Stationary Kolmogorov Solutions of the Smoluchowski Aggregation Equation with a Source Term

Colm Connaughton
Laboratoire de Physique Statistique de l'Ecole Normale Supérieur, associé au CNRS, 24 Rue Lhomond, 75231 Paris Cedex 05, France

R. Rajesh
Martin Fisher School of Physics, Brandeis University, Mailstop 057, Waltham, MA 02454-9110, USA

Oleg Zaboronski
Mathematics Institute, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, United Kingdom

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In this paper we show how the method of Zakharov transformations may be used to analyze the stationary solutions of the Smoluchowski aggregation equation with a source term for arbitrary homogeneous coagulation kernel. The resulting power-law mass distributions are of Kolmogorov type in the sense that they carry a constant flux of mass from small masses to large. They are valid for masses much larger than the characteristic mass of the source. We derive a "locality criterion", expressed in terms of the asymptotic properties of the kernel, that must be satisfied in order for the Kolmogorov spectrum to be an admissible solution. Whether a given kernel leads to a gelation transition or not can be determined by computing the mass capacity of the Kolmogorov spectrum. As an example, we compute the exact stationary state for the family of kernels, $K_\zeta(m_1, m_2) = (m_1 m_2)^{\zeta/2}$ which includes both gelling and non-gelling cases, reproducing the known solution in the case $\zeta = 0$. Surprisingly, the Kolmogorov constant is the same for all kernels in this family.

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I. INTRODUCTION

Smoluchowski's coagulation equation provides a mean field description of a variety of aggregation phenomena [1, 2, 3, 4]. The physical picture to bear in mind is one of a suspension of particles of varying masses that are moving around in d-dimensional space due to some transport mechanism. When two particles come into contact they stick together with some probability to form a new particle whose mass is the sum of the masses of the two constituent particles. Aggregation is irreversible in the sense that large aggregates are not permitted to break up into smaller ones. If it is assumed that there are no spatial correlations between aggregates then the concentration of particles of mass m, $c(m,t)$, obeys the Smoluchowski kinetic equation:

$$
\begin{aligned}
\frac{\partial c(m,t)}{\partial t} &= \frac{\lambda}{2}\int_0^\infty dm_1 dm_2 K(m_1,m_2,m) c(m_1,t) c(m_2,t) \delta(m-m_1-m_2) \\
&- \frac{\lambda}{2}\int_0^\infty dm_1 dm_2 K(m_1,m,m_2) c(m,t) c(m_1,t) \delta(m_2-m-m_1) \\
&- \frac{\lambda}{2}\int_0^\infty dm_1 dm_2 K(m,m_2,m_1) c(m,t) c(m_2,t) \delta(m_1-m_2-m) \\
&+ \frac{J_0}{m_0}\,\delta(m-m_0) - \frac{J[c]}{M}\,\delta(m-M).
\end{aligned}
\qquad (1)
$$

The kernel $K(m_1, m_2, m)$ and the constant λ control the rate at which particles of masses m_1 and m_2 react to create particles of mass $m = m_1 + m_2$. The $\delta(m - m_0)$ term provides a source of particles of mass m_0 such that the total rate of mass input is given by J_0 which we take to be constant in time. $J[c]$ represents the mass flux which is functionally dependent on the entire spectrum, $c(m,t)$. Thus the $\delta(m - M)$ term provides a sink by removing particles from the system whose masses exceed M.

The kernel must be symmetric in its first two arguments, $K(m_1, m_2, m) = K(m_2, m_1, m)$, if it is to describe a physical aggregation process. Owing to the presence of the delta functions, the kernel is effectively a function of two arguments rather than three and is usually written as $K(m_1, m_2)$. We include the explicit dependence on the third argument only for notational convenience. After writing $K(m_1, m_2, m) = K(m_1, m_2)$, some simple manipulations reduce Eq. (1) to the more "standard" form often considered in the literature:

$$\frac{\partial c(m,t)}{\partial t} = \frac{\lambda}{2}\int_0^\infty dm_1 dm_2 K(m_1,m_2)c(m_1,t)c(m_2,t) \left[\delta(m-m_1-m_2)-\delta(m-m_1)-\delta(m-m_2)\right] + \frac{J_0}{m_0}\delta(m-m_0) - \frac{J[c]}{M}\delta(m-M). \quad (2)$$

