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PERIODICITY AND QUASI-PERIODICITY FOR SUPER-INTEGRABLE HAMILTONIAN SYSTEMS

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ABSTRACT

Classical trajectories are calculated for two Hamiltonian systems with ring shaped potentials. Both systems are super-integrable, but not maximally super-integrable, having four globally defined single-valued integrals of motion each. All finite trajectories are quasi-periodical; they become truly periodical if a commensurability condition is imposed on an angular momentum component.

RÉSUMÉ

Les trajectoires classiques sont calculées pour deux systèmes hamiltoniens avec des potentiels en forme d'anneau. Les deux systèmes considérés sont super-intégrables, mais pas de façon maximale, ayant chacun quatre intégrales de mouvement uni-valuées et définies globalement. Toutes les trajectoires finies sont quasi-périodiques. Elles deviennent périodiques si l'on impose une condition de commensurabilité sur l'une des composantes du moment angulaire.

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1. Introduction

The purpose of this letter is to discuss classical solutions for two super-integrable Hamiltonian systems with ring shaped potentials. The relevant Hamiltonian (i.e., the Hamilton function in classical mechanics or the usual Hamiltonian in non-relativistic quantum mechanics) can in both cases be written as

$$H = \frac{1}{2}\vec{p}^{\,2} + V(\vec{r}), \tag{1}$$

where the two potentials are

$$V_O = \frac{1}{2}\Omega^2(x^2+y^2+z^2) + \frac{1}{2}Q\frac{1}{x^2+y^2}, \qquad \Omega > 0, \qquad Q \geq 0 \tag{2}$$

and

$$V_C = -Z\frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{1}{2}Q\frac{1}{x^2+y^2}, \qquad Z > 0, \qquad Q \geq 0. \tag{3}$$

(The reduced mass of the particle moving in V_O or V_C is taken to be equal to 1.) The potentials V_O and V_C are cylindrically symmetrical, with $O(2)$ as geometrical symmetry group. In the limiting case $Q = 0$, V_O goes over into the potential of an isotropic harmonic oscillator, whereas V_C reduces to an attractive Coulomb potential.

The two considered systems thus extend the only two three-dimensional systems with the following properties.

(i) All finite classical trajectories are closed [1].

(ii) The systems are "maximally super-integrable" [2]. This means that each of the associated classical systems has $2n - 1 = 5$ functionally independent single-valued integrals of motion, globally defined on the phase space in $2n = 6$ dimensions (n is the number of degrees of freedom). Moreover, from these integrals of motion, we can construct in each case at least two different sets of $n = 3$ integrals of motion in involution, having only the Hamilton function in common. Indeed, two such sets are, for instance,

$$\{H;\ X_1 = \frac{1}{2}p_x^2 + \frac{1}{2}\Omega^2x^2;\ X_2 = \frac{1}{2}p_y^2 + \frac{1}{2}\Omega^2y^2\}, \qquad \{H;\ \vec{L}^2;\ L_z\} \tag{4}$$

for the harmonic oscillator system and

$$\{H;\ \vec{L}^2;\ L_x^2 + \tau L_y^2,\ 0 < \tau < 1\}, \qquad \{H;\ L_z;\ L_xp_y - L_yp_x + Z\frac{z}{\sqrt{x^2+y^2+z^2}}\} \tag{5}$$

for the Coulomb system. (We use p_i and L_i to denote linear and angular momenta, respectively. All constants of motion are given in classical form ; the corresponding quantum mechanical form follow from ordinary symmetrization.)

(iii) The energy levels for each of the associated quantum mechanical systems are degenerate. The level degeneracy is described for both systems by irreducible representation classes of a dynamical symmetry group with Lie algebra of rank two : $O(4)$ for the hydrogen atom [3,4,5] and $SU(3)$ for the harmonic oscillator [6,7].

A systematic search for super-integrable Hamiltonian systems in two [8] and three [9] dimensions was initiated some time ago. The corresponding integrals of motion were restricted to being second-order polynomials in the momenta and this related the problem to that of separating variables [8,9,10] in Hamilton-Jacobi or Schrödinger equations. In the three-dimensional case, it was shown that systems that have two different pairs of commuting integrals of motion quadratic in the momenta allow the separation of variables in two systems of coordinates. The requirement that the four integrals involved be functionally independent was not imposed. Hence, while all systems obtained in ref. [9] were super-integrable (having more than $n = 3$ integrals of motion), they were not necessarily maximally super-integrable (having generally less than $2n - 1 = 5$ integrals of motion).

