

THE CORANK IS INVARIANT UNDER BLOW-NASH EQUIVALENCE

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ABSTRACT. We address the following question, raised by T. Fukui. Is the corank an invariant of the blow-analytic equivalence between real analytic function germs? We give a partial positive answer in the particular case of the blow-Nash equivalence. The proof is based on the computation of some virtual Poincaré polynomials and zeta functions associated to a Nash function germ.

INTRODUCTION

The classification of real analytic function germs is a difficult topic, notably in the choice of a good equivalence relation, between germs, to study. An interesting relation, called blow-analytic equivalence, has been introduced by T. C. Kuo [5] and studied by several authors (see [4] for a recent survey). Notably, it has been proved that such an equivalence relation does not admit moduli for a family with isolated singularities. Moreover, the proof of this result produces effective methods to prove blow-analytic triviality. On the other hand, some invariants have been introduced in order to distinguish blow-analytic types. However, because of the complexity of these invariants, it remains difficult to obtain effective classification results, at least in dimension greater than 3.

In this paper, we address the following related question, raised by T. Fukui.

Let $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic germ. Assume that 0 is singular for f , which means that the jacobian matrix of f at 0 does vanish. Let r denote the rank of the hessian of F at 0. Then f is analytically equivalent to a function of the form

$$\sum_{i=1}^s x_i^2 - \sum_{j=s+1}^{s+t} x_j^2 + F(x),$$

where $s + t = r$ (note that s or t may vanish) and the order of F is at least equal to 3. The corank of f is defined to be the corank of its hessian matrix at 0, that is $d - r$.

Question: Is the corank of an analytic function germ an invariant of the blow-analytic equivalence?

The answer to such a question would be a step toward a better understanding of the blow-analytic equivalence relation, and therefore to a better understanding of the singularities of real analytic function germs.

We will not give a complete answer to this question, but we concentrate in a particular case of the blow-analytic equivalence, where we are able to conclude. Actually, a similar situation holds in the Nash setting (that is analytic and moreover semi-algebraic). One can define the blow-Nash equivalence between Nash function germs, and this relation still has good triviality properties and effective invariants called zeta functions [3]. Recall that these zeta functions are defined using an additive and multiplicative invariant (invariant means under Nash isomorphisms) of real algebraic sets, the virtual Poincaré polynomial (cf. part 1.1).

The main result of this paper states that blow-Nash equivalent Nash function germs have the same corank. Moreover, they have the same index (the index corresponds to the integer t above).

The proof is based on the invariance of the zeta functions with respect to the blow-Nash equivalence [3], and on the computation on a significant part of these zeta functions for germs of Nash functions of the type $\sum_{i=1}^s x_i^2 - \sum_{j=s+1}^{s+t} x_j^2$ (cf part 3). To reach this aim, we need to compute some virtual Poincaré polynomials associated to these germs. Such a computation may be difficult in general, but here we manage to conclude thanks to the degeneracy of a Leray-Serre spectral sequence (cf part 2).

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1. CORANK AND BLOW-NASH EQUIVALENCE

1.1. Blow-Nash equivalence. In this section, we recall briefly the notion of blow-Nash equivalence as well as that of zeta functions. For more details, the reader is refer to [3].

Definition 1.1.

- (1) An algebraic modification of a Nash function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ is a proper birational algebraic morphism $\sigma_f : (M_f, \sigma_f^{-1}(0)) \longrightarrow (\mathbb{R}^d, 0)$, between Nash neighbourhoods of 0 in \mathbb{R}^d and the exceptional divisor $\sigma_f^{-1}(0)$ in M_f , which is an isomorphism over the complement of the zero locus of f and for which $f \circ \sigma$ is in normal crossing.
- (2) Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be germs of Nash functions. They are said to be blow-Nash equivalent if there exist two algebraic modifications

$$\sigma_f : (M_f, \sigma_f^{-1}(0)) \longrightarrow (\mathbb{R}^d, 0) \text{ and } \sigma_g : (M_g, \sigma_g^{-1}(0)) \longrightarrow (\mathbb{R}^d, 0),$$