Note that the addition of the aforementioned source and sink terms allows a time independent steady state to be reached in the limit of large time. This is the main subject of this paper. We shall be concerned with the situation where $m_0 \to 0$ and $M \to \infty$, bearing in mind that the presence of a sink at infinity may be required even at finite times in the case of the so-called "gelling" kernels. As mentioned already, we only consider here sources for which the total flux of mass into the system, J_0, is constant. In the turbulence literature the interval of masses for which $m_0 \ll m \ll M$ is called an *inertial range*. The stationary states considered in this article are valid in this range.

The details of the transport mechanism and sticking probability are assumed to be built into the kernel, $K(m_1, m_2)$, of the Smoluchowski equation. Different kernels arise in different physical contexts and determine how the solution of the equation should be interpreted physically. We refer to Refs. [3, 4] for a short list of commonly considered kernels and their physical and/or mathematical contexts.

Most of the kernels of physical interest are homogeneous functions of their arguments. We shall denote the degree of homogeneity of the kernel by ζ. That is

$$K(hm_1, hm_2, hm) = h^\zeta K(m_1, m_2, m). \quad (3)$$

This homogeneity need not be uniformly weighted between the two arguments. Following [3], we introduce exponents, μ and ν to take into account this fact:

$$K(m_1, m_2, m) \sim m_1^\mu m_2^\nu \quad \text{for } m_2 \gg m_1. \quad (4)$$

The exponents μ and ν satisfy $\mu + \nu = \zeta$. Let us consider a couple of simple examples to clarify our notation. The kernel $K(m_1, m_2) = \lambda(m_1^{1+\epsilon} + m_2^{1+\epsilon})$ has $\zeta = 1 + \epsilon$, $\mu = 0$ and $\nu = 1 + \epsilon$ whereas the kernel $K(m_1, m_2) = \lambda(m_1^{1/3} + m_2^{1/3})(m_1^{-1/3} + m_2^{-1/3})$ has $\lambda = 0$, $\mu = -1/3$ and $\nu = 1/3$. These basic properties of the kernel are all we shall require for what follows.

In this paper, we study the steady state behavior of $c(m)$ when $m_0 \ll m \ll M$. In Sec. II, using dimensional analysis, we derive the large mass dependence of $c(m)$. It is also shown that the power law spectrum corresponds to a constant flux of mass in mass space. In Sec. III, we show that the Smoluchowski equation is mathematically very similar to the kinetic equation for 3-wave turbulence. Using Zakharov transformations from 3-wave turbulence, we rederive the mass spectrum as well as compute the amplitude also known as the Kolmogorov constant. The characteristic mass of the source, m_0, does not appear in the dimensional argument. In Sec. IV, we find the conditions under which this assertion is correct when we address the question of the locality of the mass cascade. In Sec. V, we discuss the notion of mass capacity of the Kolmogorov spectrum and show how it distinguishes between gelling and non-gelling kernels. In Sec. VI, we explicitly compute the Kolmogorov spectrum for a one-parameter family of kernels given by $K_\zeta(m_1, m_2) = (m_1 m_2)^{\zeta/2}$. We find that the value of the Kolmogorov constant is the same for all models in this family. Finally, we end with a summary in Sec. VII.

II. DIMENSIONAL DERIVATION OF THE STATIONARY SPECTRUM

Before proceeding into detailed analysis of the stationary states of model Eq. (1) let us first describe intuitively what we mean by a Kolmogorov solution by employing a simple scaling argument. We shall use the simplified form Eq. (2) for brevity. The stationary energy distribution of forced hydrodynamic turbulence is described by the famous Kolmogorov 5/3 spectrum (for instance, see [5]). This spectrum, postulated from dimensional considerations, carries a constant flux of energy from large scales to small by means of vortex-vortex interactions. The analogous cascade for the Smoluchowski equation is a cascade of mass from small particles to large mediated by the coagulation of aggregates. The Kolmogorov spectrum for aggregation carries a constant flux of mass.