The potentials V_O and V_C (see (2) and (3)) correspond to such super-integrable systems, each having four (rather than five, the maximal number) functionally independent integrals of motion that happen to be second-order polynomials in the momenta. Both corresponding dynamical systems are of physical interest and have been studied before, mainly in their quantum mechanical incarnation. Thus, the potential V_C is the so-called Hartmann potential, introduced in quantum chemistry for describing ring shaped molecules like cyclic polyenes [11]. The degeneracy of its quantum energy levels has been described in terms of an underlying $SU(2)$ dynamical symmetry group [12,13]. The potential V_O might be useful in a nuclear physics context. Its levels also manifest an $SU(2)$ degeneracy [14]. It is to be noted that two systems that resemble the V_C and the V_O systems, viz., the ABC [15] and the ABO [16] systems, have been recently investigated in connection with the Aharonov-Bohm effect.

2. Classical trajectories

2.a The potential V_O

The potential (2) is a special case of the potential

$$V_3 = \alpha(x^2 + y^2 + z^2) + \beta\frac{1}{z^2} + h(\frac{y}{x})\frac{1}{x^2 + y^2} \tag{6}$$

introduced by Makarov *et al.* [9]. In the most general case where h is an arbitrary function, this potential allows the separation of variables for the Hamilton-Jacobi and the Schrödinger equations in four systems of coordinates, namely spherical, cylindrical, prolate spheroidal and oblate spheroidal coordinates ; for $h = 0$, we also have separability in Cartesian coordinates. The system corresponding to the Hamiltonian (1) with potential (2) admits the following four functionally independent integrals of motion

$$H, \qquad A_1 = L_x^2 + L_y^2 + L_z^2 + Q(1 + \frac{z^2}{x^2 + y^2}), \qquad A_2 = \frac{1}{2}(p_z^2 + \Omega^2 z^2), \qquad A_3 = L_z. \tag{7}$$

Separation of variables in the four systems mentioned above corresponds to the use of the four following sets of integrals of motion in involution

$$\{H;\ A_1;\ A_3\}, \qquad \{H;\ A_2;\ A_3\}, \qquad \{H;\ A_1 \mp 2a^2(H - A_2),\ a \in \mathbf{R};\ A_3\}, \tag{8}$$

respectively.

In order to obtain the classical trajectories, we make use of the triplet $\{H; A_2; A_3\}$ corresponding to circular cylindrical coordinates $(x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z)$. We write the integrals as

$$\begin{gathered} H - A_2 = \frac{1}{2}(\dot{\rho}^2 + \frac{m^2 + Q}{\rho^2} + \Omega^2\rho^2) \equiv E_1, \\ A_2 = \frac{1}{2}(\dot{z}^2 + \Omega^2 z^2) \equiv E_2, \\ A_3 = \rho^2\dot{\varphi} \equiv m. \end{gathered} \tag{9}$$

We see that the trajectories are always finite, satisfying

$$\begin{gathered} \rho_1 \le \rho \le \rho_2, \qquad -z_0 \le z \le z_0, \\ (\rho_{1,2})^2 = \frac{1}{\Omega^2}\left(E_1 \mp \sqrt{E_1^2 - \Omega^2 M^2}\right), \qquad z_0 = \sqrt{2}\frac{\sqrt{E_2}}{\Omega}, \end{gathered} \tag{10}$$

$$E_1 \geq \Omega|M|, \qquad |M| = \sqrt{m^2+Q}, \qquad E_2 \geq 0.$$

Solving the first order ordinary differential equations (9), we obtain the general expression for the trajectories :

$$\begin{aligned}
\rho(t) &= \frac{1}{\sqrt{2}}\sqrt{\rho_1^2+\rho_2^2+(\rho_2^2-\rho_1^2)\sin[2\Omega(t-t_0)]}, \\
z(t) &= z_0 \sin[\Omega(t-t_0')], \\
\varphi(t) &= \varphi_0 + \frac{1}{2}\frac{m}{|M|}\sin^{-1}\left\{\frac{(\rho_2^2+\rho_1^2)\sin[2\Omega(t-t_0)]-\rho_1^2+\rho_2^2}{(\rho_2^2-\rho_1^2)\sin[2\Omega(t-t_0)]+\rho_1^2+\rho_2^2}\right\},
\end{aligned} \tag{11}$$

where t_0, t_0' and φ_0 are further integration constants (in addition to the angular momentum component m and the oscillation energies E_1 and E_2).

