

# THE ODD-DIMENSIONAL GOLDBERG CONJECTURE

VESTISLAV APOSTOLOV, TEDI DRĂGHICI AND ANDREI MOROIANU

ABSTRACT. An odd-dimensional version of the Goldberg conjecture was formulated and proved in [5], by using an orbifold analogue of Sekigawa's arguments in [8], and an approximation argument of K-contact structures with quasi-regular ones. We provide here another proof of this result.

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## 1. INTRODUCTION

The celebrated Goldberg conjecture states that every compact almost Kähler Einstein manifold  $M$  is actually Kähler-Einstein. This conjecture was confirmed by Sekigawa [8] in the case when  $M$  has non-negative scalar curvature. The odd-dimensional analogues of Kähler manifolds are Sasakian manifolds, and those of almost Kähler manifolds are K-contact manifolds. In [5], Boyer and Galicki proved the following odd-dimensional analogue of Goldberg's conjecture.

**Theorem 1.1.** [5] *Any compact Einstein K-contact manifold  $(M, g, \xi)$  is Sasakian.*

Their proof goes roughly as follows. First, an Einstein K-contact manifold has prescribed (positive) Einstein constant. If the K-contact structure is quasi-regular (i.e. the orbits of the Reeb vector field  $\xi$  are closed), then the quotient of  $M$  by the flow of  $\xi$  is an almost Kähler orbifold [9] which is Einstein with positive scalar curvature by the O'Neill formulas. One then applies Sekigawa's proof to obtain that the almost Kähler structure is integrable, which in turn means that the K-contact structure is Sasakian. If the K-contact structure is not quasi-regular, the space of orbits of  $\xi$  is not an orbifold (and may not be even a tractable topological space). To overcome this difficulty, the authors of [5] provide a beautiful argument showing that the Reeb vector field  $\xi$  can be approximated (in a suitable sense) by a sequence of quasi-regular Reeb vector fields  $\xi_i$  which define K-contact structures on a sequence of (no longer Einstein) metrics  $g_i$  approaching  $g$ . Then for the sequence of orbifolds thus obtained, one can use "approximative" Sekigawa formulas and eventually show that the K-contact structure is integrable.

The aim of this note is to give another proof of Theorem 1.1 and to study further possible extensions. Instead of the quotient of  $M$  by the Reeb flow, we consider another almost Kähler manifold naturally associated to  $M$ , namely the cone over  $M$ . It is well-known that the cone is a smooth, *non-compact* Ricci-flat almost Kähler manifold which is Kähler if and only if  $M$  is Sasakian. It therefore suffices to prove the integrability of the almost Kähler cone structure. It would seem to be difficult to apply directly Sekigawa's arguments in this situation because of the non-compactness of the cone. But this can be overcome easily: we first apply a point-wise version of Sekigawa's formula on the cone manifold, and then integrate it on the level sets of the radial function (which are compact manifolds).

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The use of this approach tempted us to extend the conjecture to the more general case of *contact metric structures*, when the metric is no longer bundle-like. Indeed, one could argue that the analogue of almost Kähler manifolds in odd dimensions are the contact metric structures, since they correspond to the level sets of the radial function of almost Kähler cone metrics. The contact analogue of the Goldberg conjecture would then assert that *any compact Einstein contact metric manifold is Sasakian–Einstein*. This statement turns out to be false in general, as it follows from an example of D. Blair on the flat 3–torus, which we recall in the last section. Still, the counterexample does not generalize to higher dimensions, so the problem seems to be worth further investigation. We make a step in this direction; in the particular case when the Einstein metric admits already a compatible Sasakian structure, we use Theorem 1.1 to show that any other compatible contact metric structure is necessarily Sasakian.

**Theorem 1.2.** *Let  $(M, g, \xi)$  be a compact Sasakian–Einstein manifold of dimension  $2n + 1$ . Then any contact metric structure  $(\xi', g)$  on  $(M, g)$  is Sasakian. Moreover, if  $\xi'$  is different from  $\pm\xi$ , then the following two cases occur:*

- $(M, g)$  admits a 3–Sasakian structure and  $\xi$  and  $\xi'$  belong to the underlying  $S^2$ –family of Sasakian structures.
- $(M, g)$  is covered by the round sphere  $S^{2n+1}$ .