such that $f \circ \sigma_f$ and $\text{jac } \sigma_f$ (respectively $g \circ \sigma_g$ and $\text{jac } \sigma_g$) have only normal crossings simultaneously and a Nash isomorphism (i.e. a semi-algebraic map which is an analytic isomorphism) Φ between analytic neighbourhoods $(M_f, \sigma_f^{-1}(0))$ and $(M_g, \sigma_g^{-1}(0))$ which preserves the multiplicities of the jacobian determinants of σ_f and σ_g along the components of the exceptional divisor, and which induces a homeomorphism $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$, as illustrated by the commutative diagram:

$$\begin{array}{ccc}
 (M_f, \sigma_f^{-1}(0)) & \xrightarrow{\Phi} & (M_g, \sigma_g^{-1}(0)) \\
 \sigma_f \downarrow & & \downarrow \sigma_g \\
 (\mathbb{R}^d, 0) & \xrightarrow{\phi} & (\mathbb{R}^d, 0) \\
 & \searrow f & \swarrow g \\
 & (\mathbb{R}, 0) &
 \end{array}$$

The main results, concerning this equivalence relation between Nash function germs, are that, on one hand, it does not admit moduli for a Nash family with isolated singularity, and on the other hand, we know invariants, called zeta functions, that will be crucial in the proof of the main result of this paper. We recall now the definition of these zeta functions. To begin with, let us introduce the virtual Poincaré polynomial.

By an additive map on the category of real algebraic sets, we mean a map β such that $\beta(X) = \beta(Y) + \beta(X \setminus Y)$ where Y is an algebraic subset closed in X . Moreover β is called multiplicative if $\beta(X_1 \times X_2) = \beta(X_1) \cdot \beta(X_2)$ for real algebraic sets X_1, X_2 .

Proposition 1.2. ([3]) *There exist additive maps on the category of real algebraic sets with values in \mathbb{Z} , denoted β_i and called virtual Betti numbers, that coincide with the classical Betti numbers $\dim H_i(\cdot, \frac{\mathbb{Z}}{2\mathbb{Z}})$ on the connected component of compact nonsingular real algebraic varieties.*

Moreover $\beta(\cdot) = \sum_{i \geq 0} \beta_i(\cdot) u^i$ is multiplicative, with values in $\mathbb{Z}[u]$.

Finally, if X_1 and X_2 are Nash isomorphic real algebraic sets, then $\beta(X_1) = \beta(X_2)$.

Then we can define the zeta functions of a Nash function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ as follows. Denote by \mathcal{L} the space of arcs at the origin $0 \in \mathbb{R}^d$, that is:

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d, 0) = \{\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^d, 0) : \gamma \text{ formal}\},$$

and by \mathcal{L}_n the space of arcs truncated at the order $n + 1$:

$$\mathcal{L}_n = \mathcal{L}_n(\mathbb{R}^d, 0) = \{\gamma \in \mathcal{L} : \gamma(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n, a_i \in \mathbb{R}^d\},$$

for $n \geq 0$ an integer.

We define the naive zeta function $Z_f(T)$ of f as the following element of $\mathbb{Z}[u, u^{-1}][[T]]$:

$$Z_f(T) = \sum_{n \geq 1} \beta(A_n) u^{-nd} T^n,$$

where

$$A_n = \{\gamma \in \mathcal{L}_n : \text{ord}(f \circ \gamma) = n\} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = bt^n + \cdots, b \neq 0\}.$$

Similarly, we define zeta functions with sign by

$$Z_f^{+1}(T) = \sum_{n \geq 1} \beta(A_n^{+1}) u^{-nd} T^n \quad \text{and} \quad Z_f^{-1}(T) = \sum_{n \geq 1} \beta(A_n^{-1}) u^{-nd} T^n,$$

where

$$A_n^{+1} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = +t^n + \cdots\} \quad \text{and} \quad A_n^{-1} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = -t^n + \cdots\}.$$

The main result concerning these zeta functions is the following:

Theorem 1.3. ([3]) *Blow-Nash equivalent Nash function germs have the same naive zeta function and the same zeta functions with sign.*

1.2. Corank of a Nash function germ. The corank and the index of a Nash function germ is defined in a similar way than in the analytic case. Moreover, the same splitting lemma holds in the Nash case.

