is said to be *hyperbolic* (resp. *parabolic*, *elliptic*) if both components are hyperbolic (resp. parabolic, elliptic) elements of $\mathrm{PSL}(2, \mathbb{R})$ (we use the convention that the unit element is neither hyperbolic, nor parabolic, nor elliptic). A non-trivial element is said to be *mixed* if its two components are not of the same kind. A hyperbolic element $\gamma = (\gamma_1, \gamma_2)$ is said to be *hyper-regular* if the dominant (positive) eigenvalues of γ_1 and γ_2 are distinct (in this situation, this definition coincides with the definition of "hyper-regular" in [PR]). An example of non hyper-regular elements is given by the canonical injection of the subgroup $\mathrm{PSL}(2,\mathbb{Z})$ in a Hilbert modular lattice. The group G acts naturally on $\mathcal{F}$, but also on each factor $\partial\mathbb{H}$ by the corresponding component. For instance, a point ξ_1 in the first factor $\partial\mathbb{H}$ of $\mathcal{F} = \partial\mathbb{H} \times \partial\mathbb{H}$ is fixed by an element $\gamma = (\gamma_1, \gamma_2)$ if $\gamma_1(\xi_1) = \xi_1$.

Theorem 1.2. *Let Γ be an irreducible lattice of $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Let $g = (g_1, g_2)$ belong to the group G and x be the class of g in $\Gamma\backslash\mathrm{G}$. Then the semi-orbit $x\mathrm{A}^+$ is dense in $\Gamma\backslash\mathrm{G}$ if one of the following conditions holds:*

1) a mixed element of Γ fixes one of the points $g_1(\infty)$ and $g_2(\infty)$ of $\partial\mathbb{H}$,

2) a hyperbolic hyper-regular element of Γ fixes exactly one of the points $g_1(\infty)$ and $g_2(\infty)$.

We also describe compact orbits and retrieve conditions of compactness of [PR].

Theorem 1.3. *Let Γ be an irreducible lattice of $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Let g belong to the group G and x be the class of g in $\Gamma\backslash\mathrm{G}$. Then the following properties are equivalent:*

1) the orbit $x\mathrm{A}$ is compact,

2) the subgroup $g^{-1}\Gamma g \cap \mathrm{A}$ is isomorphic to $\mathbb{Z}^2$,

3) the points $g(\infty,\infty)$ and $g(0,0)$ of $\mathcal{F}$ are fixed by a hyperbolic hyper-regular element of Γ.

When the lattice is not uniform, there exist closed, but non-compact orbits, coming from the parabolic points on $\mathcal{F}$ (see paragraph 3 for definition of conjugate parabolic points). In [TW], a general statement is given in algebraic terms.

Theorem 1.4. *Let Γ be an irreducible (non-uniform) lattice of $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Let g belong to the group G and x be the class of g in $\Gamma\backslash\mathrm{G}$. Then the following properties are equivalent:*

1) the orbit $x\mathrm{A}$ is closed and non-compact,

2) the subgroup $g^{-1}\Gamma g \cap \mathrm{A}$ is isomorphic to $\mathbb{Z}$,

3) the points $g(\infty,\infty)$ and $g(0,0)$ (or $g(0,\infty)$ and $g(0,\infty)$) are conjugate parabolic points of $\mathcal{F}$, relatively to Γ.

Remarks 1.5. *1) The element $\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of the group $\mathrm{PSL}(2,\mathbb{R})$ normalizes the diagonal subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and exchanges the points 0 and*

∞. *Then, the previous theorem 1.2 give sufficient conditions of density for the full diagonal group* A *using the points* $g_1(0), g_1(\infty), g_2(0)$ *and* $g_2(\infty)$ *of* $\partial\mathbb{H}$.
2) If a hyperbolic hyper-regular element of Γ *fixes both points* $g_1(\infty)$ *and* $g_2(\infty)$ *then the semi-orbit* $x\mathrm{A}^+$ *is "asymptotic" to a compact orbit and hence cannot be dense.*

We obtain the following corollary for uniform lattices.

Corollary 1.6. *Let* Γ *be a uniform irreducible lattice of* $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. *Then conjecture 1.1 is true for every point* $x = \Gamma g$ *in* $\Gamma\backslash\mathrm{G}$ *such that one of the points* $g_1(0), g_1(\infty), g_2(0), g_2(\infty)$ *of* $\partial\mathbb{H}$ *is fixed by a non-trivial element of* Γ.

This article is organized as follows. Paragraph 1 contains the useful results about irreducible lattices. After some recalls about the geometry of the locally symmetric space $\Gamma\backslash\mathbb{H} \times \mathbb{H}$ (paragraph 2), theorems 1.3 and 1.4 are proved in paragraph 3. Theorem 1.2 and corollary 1.6 are proved in paragraph 4. We use classical facts about the geometry of the hyperbolic plane: geodesics, compactification, dynamics of isometries on $\mathbb{H}$ and on the boundary $\partial\mathbb{H}$. If all notations coincide with the upper half-plane model of the hyperbolic plane, we use the disc model for the figures.

2 Properties of irreducible lattices

Denote by p_i, $i = 1, 2$, the projections

$$p_i : \mathrm{G} \longrightarrow \mathrm{PSL}(2,\mathbb{R}),\ (g_1, g_2) \longmapsto g_i$$

on each factor. If Γ is an irreducible lattice of G, the subgroups $p_i(\Gamma)$ are dense in $\mathrm{PSL}(2,\mathbb{R})$ and the kernels $\Gamma \cap \ker p_i$ are central in $\ker p_i$ (which is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$) and therefore trivial (see [R]).