Note that the cone construction identifies the set of all Sasakian structures on the round sphere  $S^{2n+1}$  with the homogeneous space  $SO(2n + 2)/U(n + 1)$ .

## 2. PRELIMINARIES

Let  $(M, g)$  be a Riemannian manifold. We define the cone  $\bar{M} := M \times \mathbb{R}_+^*$  endowed with the metric  $\bar{g} = dr^2 + r^2g$ , and denote by  $\bar{\nabla}$  the covariant derivative of  $\bar{g}$ . It is well-known that the cone is a non-complete Riemannian manifold which can be completed at  $r = 0$  if and only if  $M$  is a round sphere.

Every vector field  $X$  on  $M$  induces in a canonical way a vector field  $(X, 0)$  on  $\bar{M}$ , which (with a slight abuse of notation) will still be denoted by  $X$ . Similarly, we denote by the same symbol the forms on  $M$  and their pull-backs to  $\bar{M}$  (with respect to the projection on the first factor). Let us denote by  $\partial_r$  the vector field  $\frac{\partial}{\partial r}$  on  $\bar{M}$ . The following formulas relate the covariant derivatives  $\nabla$  and  $\bar{\nabla}$ , and are immediate consequences of the definitions.

$$(1) \quad \bar{\nabla}_{\partial_r} \partial_r = 0; \quad \bar{\nabla}_X \partial_r = \bar{\nabla}_{\partial_r} X = \frac{1}{r} X; \quad \bar{\nabla}_X Y = \nabla_X Y - rg(X, Y) \partial_r.$$

Using this, we obtain for every vector  $X$  and a  $p$ –form  $\omega$  on  $M$

$$(2) \quad \bar{\nabla}_{\partial_r} \omega = -\frac{p}{r} \omega \quad \text{and} \quad \bar{\nabla}_X \omega = \nabla_X \omega - \frac{1}{r} dr \wedge X \lrcorner \omega,$$

$$(3) \quad \bar{\nabla}_{\partial_r} dr = 0 \quad \text{and} \quad \bar{\nabla}_X dr = rX^\flat.$$

The curvature tensors  $R$  and  $\bar{R}$  of  $M$  and  $\bar{M}$ , respectively, are related by

$$(4) \quad \bar{R}(\partial_r, \cdot) = 0 \quad \text{and} \quad \bar{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X.$$

**Definition 2.1.** A contact metric structure on a Riemannian manifold  $M$  is a unit length vector field  $\xi$  such that the 1-form  $\eta := \langle \xi, \cdot \rangle$  and the endomorphism  $\varphi$  associated to  $\frac{1}{2}d\eta$  are inter-related by

$$(5) \quad \varphi^2 = -1 + \eta \otimes \xi.$$

Since  $\varphi^2(\xi) = 0$ , we get  $|\varphi(\xi)|^2 = -\langle \xi, \varphi^2(\xi) \rangle = 0$ , so  $\varphi(\xi) = 0$ . In other words,  $\varphi$  defines a complex structure on the distribution orthogonal to  $\xi$ .

A contact metric structure  $(M, g, \xi, \varphi, \eta)$  is called  $K$ -contact if  $\xi$  is Killing. The  $K$ -contact structure  $(M, g, \xi, \varphi, \eta)$  is called Sasakian if

$$(6) \quad \nabla \cdot \nabla \xi = \xi \wedge \cdot.$$

Given a contact metric manifold  $(M, g, \xi, \varphi, \eta)$ , we construct a 2-form  $\Omega$  on  $\bar{M}$ , defined by

$$(7) \quad \Omega = r dr \wedge \eta + \frac{r^2}{2} d\eta.$$

This 2-form is clearly compatible with  $\bar{g}$ , and therefore defines an almost complex structure  $J$  on  $\bar{M}$  by  $\Omega(\cdot, \cdot) = \bar{g}(J\cdot, \cdot)$ . Moreover,  $\Omega$  is obviously closed, meaning that  $(\bar{M}, J)$  is almost Kähler. It is well-known that  $\Omega$  is parallel (i.e.  $(\bar{M}, J)$  is Kähler) if and only if the contact structure  $\xi$  is Sasakian.