The physical dimensions of the various quantities appearing in Eq. (2) are as follows : $[c] = \mathrm{M}^{-1}\,\mathrm{L}^{-d}$, $[J] = \mathrm{M}\,\mathrm{L}^{-d}\,\mathrm{T}^{-1}$ and $[\lambda] = \mathrm{M}^{-\zeta}\,\mathrm{L}^{d}\,\mathrm{T}^{-1}$. If we now take the combination $c \sim J^\gamma \lambda^\alpha m^\beta$, simple dimensional analysis requires that we choose $\gamma = 1/2$, $\alpha = -1/2$ and $\beta = -(\zeta + 3)/2$. Dimensional considerations therefore lead us to a spectrum of the form

$$c(m) \sim \sqrt{\frac{J_0}{\lambda}}\, m^{-\frac{\zeta+3}{2}}. \quad (5)$$

The characteristic mass of the source, m_0, does not appear in our dimensional argument on the basis that we expect this solution to be valid for masses much greater than m_0. We find the conditions under which this assertion is correct in Sec. IV when we address the question of the locality of the mass cascade.

The exponent Eq. (5) is not new. It appeared in early work by Hendriks, Ernst and Ziff [6] as the scaling of the post-gel stage of gelling systems. Their work makes an implicit connection between this scaling and the fact that there is a mass flux out of the system in the post-gel stage. It was then derived explicitly for the Smoluchowski equation with source term by Hayakawa [7] for a particular family of kernels but without making any connection with the physical role played by the mass flux.

That the spectrum Eq. (5) corresponds to a constant flux of mass in mass space is easily seen from the following scaling argument. We express the local conservation of mass by means of the continuity equation

$$\frac{\partial mc(m,t)}{\partial t} = -\frac{\partial J(m,t)}{\partial m}, \tag{6}$$

where

$$\frac{\partial J(m,t)}{\partial m} = -\frac{m\lambda}{2}\int_0^\infty dm_1 dm_2 \big[K(m_1,m_2)c(m_1,t)\times c(m_2,t)\left\{\delta(m-m_1-m_2)-\delta(m-m_1)-\delta(m-m_2)\right\}\big] \tag{7}$$

We now assume a stationary spectrum, $c(m) = Cm^{-x}$. By introducing new variables, $m_1 = m\mu_1$, $m_2 = m\mu_2$ and using the scaling properties of the kernel we deduce that

$$\frac{\partial J}{\partial m} \propto m^{2+\zeta-2x} \tag{8}$$

with the constant of proportionality being given by the integral expression which remains after scaling out the m dependence of the RHS of Eq. (7). Thus

$$J(m) \propto m^{3+\zeta-2x}. \tag{9}$$

It is clear from Eq. (6) that in order to have a stationary state, J must be independent of m which determines the exponent of the Kolmogorov spectrum as

$$x_K = \frac{3+\zeta}{2}. \tag{10}$$

Of course we cannot determine the Kolmogorov constant, C, from such a scaling argument. In addition the validity of our scaling argument depends on the convergence of the various integral expressions which have been hidden behind proportionality signs.

In this paper we address these short-comings by computing the exact stationary solutions of Eq. (2) using the method of Zakharov transformations borrowed from the theory of wave turbulence. We obtain the exponent, x_K expected from scaling considerations and the value of the Kolmogorov constant, C. An answer is obtained for arbitrary homogeneous kernels. However the analysis involves the exchange of orders of integration on the RHS of Eq. (2). It is thus necessary to check a posteriori that the RHS is convergent on the prospective spectrum in order that it be an admissible solution. This check leads to a "locality criterion", namely

$$\mu - \nu + 1 > 0, \tag{11}$$

which must be satisfied by the kernel in order that the Kolmogorov spectrum be realizable.