For $Q=0$, all trajectories are planar and closed (ellipses) as expected for the harmonic oscillator system ; the motion has the period $T_O = 2\pi/\Omega$. Interestingly enough, for strictly positive values of Q this is no longer the case. The coordinates ρ and z are "libration" coordinates [17], for which periodicity implies $\rho(t+T_\rho)=\rho(t)$ and $z(t+T_z)=z(t)$. In the case under consideration we have $T_z = 2T_\rho = T_O$, in function of the harmonic oscillator period T_O. The coordinate φ is an angular one, so periodicity means $\varphi(t+T_\varphi) = \varphi(t) \pm 2\pi$. It is clear that for arbitrary values of m, the durations T_z, T_ρ and T_φ are not necessarily commensurable. All trajectories are thus quasi-periodic : periodic in each coordinate, but not periodic in a global way. The condition for genuine periodicity is

$$\frac{|M|}{m} = \frac{k_1}{k_2} \quad \Rightarrow \quad m^2 = \frac{k_2^2}{k_1^2-k_2^2}Q, \tag{12}$$

where k_1 and k_2 are mutually prime integers (with $|k_1/k_2| \geq 1$). The overall period T then is

$$T = 2k_1T_\rho = k_1T_O. \tag{13}$$

Therefore, for fixed $Q>0$, in order for the trajectories to be periodic, rather than quasi-periodic, we obtain a "classical quantization" condition (see eq. (12)) for the angular momentum component L_z.

In general the periodic and quasi-periodic trajectories do not lie in a plane (except for $Q=0$). If $Q>0$, there are at least two cases for which the trajectories are planar.

(i) For $m=0$: then from (11) we get $\varphi=\varphi_0$ and we see that in addition to being planar the motion is periodic of period T_O.

(ii) For $E_2 = 0$: then from (10) and (11), the motion is in the xy-plane and is periodic if and only if eq. (12) is satisfied.

The planarity of the trajectories for the potential V_O (and for V_C too) will be studied in a systematic way by calculating the torsion of the relevant curves [18].

2.b The potential V_C

The potential (3) is a special case of the potential

$$V_4 = \alpha \frac{1}{r} + \beta \frac{\cos\theta}{r^2 \sin^2\theta} + h(\varphi) \frac{1}{r^2 \sin^2\theta} \tag{14}$$

found by Makarov *et al.* [9], allowing the separation of variables in spherical and parabolic coordinates. The Hamiltonian (1) with potential V_C permits four functionally independent globally defined single-valued integrals of motion, namely

$$H, \quad B_1 = L_x^2 + L_y^2 + L_z^2 + Q \frac{1}{\sin^2\theta}, \quad B_2 = L_z, \quad B_3 = L_x p_y - L_y p_x + Z \frac{z}{r} - Q \frac{z}{x^2 + y^2}. \tag{15}$$

Separation of variables in spherical or parabolic coordinates corresponds to the simultaneous diagonalization of

$$\{H; B_1; B_2\} \quad \text{or} \quad \{H; B_2; B_3\}, \tag{16}$$

respectively.

In a previous article [13] we made use of the integrals B_2 and B_3 to obtain classical trajectories in parabolic coordinates. Here, we switch to spherical coordinates ($x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta$) and use the triplet $\{H; B_1; B_2\}$ to put

$$\begin{aligned} H &= \frac{1}{2}\dot{r}^2 + \frac{1}{2} B_1 \frac{1}{r^2} - Z \frac{1}{r} \equiv E, \\ B_1 &= r^4 \dot{\theta}^2 + \frac{M^2}{\sin^2\theta} \equiv K, \\ B_2 &= r^2 \sin^2\theta\, \dot{\varphi} \equiv m, \end{aligned} \tag{17}$$

where again $M^2 = m^2 + Q$. From the equation for $\dot{r}$ in (17) we see that the motion is infinite for $E \geq 0$. Moreover, the finite trajectories correspond to