We close this section with the following

**Lemma 2.2.** (i) The codifferentials on  $M$  and  $\bar{M}$  are related by

$$(8) \quad \delta^{\bar{M}}(r^k \sigma) = r^{k-2} \delta^M \sigma, \quad \forall \sigma \in \Lambda^1 M.$$

(ii) The Laplacians on  $M$  and  $\bar{M}$  are related by

$$(9) \quad \Delta^{\bar{M}}(r^k f) = r^{k-2}(\Delta^M f - k(2n + k)f), \quad \forall f \in C^\infty(M).$$

*Proof.* (i) If  $(e_i)$  denotes a local orthonormal base on  $M$ , we have

$$\begin{aligned} \delta^{\bar{M}}(r^k \sigma) &= \sum_i \left( -\frac{e_i}{r} (r^k \sigma \left( \frac{e_i}{r} \right)) + r^k \sigma \left( \bar{\nabla}_{\frac{e_i}{r}} \frac{e_i}{r} \right) - \partial_r (r^k \sigma(\partial_r)) + r^k \sigma(\bar{\nabla}_{\partial_r} \partial_r) \right) \\ &= \sum_i -r^{k-2} e_i(\sigma(e_i)) + r^{k-2} \sigma(\nabla_{e_i} e_i) - r \partial_r = r^{k-2} \delta^M \sigma. \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} \Delta^{\bar{M}}(r^k f) &= \sum_i \left( -\frac{e_i}{r} \left( \frac{e_i}{r} (r^k f) \right) + \bar{\nabla}_{\frac{e_i}{r}} \frac{e_i}{r} (r^k f) - \partial_r (\partial_r (r^k f)) \right) \\ &= \sum_i \left( -r^{k-2} e_i(e_i(f)) + \frac{1}{r^2} (\nabla_{e_i} e_i - r \partial_r)(r^k f) - k(k-1)r^{k-2} f \right) \\ &= r^{k-2} \Delta^M f - k(2n+1)r^{k-2} f - k(k-1)r^{k-2} f \\ &= r^{k-2} (\Delta^M f - k(2n+k)f). \end{aligned}$$

□

## 3. PROOF OF THEOREM 1.1

Let  $(M^{2n+1}, g, \xi)$  be a compact K-contact Einstein manifold. By a result of Blair ([2], Theorem 7.1), a contact metric manifold is K-contact if and only if  $\text{Ric}(\xi, \xi) = 2n$ ; thus, the Einstein constant in our case must be  $2n$ .

Consider now the cone  $\bar{M}$ , which is an almost Kähler manifold. We use the following Weitzenböck-type formula, taken from [1, Prop.2.1].

**Proposition 3.1.** *For any almost Kähler manifold  $(\bar{M}, \bar{g}, J, \Omega)$  with covariant derivative denoted by  $\bar{\nabla}$  and curvature tensor  $\bar{R}$ , the following point-wise relation holds:*

$$(10) \quad \begin{aligned} \Delta(s^* - s) &= -4\delta(J\delta^{\bar{\nabla}}(J\bar{\text{Ric}}'')) + 8\delta(\langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle) + 2|\bar{R}ic''|^2 \\ &\quad - 8|\bar{R}''|^2 - |\bar{\nabla}^* \bar{\nabla} \Omega|^2 - |\phi|^2 + 4\langle \rho, \phi \rangle - 4\langle \rho, \bar{\nabla}^* \bar{\nabla} \Omega \rangle, \end{aligned}$$

where:  $\phi(X, Y) = \langle \bar{\nabla}_{JX} \Omega, \bar{\nabla}_Y \Omega \rangle$ ,  $s$  and  $s^*$  are respectively the scalar and  $*$ -scalar curvature,  $\bar{\text{Ric}}''$  is the  $J$ -anti-invariant part of the Ricci tensor  $\bar{\text{Ric}}$ ,  $\rho$  is the  $(1, 1)$ -form associated to the  $J$ -invariant part of  $\bar{\text{Ric}}$ ,  $\bar{\rho}^* := \bar{R}(\Omega)$  and  $\bar{R}''$  denotes a certain component of the curvature tensor.