Lemma 1.4. *Let $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be a Nash function germ. Assume that the jacobian matrix of f does vanish at 0. Then there exist a Nash isomorphism $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ and integers s, t (possibly equal to zero), where $s + t$ equals the rank of the hessian matrix of f at 0, such that*

$$f \circ \phi(x_1, \dots, x_d) = \sum_{i=1}^s x_i^2 - \sum_{j=s+1}^{s+t} x_j^2 + F(x_{s+t+1}, \dots, x_d),$$

where F is a Nash function germ with order at least equal to 3.

Classical proofs of lemma 1.4, in the smooth case, use a parametrized version of the Morse Lemma [1], but this is not allowed in the Nash setting since it requires integration along vector fields. However, an elementary proof, using only the implicit function theorem and the Hadamard division lemma, has been given in [6]. This method adapts to our case

since the implicit function theorem does hold in the Nash [2], whereas the Hadamard division lemma is no longer necessary because we are working with analytic functions.

Let us state now the central result of this paper.

Theorem 1.5. *Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be blow-Nash equivalent Nash function germs. Then f and g have the same corank and the same index.*

Proof. The proof of theorem 1.5 consists in comparing the T^2 coefficient of the zeta functions associated to f and g . Due to the invariance theorem 1.3, it is sufficient to compare these coefficients for the simpler Nash germs given by lemma 1.4, since they are Nash equivalent and therefore blow-Nash equivalent to f and g respectively. Now, the result follows from proposition 3.1 below. □

2. COMPUTATION OF SOME VIRTUAL POINCARÉ POLYNOMIALS

Let $X_{m,M}$ be the real algebraic subset of \mathbb{R}^{m+M} defined by the equation

$$\sum_{i=1}^m x_i^2 - \sum_{j=1}^M y_j^2 = 0.$$

In this section, we compute the value of the virtual Poincaré polynomials $\beta(X_{m,M})$ in terms of m and M . This computation is based on the degeneracy of a Leray-Serre spectral sequence associated to the projectivisation of $X_{m,M}$.

Without lost of generality, one may assume that $m \leq M$.

Proposition 2.1. *If $m \geq 1$, then $\beta(X_{m,M}) = u^{m+M-1} - u^{M-1} + u^m$.*

Remark 2.2.

- (1) If $m = 0$, $X_{0,M}$ is empty, and therefore $\beta(X_{0,M}) = 0$.
- (2) Note that if $m = 1$, then the computation of $\beta(X_{1,M})$ is easy since $X_{1,M}$ is just a cone based on a sphere. Thus $\beta(X_{1,M}) = 1 + (u-1)(1+u^{M-1}) = u^M - u^{M-1} + u$.
- (3) In the particular case where $m = M = 2$, the computation can be done in a simple way using the toric structure of $X_{2,2}$. Indeed $X_{2,2}$ is isomorphic to the toric variety given by $XY = UV$ in \mathbb{R}^4 . Therefore $X_{2,2}$ is the union of the orbits under the torus action, that is $X_{2,2}$ is the disjoint union of $(\mathbb{R}^*)^3$, one point, and four copies of $(\mathbb{R}^*)^2$ and (\mathbb{R}^*) . Therefore, by additivity of β ,

$$\beta(X_{2,2}) = (u-1)^3 + 4(u-1)^2 + 4(u-1) + 1 = u^3 + u^2 - u.$$

The proof of proposition 2.1 is based on a reduction to the projective case. As a preliminary step, we compute the virtual Poincaré polynomial of the projective subset $Z_{m,M}$ of $\mathbb{P}^{m+M-1}(\mathbb{R})$ defined by the same equation as that of $X_{m,M}$.

