Functional central limit theorems for a large network in which customers join the shortest of several queues

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Abstract. We consider N single server infinite buffer queues with service rate β. Customers arrive at rate $N\alpha$, choose L queues uniformly, and join the shortest. We study the processes $t \in \mathbb{R}_+ \mapsto R_t^N = (R_t^N(k))_{k\in\mathbb{N}}$ for large N, where $R_t^N(k)$ is the fraction of queues of length at least k at time t. Laws of large numbers (LLNs) are known, see Vvedenskaya et al. [15], Mitzenmacher [12] and Graham [5]. We consider certain Hilbert spaces with the weak topology. First, we prove a functional central limit theorem (CLT) under the *a priori* assumption that the initial data R_0^N satisfy the corresponding CLT. We use a compactness-uniqueness method, and the limit is characterized as an Ornstein-Uhlenbeck (OU) process. Then, we study the R^N in equilibrium under the stability condition $\alpha < \beta$, and prove a functional CLT with limit the OU process in equilibrium. We use ergodicity and justify the inversion of limits $\lim_{N\to\infty}\lim_{t\to\infty} = \lim_{t\to\infty}\lim_{N\to\infty}$ by a compactness-uniqueness method. We deduce *a posteriori* the CLT for R_0^N under the invariant laws, an interesting result in its own right. The main tool for proving tightness of the implicitly defined invariant laws in the CLT scaling and ergodicity of the limit OU process is a global exponential stability result for the nonlinear dynamical system obtained in the functional LLN limit.

Key-words: Mean-field interaction, load balancing, resource pooling, ergodicity, non-equilibrium fluctuations, equilibrium fluctuations, birth and death processes, spectral gap, global exponential stability

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1 Introduction

1.1 Preliminaries

We consider a Markovian network constituted of $N \geq L \geq 1$ infinite buffer single server queues. Customers arrive at rate $N\alpha$, are each allocated L distinct queues uniformly at random, and join the shortest, ties being resolved uniformly. Servers work at rate β. Arrivals, allocations, and services are independent. For $L = 1$ we have i.i.d. $M_\alpha/M_\beta/1/\infty$ queues. For $L \geq 2$ the interaction structure depends only on sampling from the empirical measure of L-tuples of queue states: in statistical mechanics terminology, the system is in L-body mean-field interaction. We continue the large N study introduced by Vvedenskaya et al. [15] and Mitzenmacher [12] and continued in Graham [5].

The process $(X_i^N)_{1\leq i\leq N}$ is Markov, where $X_i^N(t)$ denotes the length of queue i at time t in $\mathbb{R}_+$. Its empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}$ has samples in $\mathcal{P}(\mathbb{D}(\mathbb{R}_+, \mathbb{N}))$, and its marginal process

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$\bar{X}^N = (\bar{X}^N_t)_{t\geq 0}$ with $\bar{X}^N_t = \mu_t = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)}$ has sample paths in $\mathbb{D}(\mathbb{R}_+, \mathcal{P}(\mathbb{N}))$. We are interested in the tails of the marginals $\bar{X}^N_t$ and consider

$$R^N = (R^N_t)_{t\geq 0}\,, \qquad R^N_t = (R^N_t(k))_{k\in\mathbb{N}}\,, \qquad R^N_t(k) = \frac{1}{N}\sum_{i=1}^N \mathbb{I}_{X_i^N(t)\geq k}\,,$$

and $R^N_t(k)$ is the fraction of queues of length at least k at time t. For the uniform topology on

$$\mathcal{V} = \{(v(k))_{k\in\mathbb{N}} : v(0) = 1,\ v(k) \geq v(k+1),\ \lim v = 0\} \subset c_0\,, \qquad \mathcal{V}^N = \mathcal{V} \cap \frac{1}{N}\mathbb{N}^{\mathbb{N}}\,,$$

coinciding here with the product topology, the process R^N has sample paths in $\mathbb{D}\big(\mathbb{R}_+, \mathcal{V}^N\big)$.

The processes $\bar{X}^N$ and R^N are in relation through $p \in \mathcal{P}(\mathbb{N}) \longleftrightarrow v \in \mathcal{V}$ for $v(k) = p[k,\infty)$ and $p\{k\} = v(k) - v(k+1)$ for k in $\mathbb{N}$. This classical homeomorphism maps the subspace of probability measures with finite first moment onto $\mathcal{V} \cap \ell_1$, corresponding to a finite number of customers in the network. The symmetry structure implies that these processes are Markov.

The stationary regime has great practical relevance. The stability condition $\alpha < \beta$ (Theorem 5 (a) in [15], Lemma 3.1 in [12], Theorem 4.2 in [5]) is obtained from ergodicity criteria yielding little information. We study the large N asymptotics of R^N, first for transient regimes with appropriately converging initial data, and then in equilibrium using an indirect approach involving ergodicity in well-chosen transient regimes and an inversion of limits for large N and large times. Law of large numbers (LLN) results are already known, and we obtain functional central limit theorems (CLTs).