In our situation, since  $M^{2n+1}$  is Einstein with constant  $2n$ , (4) shows that  $\bar{M}$  is Ricci-flat. So the formula above becomes

$$(11) \quad \Delta^{\bar{M}} s^* - 8\delta^{\bar{M}}(\langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle) = -8|\bar{R}''|^2 - |\bar{\nabla}^* \bar{\nabla} \Omega|^2 - |\phi|^2$$

We now use Lemma 2.2 in order to express the left-hand side of this equality in terms of the codifferential and Laplacian on  $M$ . From (4) we get  $\bar{\rho}^*(X, \partial_r) = 0$  and  $\bar{\rho}^*(X, Y) = \bar{g}(\bar{R}(\frac{e_i}{r}, J\frac{e_i}{r})X, Y) = \rho^*(X, Y)$ , for some 2-form  $\rho^*$  on  $M$ . Taking the scalar product with  $\Omega$  yields

$$(12) \quad s^* = \frac{1}{r^2} f$$

for some function  $f$  on  $M$ . Note that  $f$  is everywhere positive on  $M$  since  $s^* = s^* - s = |\bar{\nabla} \Omega|^2$  on  $\bar{M}$  (see e.g. [1], p. 777).

Now, from (2), (3) and (7) we get  $\bar{\nabla}_{\partial_r} \Omega = 0$  and  $\bar{\nabla}_X \Omega = r^2 \omega + r dr \wedge \tau_X$  for some 2-form  $\omega$  and 1-form  $\tau_X$  on  $M$ . Consequently, the 1-form  $\langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle$  on  $\bar{M}$  is easily seen to be of the form

$$(13) \quad \langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle = \frac{1}{r^2} \alpha$$

for some 1-form  $\alpha$  on  $M$ . Using (12), (13) and Lemma 2.2, the equality (11) becomes

$$(14) \quad \frac{1}{r^4} (\Delta^M f + 2(2n-2)f - 8\delta^M \alpha) = -8|\bar{R}''|^2 - |\bar{\nabla}^* \bar{\nabla} \Omega|^2 - |\phi|^2.$$

Integrating this last equation on each level set  $M_r := \{r = \text{constant}\}$  of  $\bar{M}$  yields

$$\int_{M_r} \frac{2(2n-2)}{r^4} f + 8|\bar{R}''|^2 + |\bar{\nabla}^* \bar{\nabla} \Omega|^2 + |\phi|^2.$$

In particular, since  $f \geq 0$ ,  $\phi$  vanishes identically on  $\bar{M}$ , hence  $|\bar{\nabla}_X \Omega|^2 = -\phi(X, JX) = 0$  for every  $X$  on  $\bar{M}$ . Thus  $\bar{M}$  is Kähler, so  $M$  is Sasakian.

## 4. PROOF OF THEOREM 1.2

Let  $(M, g, \xi)$  be a compact Sasakian–Einstein manifold and  $\xi'$  be another  $g$ -compatible contact metric structure. Since  $\text{Ric} = 2ng$ , it follows that  $(g, \xi')$  is K-contact (see [2], Theorem 7.1), hence Sasakian according to Theorem 1.1. Since  $(M, g)$  is complete, by a result of Gallot [6], the cone  $(\bar{M}, \bar{g})$  over  $(M, g)$  is (locally) de Rham irreducible unless it is flat (i.e.  $(M, g)$  is of positive constant curvature). Therefore, Theorem 1.2 follows from the following general observation.