III. ZAKHAROV TRANSFORMATION FOR SMOLUCHOWSKI EQUATION

To find the stationary solutions of Eq. (1) in the situation $m \gg m_0$, $J_0 =$ constant, we must solve:

$$\begin{aligned} 0 = & \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, K(m_1,m_2,m)\, c(m_1)c(m_2)\, \delta(m-m_1-m_2) - \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, K(m_1,m,m_2)\, c(m)c(m_1)\, \delta(m_2-m-m_1) \\ & - \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, K(m,m_2,m_1)\, c(m)c(m_2)\, \delta(m_1-m_2-m). \end{aligned} \tag{12}$$

Structurally this equation is very similar to the kinetic equation for wave turbulence with a 3-wave interaction. For an introduction to the theory of wave turbulence see [8]. A useful trick for finding the stationary power law solutions of such equations was devised by Zakharov [9, 10] in the late 60's and is easily applied here. Restricting ourselves to power law solutions of the form $c(m) = Cm^{-x}$, we apply the following changes of variables,

$$(m_1, m_2) \rightarrow (\frac{mm_1'}{m_2'}, \frac{m^2}{m_2'}) \tag{13}$$

$$(m_1, m_2) \rightarrow (\frac{m^2}{m_1'}, \frac{mm_2'}{m_1'}). \tag{14}$$

to the second and third integrals in Eq. (12) respectively. Dropping the primes on the transformed variables and using the homogeneity and symmetry properties of the kernel we obtain the equation

$$\begin{aligned} 0 = & \frac{\lambda C^2}{2}\int_0^\infty dm_1 dm_2 \big[K(m_1,m_2,m)(m_1 m_2)^{-x} m^{2-\zeta-2x}\times \\ & \left(m^{2x-\zeta-2} - m_1^{2x-\zeta-2} - m_2^{2x-\zeta-2}\right)\delta(m-m_1-m_2)\big] \end{aligned} \tag{15}$$

It is immediately evident that the integrand is identically zero for $2x - \zeta - 2 = 1$ from which we get the same Kolmogorov exponent,

$$x_K = \frac{3+\zeta}{2}, \tag{16}$$

obtained in Sec. I by a scaling argument. The value of the Kolmogorov constant can be determined by considering the local mass flux defined from Eqs. (6) and (7). Restricting ourselves to spectra of the form $c(m) = Cm^{-x}$, the Zakharov transformation allows us to write Eq. (7) in the form

$$\begin{aligned}\frac{\partial J(m,t)}{\partial m} &= -\frac{\lambda m C^2}{2}\int_0^\infty dm_1 dm_2 K(m_1,m_2,m)(m_1m_2)^{-x} m^{2-\zeta-2x}\left(m^{2x-\zeta-2} - m_1^{2x-\zeta-2} - m_2^{2x-\zeta-2}\right)\delta(m-m_1-m_2)\\ &= \lambda C^2 m^{2+\zeta-2x} I(x)\end{aligned}$$

where

$$I(x) = -\frac{1}{2}\int_0^\infty d\mu_1 d\mu_2 \big[K(\mu_1,\mu_2,1)(\mu_1\mu_2)^{-x}\times \left(1-\mu_1^{2x-\zeta-2} - \mu_2^{2x-\zeta-2}\right)\delta(1-\mu_1-\mu_2)\big] \quad (17)$$

From this we deduce that the flux is given by

$$J(m) = \frac{\lambda C^2 I(x)}{3+\zeta-2x} m^{3+\zeta-2x}. \quad (18)$$

In the steady state, $x = x_K = (3+\zeta)/2$ and the flux must be a constant equal to J_0. Thus we have

$$J_0 = \lim_{x\to x_K} \frac{\lambda C^2 I(x)}{3+\zeta-2x} m^{3+\zeta-2x}. \quad (19)$$

We know that $I(x_K) = 0$ so we must apply l'Hopital's rule to evaluate the limit to arrive at

$$J_0 = \frac{\lambda}{2} C^2 \left.\frac{dI}{dx}\right|_{x_K}, \quad (20)$$

and hence

$$C = \sqrt{\frac{2J_0}{\lambda}\left.\frac{dI}{dx}\right|_{x_K}^{-1}}. \quad (21)$$

The Kolmogorov solution is therefore,

$$c(m) = C\, m^{-x_K}, \quad (22)$$

with C given by Eq. (21) and x_K given by Eq. (16).