Concerning the elements of irreducible lattices, we have the following. It is well-known that hyperbolic (and hyper-regular) elements exist in such a lattice. The density of the projections implies that an irreducible lattice Γ contains elements with an elliptic component. If Γ is a uniform lattice of G, then every element of Γ is semi-simple. Thus Γ doesn't contain elements with a parabolic component. If $\gamma = (\gamma_1, \gamma_2)$ belongs to a Hilbert modular lattice, then we have $\mathrm{tr}(\gamma_2) = \mathrm{tr}(\gamma_1)^\sigma$. Hence γ_1 is parabolic if and only if γ_2 is also parabolic. Therefore, if Γ is a uniform irreducible lattice or a Hilbert lattice, the non-trivial elements of Γ are of the following kinds:

- hyperbolic (and there exist hyper-regular elements),
- elliptic of finite order (if and only if Γ is not torsion free),
- mixed (one component is hyperbolic and the other is elliptic),
- parabolic (if Γ is not uniform).

For general non-uniform lattices, A. Selberg proved in [Se] his arithmeticity theorem:

Theorem 2.1 (Selberg). *A non-uniform irreducible lattice Γ of G is, up to conjugation by an element of $\mathrm{PGL}(2,\mathbb{R})\times\mathrm{PGL}(2,\mathbb{R})$, commensurable to a Hilbert modular lattice.*

Hence the previous classification of elements is valid for any irreducible lattice of G.

The following result can be seen like a refinement of the density of the projections of an irreducible lattice on each factor of G. This statement is not symmetric but also true with a permutation of indices.

Proposition 2.2. *Let Γ be an irreducible lattice of G. Let g_1 belong to $\mathrm{PSL}(2,\mathbb{R})$ and η_2^+,η_2^- belong to $\partial\mathbb{H}$. Then there exists a sequence $(\gamma_n)_n$ in the lattice Γ satisfying:*
1) $\lim_{n\to+\infty} p_1(\gamma_n)=g_1$,
2) for every point z_2 in $\mathbb{H}$: $\lim_{n\to+\infty} p_2(\gamma_n)z_2=\eta_2^+$ *and* $\lim_{n\to+\infty} p_2(\gamma_n)^{-1}z_2=\eta_2^-$,
3) for every point ξ_2 of $\partial\mathbb{H}$, distinct from η_2^-: $\lim_{n\to+\infty} p_2(\gamma_n)(\xi_2)=\eta_2^+$.

Proof. Recall that it is sufficient to prove assertion *2)* for one point z_2. Let V_1 be a neighborhood of g_1 in $\mathrm{PSL}(2,\mathbb{R})$ and V_2^+ (resp. V_2^-) be a neighborhood of η_2^+ (resp. η_2^-) in $\partial\mathbb{H}$. There exists an element α in Γ such that $p_1(\alpha)$ is elliptic and $p_2(\alpha)$ is hyperbolic. Let ξ_2^+ (resp. ξ_2^-) be the attractive (resp. repulsive) fixed point of $p_2(\alpha)$. If $(\alpha_n)_n$ is a appropriate subsequence of the positive powers of α, the sequence $(p_1(\alpha_n))_n$ converges to the point Id of $\mathrm{PSL}(2,\mathbb{R})$ and the sequence $(p_2(\alpha_n)z_2)_n$ (resp. $(p_2(\alpha_n)^{-1}z_2)_n$) of points of $\mathbb{H}$ converges to the point ξ_2^+ (resp. ξ_2^-) of $\partial\mathbb{H}$. The projection $p_1(\Gamma)$ is dense in $\mathrm{PSL}(2,\mathbb{R})$, hence there exists γ in Γ such that $p_1(\gamma)$ belongs to V_1. The points ξ_2^+ and ξ_2^- are distinct, the group $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on the set of distinct points of $\partial\mathbb{H}$ and the projection $p_2(\Gamma)$ is dense in $\mathrm{PSL}(2,\mathbb{R})$. Therefore, there exists an element δ in Γ such that

$$p_2(\delta)\xi_2^+\in V_2^+ \quad\text{and}\quad p_2(\delta)\xi_2^-\in p_2(\gamma)V_2^-.$$

The sequence $(\delta\alpha_n\delta^{-1}\gamma)_n$ of elements of Γ satisfies

$$\lim_{n\to+\infty} p_1(\delta\alpha_n\delta^{-1}\gamma)=p_1(\gamma)\in V_1,$$

$$\lim_{n\to+\infty} p_2(\delta\alpha_n\delta^{-1}\gamma)z_2 = p_2(\delta)(\xi_2^+) \in V_2^+,$$

$$\lim_{n\to+\infty} p_2(\delta\alpha_n\delta^{-1}\gamma)^{-1}z_2 = p_2(\gamma)^{-1}p_2(\delta)(\xi_2^-) \in V_2^-.$$

This proves the existence of a sequence $(\gamma_n)_n$ in the lattice Γ satisfying assertions *1)* and *2)*.
It remains to prove assertion *3)*. This is a consequence of *2)*. To prove this, we can omit the index "2" which is not useful. Consider the set $\mathcal{E}$ of points ξ in $\partial\mathbb{H}$ such that the sequence $(\gamma_n(\xi))_n$ does not converge to η^+. If this set contains two distincts points, they can be joined by a geodesic line and any point z on this geodesic cannot satisfies

$$\lim_{n\to+\infty} \gamma_n z = \eta^+.$$

Hence $\mathcal{E}$ contains at most one point ξ and it remains to show that this point (if it exists) is η^-. Assume this is false and let V be a (small) neighborhood of η^+ in $\partial\mathbb{H}$. Then, for n large enough, $\gamma_n(\eta^-)$ belongs to V, and there exists a neighborhood W of η^- in $\partial\mathbb{H}$ such that $\gamma_n W$ is contained in V, for all n. The points ξ and η^- can be joined by a geodesic line. Let z be a point of this line and for any n, let σ_n be the oriented geodesic line passing through $\gamma_n^{-1}z$ and z (in this order). Denote by $\sigma_n(-\infty)$ and $\sigma_n(+\infty)$ the extremities of this geodesic. The points $\gamma_n(\sigma_n(-\infty))$ and $\gamma_n(\sigma_n(+\infty))$ are the extremities of the geodesic line $\gamma_n\sigma_n$ passing through z and $\gamma_n z$ (in this order). The point $\gamma_n(\sigma_n(-\infty))$ belongs to V, and the fact that $(\gamma_n z)_n$ converges to η^+ implies that $\gamma_n(\sigma_n(+\infty))$ also belongs to V. This is impossible because the geodesic line $\gamma_n\sigma_n$ contains the point z. □