**Proposition 4.1.** *Suppose  $(M, g)$  is a Riemannian manifold whose cone is locally irreducible and which admits two Sasakian structures  $(g, \xi)$  and  $(g, \xi')$  with  $\xi \neq \pm\xi'$ . Then  $(M, g)$  admits a 3-Sasakian structure and  $\xi$  and  $\xi'$  belong to the underlying  $S^2$ -family of Sasakian structures.*

*Proof.* Let  $(\bar{M}, \bar{g})$  be the cone over  $(M, g)$ . The Sasakian structures  $\xi$  and  $\xi'$  give rise to two Kähler structures,  $J$  and  $J'$ , on  $(\bar{M}, \bar{g})$  with  $J \neq \pm J'$  (because  $\xi \neq \xi'$  by assumption). It suffices to show that  $(\bar{M}, \bar{g})$  must be hyperkähler and  $J$  and  $J'$  belong to the  $S^2$ -family of compatible Kähler structures. The anti-commutator  $Q = JJ' + J'J$  of  $J$  and  $J'$  is symmetric and parallel with respect to  $\bar{g}$ ; since  $(\bar{M}, \bar{g})$  is locally irreducible,  $Q = \lambda\text{Id}$  for some real constant  $\lambda$ . Since  $J$  and  $J'$  are both  $\bar{g}$  orthogonal, the Cauchy–Schwartz inequality implies  $|\lambda| \leq 2$ ; it is easy to see that equality is possible if only if  $J = \pm J'$ , a situation that we excluded. Similarly, the commutator  $A = JJ' - J'J$  of  $J$  and  $J'$  is parallel and skew-symmetric with respect to the metric  $\bar{g}$  and by using the corresponding property of  $Q$ , it verifies  $A^2 = (\lambda^2 - 4)\text{Id}$ . It follows that  $I = \frac{1}{\sqrt{4-\lambda^2}}A$  defines a parallel,  $\bar{g}$ -compatible complex structure on  $(\bar{M}, \bar{g})$  which anti-commutes with both  $J$  and  $J'$ ; therefore  $(\bar{g}, I, J, K = IJ)$  defines a hyperkähler structure. The equality  $JJ' + J'J = \lambda\text{Id}$  also shows that  $J'$  belongs to the  $S^2$ -family of Kähler structures generated by  $(\bar{g}, I, J, K)$ .  $\square$

## 5. AN EXAMPLE AND FURTHER COMMENTS

As explained in the introduction, it was tempting to ask the following question, slightly more general than Theorem 1.1: *is every compact Einstein contact metric manifold Sasakian–Einstein?* The answer is negative in general, as the following simple example of D. Blair shows (see [2], p. 23, p. 68–69 & p. 52–53).

**Example 5.1.** *The 1-form  $\eta := \cos t dx + \sin t dy$  defines a (non-regular) contact metric structure on the flat torus  $T^3$  (where  $t, x$  and  $y$  are standard coordinates on  $T^3$  of periods  $2\pi$ ), which is not K-contact (and hence not Sasakian).*

Note however that this is the only negative example to the above question in dimension 3. Indeed, in this dimension, Blair and Sharma [4] proved that a contact metric manifold of constant curvature has either curvature +1 and is Sasakian, or curvature 0 and is isometric to the above example. Note also that the example does not directly generalize to higher dimensions, as Blair [3] also shows that there are no flat contact metric structures in dimension  $\geq 5$ . More generally, there is a theorem of Olszak [7] that in dimension  $\geq 5$  there are no contact metric manifold of constant curvature, unless the curvature is +1 and the structure Sasakian. Hence, there are reasons to still investigate the above question.

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VESTISLAV APOSTOLOV, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, CASE POSTALE 8888, SUCCURSALE CENTRE-VILLE, MONTRÉAL (QUÉBEC), H3C 3P8, CANADA

*E-mail address:* `apostolo@math.uqam.ca`

TEDI DRĂGHICI, DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI FL 33199, USA

*E-mail address:* `draghici@fiu.edu`

ANDREI MOROIANU, CMAT, ÉCOLE POLYTECHNIQUE, UMR 7640 DU CNRS, 91128 PALAISEAU, FRANCE

*E-mail address:* `am@math.polytechnique.fr`