IV. LOCALITY OF THE MASS CASCADE

In the analysis of the previous section we have freely split the integrand on the RHS of Eq. (1) and exchanged orders of integration to derive the Kolmogorov spectrum. In order to justify these manipulations we must demonstrate a posteriori that the original collision integral is convergent on the Kolmogorov spectrum. We do this in this section.

The support of the integrand on the RHS of Eq. (1) is shown in Fig. 1. Since the kernel and the mass distributions which we study are scale invariant the only possible sources of divergences are at infinity and at the two points $(0,m)$ and $(m,0)$ where the contour of integration intersects the axes. Let us now study carefully the behavior of the integrand near these points for power law mass distributions.

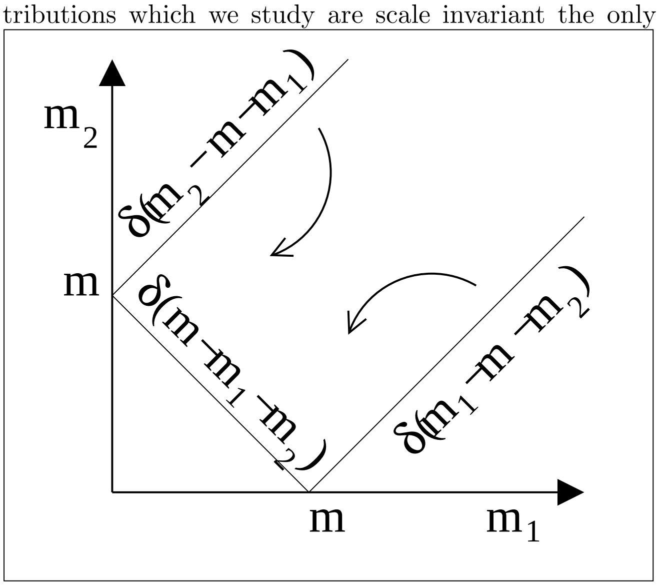

FIG. 1: Support of the integrand of Eq. (1)

The behavior at infinity is easy. As $m_2 \to \infty$ along the lower contour the integrand looks like

$$K(m, m_2, m+m_2)c(m)c(m_2) \sim m^{\mu-x} m_2^{\nu-x}. \quad (23)$$

The integral is therefore convergent as $m_2 \to \infty$ if

$$\nu - x < -1 \Rightarrow x > \nu + 1. \quad (24)$$

The same criterion is obtained along the upper contour. The convergence near zero requires a little care. Near $m_1 = 0$ the integrand looks like

$$
\begin{aligned}
& c(m_1)\left[K(m_1,m-m_1,m)c(m-m_1)-K(m_1,m,m+m_1)c(m)\right] \\
= \;& c(m_1)\left[m_1\left(\frac{\partial}{\partial\xi}\left(K(m_1,\xi,m)c(\xi)\right)|_{\xi=m}-c(m)\frac{\partial}{\partial\xi}\left(K(m_1,m,\xi)\right)|_{\xi=m}\right)+\mathrm{o}(m_1^2)\right] \\
\sim \;& m_1^{1-x+\mu}.
\end{aligned}
$$

Note the cancellation of the leading order terms in the Taylor expansion on the second line above. The corresponding integral is convergent as $m_1 \to 0$ for

$$\mu+1-x>-1 \Rightarrow x<\mu+2. \tag{25}$$

The same criterion is obtained if we look near $m_2=0$. We conclude from Eqs. (24) and (25) that a power law mass distribution, Cm^{-x}, yields a convergent collision integral if x lies in the interval $[\nu+1,\mu+2]$. The existence of such an interval of convergence puts a constraint on the asymptotic behavior of the kernel, namely

$$\mu-\nu+1>0. \tag{26}$$

We now must address the question of when the Kolmogorov spectrum derived in Sec. III lies in this interval of convergence. The answer is surprisingly simple. Remembering that $\mu+\nu=\zeta$, it is immediately evident from Eq. (16) that the Kolmogorov spectrum lies midway between the two constraints Eqs. (24) and (25). Therefore *if* an interval of convergence exists for a given kernel, the corresponding Kolmogorov spectrum is an admissible stationary solution of Eq. (1) and it lies at the midpoint of the interval of convergence.