Lemma 2.3. *Take $M \geq 2$. Let $Z_{m,M}$ be defined by $\sum_{i=1}^m x_i^2 - \sum_{j=1}^M y_j^2 = 0$ in $\mathbb{P}^{m+M-1}(\mathbb{R})$. Then*

$$\beta(Z_{m,M}) = (1 + u^{M-1})(1 + u + \dots + u^{m-1}).$$

Proof. To begin with, remark that $Z_{m,M}$ is nonsingular as a real algebraic set. Therefore the virtual Betti numbers of $Z_{m,M}$ coincide with its classical Betti numbers (cf. proposition 1.2).

In order to compute these Betti numbers, consider the projection from $Z_{m,M}$ onto $\mathbb{P}^{m-1}(\mathbb{R})$ defined by

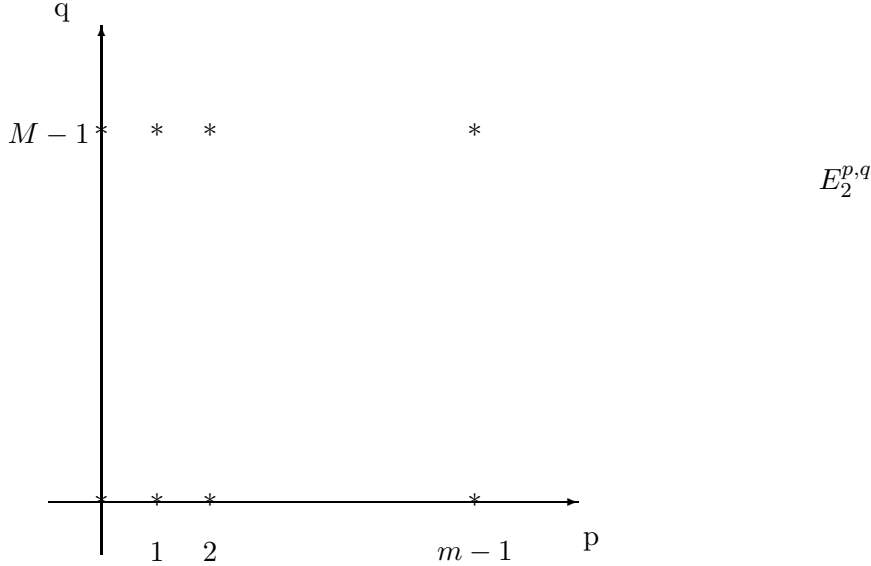
$$[x_1 : \dots : x_m : y_1 : \dots : y_M] \mapsto [x_1 : \dots : x_m].$$

It is well-defined since x_1, \dots, x_m can not vanish without cancelling y_1, \dots, y_M , and moreover it defines a fibration with fiber isomorphic to the unit sphere S^{M-1} in \mathbb{R}^M . Working

with coefficients in \mathbb{Z}_2 , the cohomological Leray-Serre spectral sequence associated with this fibration converges to the cohomology with coefficients in \mathbb{Z}_2 of $Z_{m,M}$:

$$E_2^{p,q} = H^p(\mathbb{P}^{m-1}(\mathbb{R}), H^q(S^{M-1}, \mathbb{Z}_2)) \Rightarrow H^{p+q}(Z_{m,M}, \mathbb{Z}_2).$$

However $H^q(S^{M-1}, \mathbb{Z}_2)$ is zero unless $q = 0$ and $q = M - 1$ for which it equals \mathbb{Z}_2 . Therefore the nonzero terms (that equals \mathbb{Z}_2) of $E_2^{p,q}$, shown in the figure below, are localized in two lines.