We obtain the following corollary which allows to simplify the setting of Margulis' conjecture and the proof of Lindenstrauss-Weiss' result in this situation (see [F] for details).

Corollary 2.3. *Let* Γ *be an irreducible lattice of* $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ *and* F *be a closed connected subgroup of* G *strictly containing the group* A. *Then, in* $\Gamma\backslash\mathrm{G}$, *every orbit of* F *is dense.*

Proof. Let U^+ (resp. U^-) be the upper (resp. lower) unipotent subgroup of G and $\mathrm{A}_1 \times \mathrm{A}_2$, $\mathrm{U}_1^+ \times \mathrm{U}_2^+$ and $\mathrm{U}_1^- \times \mathrm{U}_2^-$ be the canonical decomposition of A, U^+ and U^- in $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. If F is a connected subgroup of G strictly containing the group A, then its Lie algebra is invariant under the adjoint action of A. Thus F contains one of the four unipotent triangular subgroups $\mathrm{U}_1^+, \mathrm{U}_1^-, \mathrm{U}_2^+, \mathrm{U}_2^-$ of G. It is therefore sufficient (up to transposition and permutation of indices) to prove that the subgroup $A_2U_2^+$ acts minimally on $\Gamma\backslash G$. We consider the "dual" action of Γ on $G/A_2U_2^+$. This coset can be identified with $PSL(2,\mathbb{R}) \times \partial\mathbb{H}$, on which the action of Γ is given by

$$\gamma(h_1, \xi_2) = (p_1(\gamma)h_1, p_2(\gamma)\xi_2).$$

The minimality of the action is therefore an easy consequence of the previous proposition (assertions *1* and *3*). □

3 Parabolic points and horoballs

Let $\beta_.(.,.)$ be the classical Busemann cocycle on $\mathbb{H}$ whose sign is fixed by the following equality

$$\beta_\infty(x,y) = \ln\left(\frac{\mathrm{Im}x}{\mathrm{Im}y}\right) \qquad \text{for } x, y \text{ in } \mathbb{H}.$$

We define on $\mathbb{H}\times\mathbb{H}$ the cocycle $\beta_\xi(.,.)$ as the sum of the Busemann cocycles on each factor: if $\xi = (\xi_1, \xi_2)$ belongs to $\mathcal{F} = \partial\mathbb{H}\times\partial\mathbb{H}$ and $z = (z_1, z_2), z' = (z_1', z_2')$ belong to $\mathbb{H}\times\mathbb{H}$,

$$\beta_\xi(z,z') = \beta_{\xi_1}(z_1, z_1') + \beta_{\xi_2}(z_2, z_2').$$

When the point ξ is (∞,∞), the cocycle is then given by

$$\beta_\infty(z,z') = \ln\left(\frac{\mathrm{Im}z_1\mathrm{Im}z_2}{\mathrm{Im}z_1'\mathrm{Im}z_2'}\right).$$

Definition. Fix a point z_o in $\mathbb{H}\times\mathbb{H}$. Let ξ be a point in the boundary $\mathcal{F} = \partial\mathbb{H}\times\partial\mathbb{H}$ and T be a real. The *horoball* based at ξ and of level T is the subset

$$\mathrm{HB}(\xi, T) = \{z \in \mathbb{H}\times\mathbb{H} \ : \ \beta_\xi(z, z_o) > T\}.$$

The following lemma will be used in the proof of theorem 1.4.

Lemma 3.1. *Fix a point $z = (z_1, z_2)$ in $\mathbb{H}\times\mathbb{H}$. Let $\xi = (\xi_1, \xi_2)$ be in $\mathcal{F}$, $g = (g_1, g_2)$ be in G and $a = (a_1, a_2)$ be a non-trivial element of* A. *Assume that a horoball* $\mathrm{HB}(\xi, T)$ *based at ξ contains infinitely many points ga^nz, where n is a positive integer. Then*

$$\xi_1 = g_1a_1^+ \text{ if } a_2 = \mathrm{Id} \qquad (\text{resp. } \xi_2 = g_2a_2^+ \text{ if } a_1 = \mathrm{Id}),$$

where a_i^+ (which equals 0 or ∞) is the attractive point in $\partial\mathbb{H}$ of the hyperbolic isometry a_i.

Proof. It is sufficient to prove the first assertion. If $a_2 = \mathrm{Id}$, a_1 is non-trivial and the sequence $(a_1^n z_1)_n$ of $\mathbb{H}$ goes to a_1^+ when n goes to infinity. Hence $(\beta_{g_1^{-1}\xi_1}(a_1^n z_1, g_1^{-1} z_1))_n$ goes to $-\infty$ if $g_1^{-1}\xi_1$ is different from a_1^+. The point $ga^n z$ belongs to $\mathrm{HB}(\xi, T)$, therefore we have

$$\beta_{g_1^{-1}\xi_1}(a_1^n z_1, g_1^{-1} z_1) + \beta_{g_2^{-1}\xi_2}(z_2, g_2^{-1} z_2) = \beta_{g^{-1}\xi}(a^n z, g^{-1} z) = \beta_\xi(ga^n z, z) > T.$$

Consequently the point $g_1^{-1}\xi_1$ is equal to a_1^+. □

Definition. A point ξ of the boundary $\mathcal{F}$ is said to be *parabolic* (with respect to a subgroup Γ of G) if it is fixed by a parabolic element of Γ.