We call Eq. (26) a *locality* criterion since systems for which it is satisfied can be characterized in the stationary state by a local mass flux, J_0. When the spectrum is local, the details of how we take the limits $m_0 \to 0$ and $M \to \infty$ to produce a large inertial range are inconsequential since all integrals converge. If Eq. (26) is not satisfied then presumably the final stationary state depends on the details of the source/sink and is therefore non-universal.

We note that the kernel $m_1^{1+\epsilon}+m_2^{1+\epsilon}$, mentioned in the introduction, is marginal in the sense that it violates the locality criterion for any finite ϵ. It has been shown[11] that this kernel undergoes instantaneous gelation so perhaps there is some connection between this phenomenon and the concept of locality. In addition, the generalised sum kernel, $K(m_1,m_2)=m_1^{-\mu}+m_2^{-\mu}$, which violates the locality condition for $\mu \geq 1$, was studied extensively by Krapivsky et al. [12, 13]. They found that in this case, the system does not reach a steady state but rather continues to evolve very slowly on a logarithmic timescale for all time.

In closing this section it should be noted that a rigorous understanding of the conditions under which the stationary state depends only on the local flux is one of the missing pieces in the theory of hydrodynamic turbulence.

V. FINITE AND INFINITE CAPACITY CASES – GELLING AND NON-GELLING KERNELS

It was found in the 60's [14] that the solution of Eq. (2) for certain kernels violates mass conservation within a finite time, t^*. When this violation occurs, $\lim_{m\to\infty} P(m)$ becomes finite. In the late 70's it was found that meaningful solutions exist post-t^* and the violation of mass conservation was given a physical interpretation in terms of what is now termed a "gelation transition" [15, 16, 17]. Gelation occurs when there is a finite flow of mass to an infinite mass cluster ("gel"). As a consequence, the total mass of the normal ("sol") particles is no longer conserved. In order to avoid inconsistencies, the gel particles must be considered as those clusters whose mass diverges as the size of a finite system is taken to infinity. It is now well known [3] that the solutions of Eq. (2) undergo gelation for kernels having $\zeta > 1$.

The gelation criterion, $\zeta>1$, can be given a very simple physical interpretation by examining the mass capacity of the Kolmogorov spectrum. If we continue to add mass to the system at a constant rate J_0 then we know that the final steady state is given by the Kolmogorov spectrum Eq. (22). When the total mass contained in this solution is finite then mass conservation must be violated at some time since the total mass supplied to the system grows linearly in time. The total mass capacity of the Kolmogorov spectrum is finite when

$$\int_{m_0}^{\infty} dm\, m\, C\, m^{-\frac{3+\zeta}{2}} < \infty.$$

This integral in convergent at its upper limit when $1-(3+\zeta)/2<-1$ or $\zeta>1$. Thus gelation can be seen as a kind of safety valve which allows mass to flow out of the system when the Kolmogorov spectrum is incapable of absorbing all of the mass supplied to the system. Conversely, one would expect intuitively that infinite capacity systems should not exhibit gelation.

VI. EXAMPLE : THE FAMILY OF KERNELS, $K_\zeta(m_1,m_2,m)=(m_1m_2)^{\zeta/2}$

In this section we explicitly evaluate the Kolmogorov constant, C, for the family of kernels, $K_\zeta(m_1,m_2,m)=(m_1m_2)^{\zeta/2}$. These kernels have $\mu=\nu=\zeta/2$ so that the corresponding Kolmogorov spectrum always satisfies the locality criterion, Eq. (26). The family includes both gelling and non-gelling kernels. In general to compute, C, we need to evaluate the following integral at $x = (3+\zeta)/2$:

$$\frac{dI}{dx} = \frac{1}{2}\int_0^1 d\mu_1 K(\mu_1, 1-\mu_1, 1)\left\{(\mu_1(1-\mu_1))^{-x}\left[-2\mu_1^{2x-\zeta-2}\log\mu_1 - 2(1-\mu_1)^{2x-\zeta-2}\log(1-\mu_1)\right.\right.$$
$$\left.\left. - \log(\mu_1(1-\mu_1))(1-\mu_1^{2x-\zeta-2} - (1-\mu_1)^{2x-\zeta-2})\right]\right\}. \quad (27)$$