1.2 Previous results: laws of large numbers

We relate results found in essence in Vvedenskaya et al. [15]. We follow Graham [5] which extends these results, notably by considering the empirical measures on path space μ^N and thus yielding chaoticity results (asymptotic independence of queues). Chapter 3 in Mitzenmacher [12] gives related results. (The rates α and β correspond to λ and 1 in [15, 12] and ν and λ in [5].)

Consider the mappings with values in c_0^0 given for v in c_0 by

$$F_+(v)(k) = \alpha\big(v(k-1)^L - v(k)^L\big)\,, \quad F_-(v)(k) = \beta(v(k) - v(k+1))\,, \qquad k \geq 1\,, \tag{1.1}$$

and $F = F_+ - F_-$, and the nonlinear differential equation $\dot{u} = F(u)$ on $\mathcal{V}$, given for $t \geq 0$ by

$$\begin{aligned}\dot{u}_t(k) &= \alpha\big(u_t(k-1)^L - u_t(k)^L\big) - \beta(u_t(k) - u_t(k+1)) \\ &= \alpha u_t(k-1)^L - \big(\alpha u_t(k)^L + \beta u_t(k)\big) - \beta u_t(k+1)\,, \qquad k \geq 1\,. \end{aligned} \tag{1.2}$$

This corresponds to the systems (1.6) in [15], (3.5) in [12] and (3.9) in [5]. Note that F_- is linear.

Theorem 1.1 *There exists a unique solution $u = (u_t)_{t\geq 0}$ taking values in $\mathcal{V}$ for (1.2), and u is in $C(\mathbb{R}_+, \mathcal{V})$. If u_0 is in $\mathcal{V} \cap \ell_1$ then u takes values in $\mathcal{V} \cap \ell_1$.*

Proof. We use Theorem 3.3 and Proposition 2.3 in [5]. These exploit the homeomorphism between $\mathcal{P}(\mathbb{N})$ with the weak topology and $\mathcal{V}$ with the product topology. Then (1.2) corresponds to a non-linear forward Kolmogorov equation for a pure jump process with uniformly bounded (time-dependent) jump rates. Uniqueness within the class of bounded measures and existence of a probability-measure valued solution are obtained using the total variation norm. Theorem 1 (a) in [15] yields existence (and uniqueness) in $\mathcal{V} \cap \ell_1$. □

Firstly, a functional LLN for initial conditions satisfying a LLN is part of Theorem 3.4 in [5] and can be deduced from Theorem 2 in [15].

Theorem 1.2 *Assume that $(R_0^N)_{N\geq L}$ converges in law to u_0 in $\mathcal{V}$. Then $(R^N)_{N\geq L}$ converges in law in $\mathbb{D}(\mathbb{R}_+, \mathcal{V})$ to the unique solution $u = (u_t)_{t\geq 0}$ starting at u_0 for (1.2).*

Secondly, for $\alpha < \beta$ the limit equation (1.2) has a globally attractive stable point $\tilde{u}$ in $\mathcal{V} \cap \ell_1$.

Theorem 1.3 *Let $\rho = \alpha/\beta < 1$. The equation (1.2) has a unique stable point in $\mathcal{V}$ given by*

$$\tilde{u} = (\tilde{u}(k)_{k\in\mathbb{N}}\,, \qquad \tilde{u}(k) = \rho^{(L^k-1)/(L-1)} = \rho^{L^{k-1}+L^{k-2}+\cdots+1}\,,$$

and the solution u of (1.2) starting at any u_0 in $\mathcal{V} \cap \ell_1$ is such that $\lim_{t\to\infty} u_t = \tilde{u}$.

Proof. Theorem 1 (b) in [15] yields that $\tilde{u}$ is globally asymptotically stable in $\mathcal{V} \cap \ell_1$. A stable point u in $\mathcal{V}$ satisfies $\beta u(k+1) - \alpha u(k)^L = \beta u(k) - \alpha u(k-1)^L = \cdots = \beta u(1) - \alpha$ and converges to 0, hence $u(1) = \alpha/\beta$ and $u(2), u(3), \ldots$ are successively determined uniquely. □

Lastly, a compactness-uniqueness argument justifies the inversion of limits $\lim_{N\to\infty}\lim_{t\to\infty} = \lim_{t\to\infty}\lim_{N\to\infty}$, which yields a result in equilibrium. This method, used by Whitt [16] for the star-shaped loss network, is detailed in Graham [6] Sections 9.5 and 9.7.3. The following functional LLN in equilibrium (Theorem 4.4 in [5]) can be deduced from [15] but is not stated there, and implies that under the invariant laws $\lim_{N\to\infty} \mathbf{E}(R_0^N(k)) = \tilde{u}(k)$ for $k \in \mathbb{N}$ (Theorem 5 (c) in [15]).