For instance, the point (∞, ∞) is parabolic with respect to any Hilbert modular lattice of G and its stabilizer in such a lattice is given by the set of matrices

$$\left\{ \pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \; : \; a \in \mathcal{O}^\times, b \in \mathcal{O} \right\}.$$

The Galois conjugate a^σ of an element a of the unit group $\mathcal{O}^\times$ is equal to $\pm a^{-1}$ because $|aa^\sigma| = 1$ hence every hyperbolic element of the stabilizer is conjugate to an element of the subgroup

$$\mathrm{A}' = \left\{ \left(\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \pm \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \right) \; : \; \lambda \in \mathbb{R}^* \right\}$$

which preserves each horoball based on (∞, ∞) and all of whose non-trivial element is hyperbolic but non hyper-regular. The following results on stabilizers of parabolic points are proved in [Sh].

Proposition 3.2. *Let Γ be an irreducible lattice of* G*, ξ be a parabolic point in $\mathcal{F}$ and Γ_ξ the stabilizer of ξ in Γ.*

- There exists a real number T such that:

$$\gamma\mathrm{HB}(\xi, T) \cap \mathrm{HB}(\xi, T) = \emptyset \quad \textit{for every element } \gamma \textit{ in } \Gamma - \Gamma_\xi.$$

- If $\xi = g(\infty, \infty)$, the stabilizer Γ_ξ is a (uniform) lattice in the solvable subgroup $g\mathrm{A}'\mathrm{U}^+ g^{-1}$. In particular this subgroup does not contain hyper-regular element and globally preserves each horoball based on ξ.

Hence a parabolic point is fixed by a hyperbolic element. Two parabolic points are *conjugate* if they are fixed by the same hyperbolic element.

The following theorem (called *Property (F)* in [Sh]) which is obvious for uniform lattices and known for Hilbert modular lattices is true for any irreducible lattice of G, using the arithmeticity theorem 2.1.

Theorem 3.3. *Let Γ an irreducible lattice in* G*. Then there exists a real number T and some representatives $\xi^1, \ldots, \xi^r$ in $\mathcal{F}$ of all equivalence classes of parabolic points such that Γ has a fundamental domain $\mathcal{D}$ in $\mathbb{H} \times \mathbb{H}$ which is a disjoint union*

$$\mathcal{D} = K \sqcup V_1 \sqcup \ldots \sqcup V_r,$$

where K is a compact subset and V_i is a fundamental domain for the action of Γ_{ξ^i} on the horoball $\mathrm{HB}(\xi^i, T)$.

Corollary 3.4. *The complement of ΓK in $\mathbb{H} \times \mathbb{H}$ is a disjoint union of horoballs of level T based on all parabolic points of $\mathcal{F}$.*

4 Closed orbits

In this section, we prove theorems 1.4 and 1.3 using the following classical result. Let Γ be a discrete subgroup of G and A be the maximal diagonal subgroup of G. Let g be an element of G and x the class of g in $\Gamma\backslash$G. Then the (well-defined) map

$$\Psi_x : (g^{-1}\Gamma g \cap \mathrm{A})\backslash\mathrm{A} \longrightarrow \Gamma\backslash\mathrm{G},\ (g^{-1}\Gamma g \cap \mathrm{A})a \longmapsto xa$$

is continuous, injective and its image is precisely the orbit xA.

Lemma 4.1. *Assume moreover that Γ is an irreducible lattice of* G*. With these notations:*
1) The orbit xA is closed in $\Gamma\backslash$G if and only if the application Ψ_x is proper,
2) If the subgroup $g^{-1}\Gamma g \cap \mathrm{A}$ is non-trivial, 1) holds.

Proof. 1) If Ψ_x is proper, then xA is closed. Conversely, assume that xA is closed and Ψ_x is not proper: there exists a sequence $(a_n)_n$ in A diverging in $(g^{-1}\Gamma g \cap \mathrm{A})\backslash\mathrm{A}$ but such that the sequence $(xa_n)_n$ converges in $\Gamma\backslash$G to a point xa, with a in A. Then there exists a sequence $(\gamma_n)_n$ in Γ such that $(\gamma_n x a_n)_n$ converges in G to ga, hence in G/A to the class gA of g in G/A. But the set ΓgA is assumed to be closed in G, therefore the Γ-orbit of the class gA is closed hence discrete (a consequence of the countability of Γ and the Baire's property of the quotient set G/A). So the sequence $(\gamma_n g\mathrm{A})_n$ is stationary in G/A: there exists a' in A and indices $m > n$ such that

$$\gamma_m \neq \gamma_n \quad \text{and} \quad \gamma_m g = \gamma_n g a'.$$

Hence $g^{-1}\Gamma g \cap \mathrm{A}$ is non-trivial and it remains to prove the second part of the lemma.