This is obtained from Eq. (17) by integrating out μ_2 and differentiating with respect to x. When we set $K(\mu_1, \mu_2, 1) = (\mu_1\mu_2)^{\zeta/2}$ and $x = (3+\zeta)/2$ in this expression we find, rather surprisingly, that all dependence on ζ cancels out and we are left with

$$\left.\frac{dI}{dx}\right|_{(3+\zeta)/2} = -\int_0^1 d\mu_1 \frac{\mu_1\log\mu_1 + (1-\mu_1)\log(1-\mu_1)}{\mu_1^{3/2}(1-\mu_1)^{3/2}} = 4\pi \qquad \textit{(Mathematica)} \quad (28)$$

Hence the Kolmogorov solution for all kernels in this family is

$$c(m) = \sqrt{\frac{J_0}{2\pi\lambda}}\, m^{-\frac{3+\zeta}{2}}. \quad (29)$$

To close we note that we can check our answer independently for at least one case. For the constant kernel with zero initial concentration, an exact solution of Eq. (2) has been known for some time. The details can be found in [18]. This solution is

$$c_m(t) = \sum_{k=1}^{\infty} c_k(t)\,\delta(m - km_0) \quad (30)$$

with

$$c_k(t) = \frac{m_0\pi^2}{\lambda^2 J_0 t^3}\sum_{j=-\infty}^{j=\infty}(2j+1)^2\left[1 + \frac{(2j+1)^2 m_0\pi^2}{2\lambda J_0 t^2}\right]^{-k-1}. \quad (31)$$

The $t \to \infty$ limit of this expression exists can be calculated by replacing the sum by an integral in the limit of large t. This integral can be expressed in terms of gamma functions. One finds

$$\lim_{t\to\infty} c_k(t) = \sqrt{\frac{J_0}{2\pi\lambda m_0}}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+1)} \sim \sqrt{\frac{J_0}{2\pi\lambda m_0}} k^{-\frac{3}{2}} \quad \text{for } k \gg 1.$$

Setting $m_0 = 1$ we recover the result of our earlier computation of the Kolmogorov spectrum for $m \gg 1$.

As pointed out to us by one of our referees, the constant, C, has also been computed [12, 13] for the generalised sum kernel

$$K(m_1, m_2) = m_1^{-\zeta} + m_2^{-\zeta}. \quad (32)$$

We computed the integral (27) for this kernel using *Mathematica* and found the Kolmogorov constant to be

$$C = \sqrt{\frac{J_0(1-\zeta^2)\cos\frac{\pi\zeta}{2}}{4\lambda\pi}} \quad (33)$$

as found in [12, 13] using completely different methods.

VII. CONCLUSION

To summarize, we have shown how the notion of a mass cascade analogous to the Kolmogorov energy cascade of hydrodynamic turbulence is relevant to understanding the stationary state of the Smoluchowski equation with constant mass production term. Furthermore we have shown how the exact stationary spectrum may be computed using the method of Zakharov transformations and given some criteria for assessing the physical validity of this solution. We have not made any attempt to address the important question of the validity of the Smoluchowski equation itself in describing the statistics of particular aggregation models. The mean field assumption leading to Eq. (1) can be violated in several ways as discussed in [18]. Of particular relevance to lattice aggregation models is the case where fluctuations dominate the statistics and invalidate the mean field Smoluchowski equation [19, 20]. In a future publication [21] we shall address this issue for the particular case of constant kernel stochastic aggregation where the presence of fluctuations leads to a renormalization of the constant λ. The techniques developed in this paper will allow us to find the renormalized Kolmogorov spectrum as the stationary solution of a modified Smoluchowski equation.

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Note added in proof

We have found that V.M. Kontorovich has recently applied the Zakharov Transformation to aggregation problems [22] for a class of kernels arising from the study of galactic mergers in astrophysics.

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