But, as $m \leq M$ by assumption, the spectral sequence degenerates and gives the Betti numbers of $Z_{m,M}$. More precisely,

$$\dim H_i(Z_{m,M}, \mathbb{Z}_2) = \begin{cases} 1 & \text{if } i \in \{0, \dots, m-1, M-1, \dots, m+M-2\}, \\ 0 & \text{otherwise,} \end{cases}$$

if $m < M$, and in the particular case where $m = M$, then

$$\dim H_i(Z_{m,m}, \mathbb{Z}_2) = \begin{cases} 1 & \text{if } i \in \{0, \dots, m-2, m, \dots, 2m-2\}, \\ 2 & \text{if } i = m-1, \\ 0 & \text{otherwise.} \end{cases}$$

So in general

$$\beta(Z_{m,M}) = (1+u+\dots+u^{m-1})+(u^{M-1}+\dots+u^{m+M-2}) = (1+u^{M-1})(1+u+\dots+u^{m-1}).$$

□

Now we explain how to compute the virtual Poincaré polynomial of $X_{m,M}$ in terms of that of $Z_{m,M}$.

Proof of proposition 2.1. It suffices to notice that the projection from $X_{m,M} \setminus \{0\}$ onto $Z_{m,M}$ is a piecewise algebraically trivial fibration with fiber \mathbb{R}^* . Therefore

$$\beta(X_{m,M}) = 1 + (u-1)\beta(Z_{m,M})$$

by additivity and multiplicativity of the virtual Poincaré polynomial β .

Now, remark that $\beta(Z_{m,M}) = (1+u^{M-1})\frac{u^m-1}{u-1}$, and so

$$\beta(X_{m,M}) = 1 + (1+u^{M-1})(u^m-1) = u^{m+M-1} - u^{M-1} + u^m.$$

□

The following corollaries, which also specify the virtual Poincaré polynomial of some algebraic sets, will be useful for computing zeta functions with sign in section 3.

Corollary 2.4. *Let $X_{s,t}^1$ be the real algebraic subset of \mathbb{R}^{s+t} defined by the equation*

$$\sum_{i=1}^s x_i^2 - \sum_{j=1}^t y_j^2 = 1.$$

Assume that $s, t > 0$.

- *If $s \leq t$, then $\beta(X_{s,t}^1) = u^{t-1}(u^s - 1)$.*
- *If $s > t$, then $\beta(X_{s,t}^1) = u^t(u^{s-1} + 1)$*

Moreover $\beta(X_{0,t}^1) = 0$ and $\beta(X_{s,0}^1) = 1 + \dots + u^{s-1}$ if $s \geq 1$.

Proof. Let us begin with the case $s \leq t$. Let homogenize the equation defining $X_{s,t}^1$. Then, we obtain a projective subset of $\mathbb{P}(\mathbb{R})^{s+t}$, denoted $Z_{s,t+1}$ in lemma 2.3, whose affine part is isomorphic to $X_{s,t}^1$, and whose part at infinity is isomorphic to $Z_{s,t}$. Therefore

$$\begin{aligned} \beta(X_{s,t}^1) &= \beta(Z_{s,t+1}) - \beta(Z_{s,t}) = (1 + u^t)(1 + \dots + u^{s-1}) - (1 + u^{t-1})(1 + \dots + u^{s-1}) \\ &= u^{t-1}(u - 1) \frac{u^s - 1}{u - 1} = u^{t-1}(u^s - 1). \end{aligned}$$

Now, let us turn to the case $s > t$. In the same way, by homogenization of the equation defining $X_{s,t}^1$, we obtain a projective subset of $\mathbb{P}(\mathbb{R})^{s+t}$, denoted $Z_{t+1,s}$ (and not $Z_{s,t+1}$ because $s \geq t + 1$), whose affine part is isomorphic to $X_{s,t}^1$. Moreover, the part at infinity is isomorphic to $Z_{t,s}$, therefore

$$\beta(X_{s,t}^1) = \beta(Z_{t+1,s}) - \beta(Z_{t,s}),$$

and the second member can be computed thanks to lemma 2.3. More precisely:

$$\beta(X_{s,t}^1) = (1 + u^{s-1})(1 + u + \dots + u^t) - (1 + u^{s-1})(1 + u + \dots + u^{t-1}) = u^t(1 + u^{s-1}).$$