2) Assume there exists a non-trivial element γ in Γ such that $g^{-1}\gamma g$ belongs to A. By the hypothesis of irreducibility, the centralizer of $g^{-1}\gamma g$ in G is precisely A. The properness of Ψ_x follows from the following. Assume that the sequence $(ga_n)_n$ is convergent modulo Γ: there exists a point h in G and a sequence $(\gamma_n)_n$ in Γ such that $(\gamma_n g a_n)_n$ converges to h. The sequence of elements

$$\gamma_n \gamma \gamma_n^{-1} = \gamma_n g a_n (g^{-1}\gamma g) a_n^{-1} g^{-1} \gamma_n^{-1}$$

converges to $hg^{-1}\gamma g h^{-1}$. Thus the sequence $(\gamma_n \gamma \gamma_n^{-1})_n$ is stationary: there exists a index n_o such that

$$\gamma_n \gamma \gamma_n^{-1} = \gamma_{n_o} \gamma \gamma_{n_o}^{-1}.$$

Consequently, each element $\gamma_{n_o}^{-1}\gamma_n$ commutes with γ. Therefore $g^{-1}\gamma_{n_o}^{-1}\gamma_n g$ belongs to $g^{-1}\Gamma g \cap \mathrm{A}$. The sequence $(g^{-1}\gamma_{n_o}^{-1}\gamma_n g a_n)_n$ converges to $g^{-1}\gamma_{n_o}^{-1} h$ thus $(a_n)_n$ is convergent modulo $g^{-1}\Gamma g \cap \mathrm{A}$ □

Proof of theorem 1.4. We denote by Λ the discrete subgroup $g^{-1}\Gamma g \cap \mathrm{A}$ of A.

1) $\Rightarrow$ *3)* We assume that $x\mathrm{A}$ is closed but non-compact. Then Λ is not a uniform lattice in A. Let z_o be a point in $\mathbb{H} \times \mathbb{H}$ and consider the following proper map:

$$\pi : \Gamma\backslash \mathrm{G} \to \Gamma\backslash \mathbb{H} \times \mathbb{H},\ \Gamma g \longmapsto \Gamma g z_0.$$

The orbit $x\mathrm{A}$ is closed and non-compact, thus by lemma 4.1 the map Ψ_x is proper and the subset $(\pi \circ \Psi_x)^{-1}(K)$ is a compact subset of the non-compact set $\Lambda\backslash\mathrm{A}$.

Assume Λ is trivial. Then $(\pi \circ \Psi_x)^{-1}(K)$ is a compact subset of A. Let C be the unique unbounded connected component of $\mathrm{A} - (\pi \circ \Psi_x)^{-1}(K)$; thus the subset gCz_o of $\mathbb{H} \times \mathbb{H}$ is connected and disjoint from ΓK. Therefore, by corollary 3.4, it is contained in a horoball $\mathrm{HB}(\xi, T)$ based at a parabolic point. There exists an element $a = (a_1, \mathrm{Id})$ in C, $a_1 \neq \mathrm{Id}$, such that the elements a^n and a^{-n} belong to C for every positive integer n. The points $ga^n z_o$ and $ga^{-n} z_o$ therefore belong to $\mathrm{HB}(\xi, T)$. The lemma 3.1 imply that the point ξ_1 is equal to $g_1(\infty)$ and $g_1(0)$. This is a contradiction. Consequently the discrete subgroup Λ of A is not trivial. It is therefore isomorphic to $\mathbb{Z}$ because $\Lambda\backslash\mathrm{A}$ is not compact.

The set $(\pi \circ \Psi_x)^{-1}(K)$ is a compact subset of the cylinder $\Lambda\backslash\mathrm{A}$, thus it is contained in a subset $\Lambda\backslash B$ of $\Lambda\backslash\mathrm{A}$ where B is a "band" in A, invariant by the subgroup Λ. The connected components C^+ and C^- of $\mathrm{A} - B$ satisfy

$$gC^+ z_o \subseteq \mathrm{HB}(\xi^+, T) \quad \text{and} \quad gC^- z_o \subseteq \mathrm{HB}(\xi^-, T)$$

for some parabolic points ξ^+ and ξ^-. The group Λ is generated by an element $g^{-1}\gamma g$ where γ belongs to Γ. Its components are non-trivial (by irreducibility of Γ), therefore C^+ and C^- contain non-trivial elements of the form (a_1, Id) and (Id, a_2). Applying once again the lemma 3.1 with such elements, we obtain

$$\{\xi^+, \xi^-\} = \{g(0,0), g(\infty,\infty)\} \quad \text{or} \quad \{g(0,\infty), g(0,\infty)\},$$

according to the position of C^+ and C^-. Moreover, the points ξ^+ and ξ^- are fixed by $\gamma = gag^{-1}$.

3) $\Rightarrow$ *2)* If an element γ of Γ fixes the parabolic points $g(0,0)$ and $g(\infty,\infty)$ (or $g(0,\infty)$ and $g(0,\infty)$), the diagonalizable subgroup $\Gamma \cap g\mathrm{A}g^{-1}$ is non-trivial and it cannot be isomorphic to $\mathbb{Z}^2$ since it is contained in the stabilizer of a parabolic point (proposition 3.2).

2) $\Rightarrow$ *1)* Under the assumption *2)*, the map Ψ_x is proper by lemma 4.1. Thus $x\mathrm{A}$ is closed, but not compact since Λ is not a lattice in A. □

Proof of the theorem 1.3. Recall that Λ denote the subgroup $g^{-1}\Gamma g \cap \mathrm{A}$.

1) $\Rightarrow$ *2)* The orbit is compact, therefore the map Ψ_x is proper and $\Lambda\backslash\mathrm{A}$ is compact. The discrete subgroup Λ has to be a lattice in A.

2) $\Rightarrow$ *3)* Since any lattice in A contains a hyper-regular hyperbolic element, the lattice Λ contains an element $g^{-1}\gamma g$ where γ is a hyper-regular element of Γ. Thus γ fixes both points $g(\infty,\infty)$ and $g(0,0)$.

