About "On certain incomplete statistics" by Lima et al

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Abstract

Lima et al recently claim that (*Chaos, Solitons* & *Fractals,* 2004;19:1005) the entropy for the incomplete statistics based on the normalization $\sum_i p_i^q = 1$ should be $S = -\sum_i p_i^{2q-1} \ln_q p_i$ instead of $S = -\sum_i p_i^q \ln_q p_i$ initially proposed by Wang. We indicate here that this conclusion is a result of erroneous use of temperature definition for the incomplete statistics.

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In a recent work[1] addressing the incomplete statistics (IS) based upon the normalization $\sum_i p_i^q = 1$ proposed by Wang[2, 3, 4], Lima et al calculated the entropy for IS with a method using the usual thermodynamic relations preserved within the nonextensive statistical mechanics (NSM)[5, 6, 7] which is derived from Tsallis entropy $S = -\sum_i p_i^q \ln_q p_i$ (let Boltzmann constant k be equal to unity) where $\ln_q p_i = \frac{p_i^{1-q}-1}{1-q}$ is the so called generalized logarithm, q is the nonadditive index and p_i is the probability that the system of interest be found at the state i. They ended with an entropy $S_{IS} = -\sum_i p_i^{2q-1} \ln_q p_i$

and concluded that S was intrinsically connected to the complete distribution normalization $\sum_i p_i = 1$.

Their calculation is based on the following relationships. 1) The first law : $dU = TdS - PdV$ or $dF = -SdT - PdV$, $F = U - TS = -T \ln_q Z$, and $S = -\left(\frac{\partial F}{\partial T}\right)_V$ where $T = 1/\beta = \left(\frac{\partial U}{\partial S}\right)_V$ is the temperature and Z is the partition function of IS. This method is correct as long as the above relationships hold, as indicated by Lima et al. By this comment, we would like to indicate that these conventional thermodynamic relationships are formally preserved within IS but with a deformed entropy and a physical or generalized temperature T_p different from the T given above. So the conclusion of [1] should be modified.

Due to the fact that there are several versions for NSM proposed from different statistics or information considerations, thermodynamic functions do not in general have the same nonadditive nature in different versions of NSM. This has led to different definitions of physical temperature β_p which is sometimes equal to β[6, 8], sometimes equal to β multiplied by a function of the partition function Z^{q-1} or Z^{1-q}[3]. Within IS using the energy expectation $U = \sum_i p_i^q E_i$ in respecting its nonadditivity[9], the maximum of S leads to the distribution function $p_i = \frac{1}{Z}[1-(1-q)\beta_p E_i]^{\frac{1}{1-q}}$ with the partition function $Z^q = \sum_i [1-(1-q)\beta_p E_i]^{\frac{q}{1-q}}$ where $\beta_p = 1/T_p$ is given by[3] :

$$\beta_p = Z^{1-q}\frac{\partial S}{\partial U} = Z^{1-q}\beta. \tag{1}$$

On the other hand, the introduction of the distribution p_i into Tsallis entropy gives

$$S = \frac{Z^{q-1}-1}{q-1} + \beta_p Z^{q-1} U \tag{2}$$

or $S_p = Z^{1-q}S = \ln_q Z + \beta_p U$ where S_p is a "deformed entropy" introduced in [3] to write the heat as $dQ = T_p dS_p$ and the first law as $dU = T_p dS_p + dW$ or $dF = -S_p dT_p + dW$ where

$$F = U - T_p S_p = -T_p \ln_q Z \tag{3}$$

is the Helmholtz free energy and dW is the work done to the system. We get

$$S_p = -\left(\frac{\partial F}{\partial T_p}\right). \tag{4}$$

Then, thanks to the techniques of [1], this mathematically useful "entropy" whose probability dependence was unknown[3] can be calculated and given by :

$$S_p = -\sum_i p_i^{2q-1} \ln_q p_i = -\sum_i p_i^q \frac{p_i^{q-1} - 1}{q-1}. \tag{5}$$

which is by definition not the original entropy S of IS[2]. S_p is concave only for $q > 1/2$ so that not to be maximized to get distribution functions for NSM although its maximum formally leads to $p_i' \propto [1-(q-1)\beta_p E_i]^{\frac{1}{q-1}}$. Notice that this latter is not the original distribution function p_i of IS. It should also be noticed that the method for calculating S_p cannot be used for calculating S by using β or T within this version of IS because $S \neq -\left(\frac{\partial F}{\partial T}\right)$ although we can formally write $F = U - T_p S_p = U - TS$. In addition, Z is not derivable with respect to β since it is a self-referential function when written as a function of β.

S can be calculated in this way only when $\beta_p = \beta$, which is possible within NSM only when "unnormalized expectation" is used[8]. For IS, if one use the unnormalized expectation $U = \sum_i p_i E_i$ (or normalized by $U = \frac{\sum_i p_i E_i}{\sum_i p_i}$), the maximum of S will lead to the distribution function $p_i = \frac{1}{Z}[1-(q-1)\beta_p E_i]^{\frac{1}{q-1}}$ with the partition function $Z^q = \sum_i [1-(q-1)\beta_p E_i]^{\frac{q}{q-1}}$. It is easy to prove $\sum_i p_i = Z^{q-1} + (q-1)\beta_p U$ leading to

$$S = \frac{Z^{q-1} - 1}{q-1} + \beta_p U \tag{6}$$

which implies $\beta_p = \frac{\partial S}{\partial U} = \beta$. Now we can naturally write $dQ = TdS$ and $dU = TdS + dW$ or $dF = -SdT + dW$ with $F = U - TS = -T \ln_q' Z$ where $\ln_q' x = \frac{x^{q-1}-1}{q-1}$ is another generalized logarithm. It is easy to prove that $S = -\left(\frac{\partial F}{\partial T}\right)$ with the normalization $\sum_i p_i^q = 1$ leads to Tsallis entropy.

Let us give here a summary of the definitions of temperature for different versions of NSM found in the literature. We have $\beta_p = Z^{q-1}\beta = Z^{q-1}\frac{\partial S}{\partial U}$ for the normalized expectations $U = \sum_i p_i E_i$ or $U = \sum_i p_i^q E_i / \sum_i p_i^q$ with $\sum_i p_i = 1$; $\beta_p = Z^{1-q}\beta = Z^{1-q}\frac{\partial S}{\partial U}$ for the normalized expectations $U = \sum_i p_i^q E_i$ with $\sum_i p_i^q = 1$; and $\beta_p = \beta = \frac{\partial S}{\partial U}$ if and only if unnormalized expectation $U = \sum_i p_i^q E_i$ with $\sum_i p_i = 1$ or $U = \sum_i p_i E_i$ with $\sum_i p_i^q = 1$ is used. The conclusion of this comment is that $S = -\left(\frac{\partial F}{\partial T}\right)$ is valid only for $\beta_p = \beta$ and that Tsallis entropy can be derived with this formula for both $\sum_i p_i = 1$ and $\sum_i p_i^q = 1$.

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