$$r_1 \leq r \leq r_2, \qquad \theta_0 \leq \theta \leq \pi - \theta_0,$$

$$r_{1,2} = \frac{1}{-2E}\left(Z \mp \sqrt{Z^2+2EK}\right), \qquad \sin\theta_0 = \frac{|M|}{\sqrt{K}}, \tag{18}$$

$$-\frac{Z^2}{2K} \le E < 0, \qquad K \ge M^2.$$

Equations (17) can be integrated to give

$$t - t_0 = -(-2E)^{-1/2}\sqrt{(r-r_1)(r_2-r)} + Z(-2E)^{-3/2}\sin^{-1}\left(\frac{2r}{r_2-r_1} - \frac{r_2+r_1}{r_2-r_1}\right),$$

$$r\cos\theta = \cos\theta_0 \frac{1}{r_2-r_1}\left\{[2r_1r_2 - (r_1+r_2)r]\cos\beta_0 + 2\sqrt{r_1r_2}\sqrt{(r-r_1)(r_2-r)}\sin\beta_0\right\}, \tag{19}$$

$$\varphi = \varphi_0 + \frac{1}{2}\frac{m}{|M|}\left\{\sin^{-1}\left[\frac{1}{\cos\theta_0}\left(-1+\frac{\sin^2\theta_0}{1+\cos\theta}\right)\right] - \sin^{-1}\left[\frac{1}{\cos\theta_0}\left(-1+\frac{\sin^2\theta_0}{1-\cos\theta}\right)\right]\right\},$$

where t_0, φ_0 and β_0 are integration contants (in addition to the constants of motion E, K and m).

For $Q = 0$, we recover the well-known result for the Coulomb system : all finite trajectories are planar and closed (ellipses) and the motion has the Kepler period $T_C = 2\pi Z(-2E)^{-3/2}$. For $Q > 0$, the coordinates r and z have the period $T_r = T_z = T_C$ which is generally not commensurable with T_φ. Thus for the overall motion to be periodic, rather than only quasi-periodic, we again arrive at the commensurability condition (12) (an important qualification not mentioned in our previous article [13]). The overall period T in this case is

$$T = k_1 T_C. \tag{20}$$

Therefore, the finite trajectories for V_C are not closed in general ; however, all trajectories are quasi-periodic and the periodicity evocated in ref. [13] is actually a quasi-periodicity.

3. Conclusions

In this paper we have concentrated on the properties of super-integrable, but not maximally super-integrable, systems in classical mechanics. We have used the examples of two ring shaped potentials, V_O (eq. (2)) and V_C (eq. (3)), to show that the classical trajectories are always quasi-periodic and that they are periodic if a commensurability condition (eq. (12)) is satisfied. (It is remarkable that the same closure condition (12) be obtained for V_O and V_C.) For both potentials, periodicity depends on only one of the three constants of the motion, namely the constant m, the value of the angular momentum component L_z.

Whether the trajectory lies in a plane or not may depend on the values of other integrals of motion, a fact to be explored in a publication in preparation [18].

Periodic trajectories exist in Hamiltonian systems that are not super-integrable, but they are rare. Here, we have infinitely many periodic trajectories. Moreover, since an irrational number can be approximated with any chosen accuracy by a rational one, infinitely many periodic trajectories lie in the neighbourhood of any non-periodic one.

Turning to the quantum mechanical systems corresponding to the potentials V_O and V_C, we note that, as usual, super-integrability manifests itself in the degeneracy of energy levels. The fact that these systems are not maximally super-integrable has the consequence that the degeneracy is not maximal either. Indeed, the wave functions depend on three quantum numbers, say the eigenvalues of the quantum versions of H, A_2 and A_3 or H, B_1 and B_2 for V_O or V_C, respectively. The quantization conditions then imply that states with different eigenvalues of A_2, repectively B_1, have the same energy, which does however depend (via M) on the other quantum number m. The Lie algebra ($su(2)$) explaining the degeneracy is of rank one in both cases [13,14]. For a subset of states satisfying the periodicity condition (12), a higher degeneracy, as yet not investigated, will occur.

Let us mention that new super-integrable Hamiltonian systems have recently been found by Evans [19] and that it would be of interest to study the corresponding classical and quantum solutions.

We plan to further investigate properties of super-integrable Hamiltonian systems, partly in view of the possible application of such systems with infinitely many periodic trajectories in particle accelerators and storage rings. The classical trajectories in the ring shaped potentials V_O and V_C will be given in graphical form in ref. [18].

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