3) $\Rightarrow$ *1)* Let γ be a hyper-regular element fixing $g(\infty,\infty)$ and $g(0,0)$. Then $g^{-1}\gamma g$ belongs to Λ and the orbit $x\mathrm{A}$ is closed by lemma 4.1. This orbit cannot be non-compact by theorem 1.4 since γ is not contained in the stabilizer of a parabolic point. □

5 Dense orbits

In this section, we prove theorem 1.2. We use here the "geometric language": an element of $\mathrm{PSL}(2,\mathbb{R})$ can be seen as a unit tangent vector (or a geodesic ray) of the hyperbolic plane. Therefore an element of $\mathrm{G} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ can be seen as a product of two unit tangent vectors of $\mathbb{H}$ (or a Weyl chamber in $\mathbb{H} \times \mathbb{H}$). The action (by right translation) of the diagonal group A (resp. the diagonal semigroup A^+) on G is then conjugated to the product of the geodesic flows (resp. in positive time). We will use the notation ϕ^t for the geodesic flow at time t. This is in fact the diagonal matrix

$$\pm \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$$

(for the appropriate normalization of the metric).

Proof of theorem 1.2. Let $g = (g_1, g_2)$ belong to G and denote by $z = (z_1, z_2)$ the basepoint in $\mathbb{H} \times \mathbb{H}$ of the Weyl chamber g.

1) We can assume (up to a permutation) that the fixed point is $g_1(\infty)$. This point is fixed by $h_1 = p_1(\gamma)$ where $\gamma = (h_1, e_2)$ is a mixed element of Γ. The hyperbolic isometry h_1 of $\mathbb{H}$ fixes the points h_1^+ and h_1^- of $\partial\mathbb{H}$ and the elliptic isometry e_2 of $\mathbb{H}$ fixes a point x_2 in $\mathbb{H}$. Replacing, if necessary, γ by γ^{-1} we assume that $g_1(\infty)$ is the attractive fixed point h_1^+ of h_1.
We prove this first part in three steps, decreasing assumptions in each step.
Step 1: We add here the following assumptions: $g_1(0) = h_1^-$ and $z_2 = x_2$.
Since the element $\gamma = (h_1, e_2)$ belongs to an irreducible lattice, the component e_2 is of infinite order. Therefore it generates a dense semi-subgroup of the stabilizer (conjugated to $\mathrm{PSO}(2, \mathbb{R})$) of the point x_2. If k is any (elliptic) isometry of $\mathbb{H}$ fixing the point x_2, there then exists a sequence $(m_n)_n$ of positive integers, going to $+\infty$, such that

$$\lim_{n \to +\infty} e_2^{-m_n} = k.$$

The point $g_1(\infty)$ (resp. $g_1(0)$) is the attractive (resp. repulsive) fixed point of the hyperbolic isometry h_1, therefore there exists an element a_1 in the diagonal semi-subgroup A_1^+ of $\mathrm{PSL}(2, \mathbb{R})$ such that

$$h_1 g_1 = g_1 a_1.$$

The element

$$(g_1, k g_2) = \lim_{n \to +\infty} \gamma^{-m_n} g(a_1^{m_n}, \mathrm{Id})$$

belongs to the set $\overline{\Gamma g \mathrm{A}^+}$. Let $\tilde{g}_1$ be in $\mathrm{PSL}(2, \mathbb{R})$. By proposition 2.2, there exists a sequence $(\gamma_n)_n$ in Γ such that $(p_1(\gamma_n))_n$ converges to $\tilde{g}_1 g_1$ and $(p_2(\gamma_n) z_2)_n$ converges to the point $g_2(0)$. For each n, there exists an elliptic oriented isometry k_n of $\mathbb{H}$ fixing z_2 such that $p_2(\gamma_n) k_n g_2$ defines the unit tangent vector of $\mathbb{H}$ based on the point $p_2(\gamma_n) z_2$ and tangent to the geodesic line passing (in this order) through the points $p_2(\gamma_n) z_2$ and z_2. Therefore, for every n, the unit tangent vector

$$u_n = p_2(\gamma_n) k_n g_2 \phi^{d(z_2, p_2(\gamma_n) z_2)}$$

is based on z_2 and defines an oriented geodesic line containing the point $p_2(\gamma_n) z_2$. Since this last point converges to $g_2(0)$, the sequence $(u_n)_n$ converges to the unit tangent vector defined by g_2. We have

$$\lim_{n \to +\infty} \gamma_n (g_1, k_n g_2)(\mathrm{Id}, \phi^{d(z_2, p_2(\gamma_n) z_2)}) = (\tilde{g}_1, g_2)$$

which therefore belongs to $\overline{\Gamma g \mathrm{A}^+}$. We proved that $\mathrm{PSL}(2,\mathbb{R}) \times \{g_2\}$ is contained in $\overline{\Gamma g \mathrm{A}^+}$. The fact that the second projection $p_2(\Gamma)$ is dense in G_2 implies that $\overline{\Gamma g \mathrm{A}^+}$ equals G, that is to say $x\mathrm{A}^+$ is dense in $\Gamma\backslash\mathrm{G}$.
Step 2: Here we add only the assumption that $g_1(0) = h_1^-$.
By the same argument as in step 1, for every isometry k fixing the point x_2, the element (g_1, kg_2) belongs to the set $\overline{\Gamma g \mathrm{A}^+}$. By proposition 2.2, there exists a sequence $(\gamma_n)_n$ in Γ such that

$$\lim_{n\to+\infty} p_1(\gamma_n) = \mathrm{Id} \quad \text{and} \quad \lim_{n\to+\infty} p_2(\gamma_n)y_2 = g_2(\infty) \text{ for every point } y_2 \text{ in } \mathbb{H}.$$

The sequence $(p_2(\gamma_n))_n$ is divergent in $\mathrm{PSL}(2,\mathbb{R})$, therefore the sequence $(p_2(\gamma_n)^{-1}x_2)_n$ is divergent in $\mathbb{H}$. For large enough n, the point z_2 is contained in the open hyperbolic disc of center x_2 and of radius $d(x_2, p_2(\gamma_n)^{-1}x_2)$. Thus the geodesic ray (defined by g_2) based on z_2 and directed to $g_2(\infty)$ intersects the hyperbolic circle of center x_2 and passing through the point $p_2(\gamma_n)^{-1}x_2$. This intersection is a point $k_n p_2(\gamma_n)^{-1}x_2$ where k_n belongs to the stabilizer of x_2 in $\mathrm{PSL}(2,\mathbb{R})$ (Fig.1).