Finally, remark that in the case $s = 0$, then the sets considered are either empty or isomorphic to a sphere. \square

Corollary 2.5. *Let $X_{s,t}^{-1}$ be the real algebraic subset of \mathbb{R}^{s+t} defined by the equation*

$$\sum_{i=1}^s x_i^2 - \sum_{j=1}^t y_j^2 = -1.$$

Assume that $s, t > 0$.

- *If $s \geq t$, then $\beta(X_{s,t}^{-1}) = u^{s-1}(u^t - 1)$.*
- *If $s < t$, then $\beta(X_{s,t}^{-1}) = u^s(u^{t-1} + 1)$.*

Moreover $\beta(X_{s,0}^{-1}) = 0$ and $\beta(X_{0,t}^{-1}) = 1 + \dots + u^{t-1}$ if $t \geq 1$.

Remark 2.6. This is just a rewriting of corollary 2.4 after noticing that $X_{s,t}^{-1} = X_{t,s}^1$

3. PROOF OF THEOREM 1.5

We study the behaviour of the zeta functions of a germ of functions $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ of the form

$$f(x_1, \dots, x_s, y_1, \dots, y_t, z_1, \dots, z_{d-s-t}) = \sum_{i=1}^s x_i^2 - \sum_{j=1}^t y_j^2.$$

In particular, we compute the coefficient of T^2 of the naive zeta function and of the zeta functions with sign.

The main result, stated in proposition 3.1 below, is that the corresponding coefficients of the zeta functions with sign determine the integers s and t . It completes the proof of theorem 1.5.

Proposition 3.1. *The coefficients of T^2 of the zeta functions with sign of a germ of analytic functions $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ of the form*

$$f(x_1, \dots, x_s, y_1, \dots, y_t, z_1, \dots, z_{d-s-t}) = \sum_{i=1}^s x_i^2 - \sum_{j=1}^t y_j^2$$

determine s and t .

Proof. The space of truncated arcs $A_2^{+1}(f)$ is isomorphic to

$$\mathbb{R}^{2(d-s-t)} \times \mathbb{R}^{s+t} \times X_{s,t}^1.$$

Actually, for an arc $(a_1 t +_1 t^2, \dots, a_d t + b_d t^2)$ in $A_2^{+1}(f)$, the first term of the product corresponds to the choice of the coefficients $a_{s+t+1}, b_{s+t+1}, \dots, a_d, b_d$, the second to the choice of b_1, \dots, b_{s+t} and finally $X_{s,t}^1$ to the choice of a_1, \dots, a_{s+t} .

Now, putting into factor in $\beta(A_2^{+1}(f))$ the maximal power of u , we remark that, because of the possible forms of this polynomial as specified in corollary 2.4, it remains a polynomial of the form $u^k + 1$ or $u^k - 1$. Now, due to corollary 2.4 again, it follows that $s = k + 1$ in the former case, and $s = k$ in the latter one.

Similarly, $A_2^{-1}(f)$ is isomorphic to

$$\mathbb{R}^{2(d-s-t)} \times \mathbb{R}^{s+t} \times X_{s,t}^{-1},$$

and once more, after dividing $\beta(A_2^{-1}(f))$ by the maximal power of u , we obtain a polynomial of the type $u^l + 1$ or $u^l - 1$. In the former case, then $t = l + 1$ whereas in the latter one $t = l$. \square

Remark 3.2. The same method, applied to the coefficient of T^2 in the naive zeta function instead of the zeta functions with sign, no longer gives such a determination. More precisely, what we still determine is $m = \min\{s, t\}$ and $M = \max\{s, t\}$, unless $m = 0$ or $M = m + 1$ where we can even not specify the value of M and m, M respectively.

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