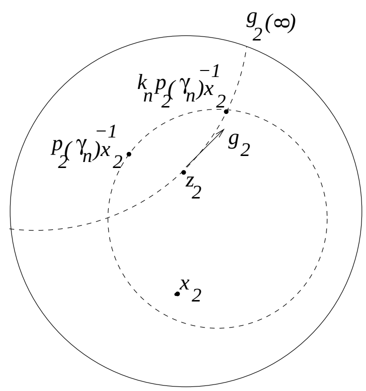

Fig. 1

The geodesic ray defined by $p_2(\gamma_n)k_n^{-1}g_2$ contains the point x_2. If t_n equals the distance $d(x_2, p_2(\gamma_n)k_n^{-1}z_2)$, then the element $p_2(\gamma_n)k_n^{-1}g_2\phi^{t_n}$ defines for each n a unit tangent vector based on the point x_2. Passing to a subsequence, it converges to a unit tangent vector $\tilde{g}_2$ based on x_2 and satisfying

$$\lim_{n\to+\infty} \gamma_n(g_1, k_n^{-1}g_2)(\mathrm{Id}, \phi^{t_n}) = (g_1, \tilde{g}_2).$$

Therefore $\overline{\Gamma g \mathrm{A}^+}$ contains $\Gamma(g_1, \tilde{g}_2)\mathrm{A}^+$ which is dense in G by step 1.
Step 3: No additional assumption, we only assume that $g_1(\infty) = h_1^+$.

Since the point $g_1(\infty)$ is distinct from the point h_1^-, $s = g_1^{-1}(h_1^-)$ is a real number and the element $u = \pm \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ of $\mathrm{PSL}(2,\mathbb{R})$ satisfies the following:

$$u^{-1}g_1^{-1}h_1g_1u = a_1 = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $\lambda > 1$ is the greatest eigenvalue of h_1. We have

$$u^{-1}g_1^{-1}h_1^{-n}g_1u = a_1^{-n}, \quad h_1^{-n}g_1a_1^n = g_1ua_1^{-n}u^{-1}a_1^n,$$

$$\text{and} \quad \lim_{n\to+\infty} h_1^{-n}g_1a_1^n = g_1u \quad \text{because} \quad \lim_{n\to+\infty} a_1^{-n}u^{-1}a_1^n = \mathrm{Id}.$$

Since the isometry $p_2(\gamma) = e_2$ is elliptic, there exists a divergent sequence $(m_n)_n$ of positive integers such that

$$\lim_{n\to+\infty} e_2^{-m_n} = \mathrm{Id}.$$

We have

$$\lim_{n\to+\infty} \gamma^{-m_n}g(a_1^n, \mathrm{Id}) = (g_1u, g_2)$$

which is therefore an element of $\overline{\Gamma g \mathrm{A}^+}$. But this element (g_1u, g_2) satisfies: $g_1u(\infty)$ and $g_1u(0)$ are fixed by $p_1(\gamma)$. Therefore, by step 2, $\Gamma(g_1u, g_2)\mathrm{A}^+$ is dense in G and so is $\Gamma g\mathrm{A}^+$.

2) We prove here the second part of the theorem. Now we assume that exactly one of the points $g_1(\infty)$ and $g_2(\infty)$ is fixed by a hyper-regular element $\gamma = (h_1, h_2)$ of Γ. Let $h_i^\pm$ be the fixed points of h_i. We may assume that $g_1(\infty)$ is fixed by (the first component h_1 of) γ and $g_2(\infty)$ is not fixed by (the second component h_2 of) γ. The centralizer of γ in G is a conjugate of the subgroup A; we denote it by $\mathrm{Z} = \mathrm{Z}_1 \times \mathrm{Z}_2$. Since the element γ of the irreducible lattice Γ is hyper-regular, the discrete subgroup $\mathrm{Z} \cap \Gamma$ is a lattice in Z by theorem 1.3. The element γ is a positive power of a primitive element in the lattice $\mathrm{Z} \cap \Gamma$. Thus we can assume that γ is primitive: there exists another element $\gamma' = (h_1', h_2')$ in the lattice $\mathrm{Z} \cap \Gamma$ such that $\mathrm{Z} \cap \Gamma$ is generated by γ and γ'. For $i = 1, 2$, h_i and h_i' have the same fixed points $h_i^\pm$. Moreover we can assume they have the same attractive (resp. repulsive) point and their eigenvalues are such that the elements γ and γ' have the following position in (the Lie algebra of) Z (see Fig.2).

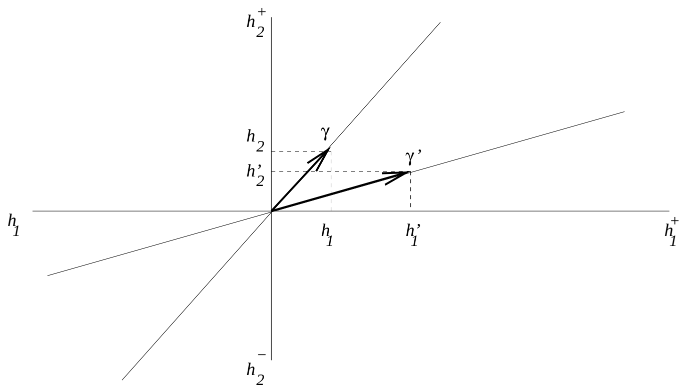

Fig. 2

By irreducibility, the restriction to the lattice $\mathrm{Z} \cap \Gamma$ of each projection

$$p_i \; : \; \mathrm{Z} \longrightarrow \mathrm{Z}_i \qquad i = 1, 2,$$

is injective. Consequently the images $p_i(\mathrm{Z} \cap \Gamma)$ are dense in Z_i and the closure of the sub-semigroup

$$\left\{ h_2^m {h'_2}^{-m'} \mid m, m' \in \mathbb{Z}, m \geq 0, m' \geq 0 \right\}$$

of the group Z_2 has a non-empty interior. In the second factor $\partial\mathbb{H}$ of $\mathcal{F}$, the set of points fixed by a mixed element of the lattice Γ is dense (it is non-empty and invariant under $p_2(\Gamma)$). Therefore there exists a mixed element $f = (f_1, f_2)$ in Γ, where f_1 is elliptic and f_2 is hyperbolic fixing a point f_2^+ which belongs to the set

$$\overline{\left\{ h_2^m {h'_2}^{-m'} g_2(\infty) \mid m, m' \in \mathbb{Z}, m \geq 0, m' \geq 0 \right\}},$$

(which has non-empty interior since $g_2(\infty)$ is fixed neither by h_2 nor by h'_2). There exist sequences $(m_n)_n$ and $(m'_n)_n$ of positive integers such that the sequence $(h_2^{m_n} {h'_2}^{-m'_n})_n$ converges to an element c of Z_2 satisfying:

$$c g_2(\infty) = f_2^+.$$

By discreteness of Γ, the sequence of isometries $(h_1^{-m_n} {h'_1}^{m'_n})_n$ diverges (in Z_1). The choice of γ and γ' implies moreover that, for n large enough, the attractive fixed point of $(h_1^{-m_n} {h'_1}^{m'_n})_n$ is h_1^+. We now apply the same kind of argument than in step 3. The point $g_1(\infty)$ is distinct from the point

h_1^-, thus $s = g_1^{-1}(h_1^-)$ is a real number and the element $u = \pm\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$ of $\mathrm{PSL}(2,\mathbb{R})$ satisfies the following:

$$u^{-1}g_1^{-1}h_1^{-m_n}h_1'^{\,m_n'}g_1u = \pm\begin{pmatrix}\lambda_n & 0\\ 0 & \lambda_n^{-1}\end{pmatrix} = \phi^{2\ln\lambda_n}$$

where $(\lambda_n)_n$ is the sequence of the greatest eigenvalue of $h_1^{-m_n}h_1'^{\,m_n'}$. This sequence goes to $+\infty$ and we have

$$\lim_{n\to+\infty}(h_1^{-m_n}h_1'^{\,m_n'})^{-1}g_1\phi^{2\ln\lambda_n} = g_1u \quad \text{because} \quad \lim_{n\to+\infty}\phi^{-2\ln\lambda_n}u^{-1}\phi^{2\ln\lambda_n} = \mathrm{Id}.$$

Therefore we obtain

$$\lim_{n\to+\infty}\gamma^{m_n}\gamma'^{-m_n'}g(\phi^{2\ln\lambda_n},\mathrm{Id}) = (g_1u, cg_2)$$

and this limit belongs to the closure $\overline{\Gamma g\mathrm{A}^+}$. Since the point $cg_2(\infty)$ is fixed by the mixed element f of Γ, then the point (g_1u, cg_2) has a dense semi-orbit by 1). Therefore $\Gamma g\mathrm{A}^+$ is also dense in G. □

Remark 5.1. *In step 1, we used the property that, if an elliptic isometry of $\mathbb{H}$ fixing a point z is of infinite order, it generates a dense subgroup in the stabilizer of z. This property is not true for elliptic isometries of higher-dimensional hyperbolic spaces. This is the obstruction to generalizing theorem 1.2 to the Weyl chamber flow on irreducible quotients $\Gamma\backslash\mathbb{H}^n\times\mathbb{H}^n$ when $n\geq 3$.*

Proof of the corollary 1.6. Here we assume that Γ is a uniform irreducible lattice and contains a non-trivial element γ which fixes one of the points $g_1(0)$,$g_1(\infty)$,$g_2(0)$ or $g_2(\infty)$. This element γ is semi-simple (because Γ is uniform) but cannot be elliptic. Thus it is mixed or hyperbolic. If it is mixed, the orbit $x\mathrm{A}$ of the point $x = \Gamma g$ is dense by theorem 1.2 and remark 1.5 *1)*. If it is hyperbolic, there exists an element $h = (h_1, h_2)$ in G such that $h^{-1}\gamma h$ belongs to the subgroup A. The orbit $y\mathrm{A}$ of the point $y = \Gamma h$ is closed by lemma 4.1. Therefore it is compact and theorem 1.3 implies the existence of a hyper-regular element γ' in Γ which commutes with γ. Thus γ' fixes the same points in $\mathcal{F}$. If all the points $g_1(0), g_1(\infty), g_2(0)$ and $g_2(\infty)$ are fixed by γ', the orbit is compact by theorem 1.3, otherwise the orbit is dense by theorem 1.2 and remark 1.5 *1)*. □

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Damien Ferte
IRMAR – Université Rennes I
35042 RENNES CEDEX
FRANCE
damien.ferte@math.univ-rennes1.fr