Theorem 1.4 *Let $\rho = \alpha/\beta < 1$ and the networks of size N be in equilibrium. Then $(R^N)_{N\geq L}$ converges in probability in $\mathbb{D}(\mathbb{R}_+, \mathcal{V})$ to $\tilde{u}$.*

Note that $\tilde{u}(k)$ decays hyper-exponentially in k for $L \geq 2$ instead of the exponential decay ρ^k corresponding to i.i.d. queues in equilibrium ($L = 1$). For finite networks in equilibrium there is at most exponential decay since $\mathbf{P}\big(X_1^N + \cdots + X_N^N \geq Nk\big) \leq \mathbf{P}\big(X_1^N \geq k\big) + \cdots + \mathbf{P}\big(X_N^N \geq k\big)$ and by comparison with an $M_{N\alpha}/M_{N\beta}/1$ queue

$$\mathbf{E}\big(R_t^N(k)\big) = \mathbf{P}\big(X_i^N(t) \geq k\big) \geq \frac{1}{N}\rho^{Nk}\,, \qquad k \geq 0\,. \tag{1.3}$$

The asymptotic queue sizes are dramatically decreased by this simple load balancing (or resource pooling) procedure, which carries little overhead even for large N since L is fixed (for instance $L = 2$). This feature is quite robust and true for many systems, as was illustrated on several examples by Mitzenmacher [12] and Turner [14] using proofs as well as simulations. It can be used as a guideline for designing practical networks. In contrast, the bound (1.3) assumes the best utilization of the N servers, fully collaborating even for a single customer.

Theorem 3.5 in Graham [5] gives convergence bounds on bounded time intervals $[0, T]$ for i.i.d. $(X_i^N(0))_{1\le i\le N}$ using results in Graham and Méléard [7]. This can be extended if the initial laws satisfy *a priori* controls, but it is not so in equilibrium (the bounds are exponentially large in T).

1.3 The outline of this paper

The study of the fluctuations around the functional LLN will yield for instance asymptotically tight confidence intervals for the process $t \mapsto N^{-1}\mathrm{Card}\{i = 1, \dots, N : X_i^N(t) \in A\}$. In a realistic setting (finite number of finite buffer queues) such confidence intervals would allow network evaluation or dimensioning in function of quality of service requirements on delays and overflows. The LLN on path space concerns objects such as $N^{-1}\mathrm{Card}\{i = 1, \dots, N : \big(t \mapsto X_i^N(t)\big) \in B\}$ with a richer temporal structure, but topological difficulties usually block the corresponding fluctuation study.

We consider the process R^N and solution u for (1.2) starting at R_0^N in $\mathcal{V}^N$ and u_0 in $\mathcal{V}$, and

$$Z^N = \sqrt{N}(R^N - u). \tag{1.4}$$

The processes $Z^N = (Z_t^N)_{t\ge 0}$ will be studied in the Skorokhod spaces on appropriate Hilbert spaces with the weak topology. These spaces are not metrizable and require appropriate tightness criteria.

We first consider a wide class of R_0^N and u_0 under the *assumption* that $(Z_0^N)_{N\ge L}$ converges in law (for instance satisfies a CLT). We obtain a functional CLT in relation to Theorem 1.2, with limit given by an Ornstein-Uhlenbeck (OU) process starting at the limit of the $(Z_0^N)_{N\ge L}$. This covers without constraints on α and β many *transient* regimes with *explicit* initial conditions, such as initially empty networks, or more generally i.i.d. initial queue sizes.

We then focus on the *stationary* regime for $\alpha < \beta$. The initial data is now *implicit*: the law of R_0^N is the invariant law for R^N and $u_0 = \tilde{u}$. We prove tightness for $(Z_0^N)_{N\ge L}$ using the ergodicity of Z^N for fixed N and intricate fine studies of the long-time behavior of the nonlinear dynamics appearing at the large N limit. The main result in this paper is a functional CLT in equilibrium for $(Z^N)_{N\ge L}$ with limit the OU process in equilibrium. This *implies* a CLT under the invariant laws for $(Z_0^N)_{N\ge L}$, an important result which seems difficult to obtain directly.

Section 2 introduces without proof the main notions and results. Section 3 gives the proof of the functional CLT for converging initial data by compactness-uniqueness and martingale techniques.

We then consider $u_0 = \tilde{u}$. We study the OU process in Section 4, derive a spectral representation for the linear operator in the drift, and prove the existence of a spectral gap. A main difficulty is that the scalar product for which the operator is self-adjoint is *too strong* for the limit dynamical system and the invariant laws for finite N. We consider appropriate Hilbert spaces in which the operator is *not* self-adjoint and prove exponential stability.

In Section 5 we likewise prove that $\tilde{u}$ is globally exponentially stable for the non-linear dynamical system. In Section 6 we obtain bounds for Z_t^N uniform for $t \geq 0$ and large N, using the preceding stability result in order to iterate the bounds on intervals of length T. Bounds on the invariant laws of Z^N follow using ergodicity. The proof for the functional CLT in equilibrium follows from a compactness-uniqueness argument involving the functional CLT for converging initial data.

2 The functional central limit theorems

2.1 Preliminaries

The exponential of a bounded linear operator is given by the usual series expansion. Let c_0^0 and ℓ_p^0 for $p \geq 1$ be the subspaces of sequences vanishing at 0 of the classical sequence spaces c_0 (with limit 0) and ℓ_p (with summable p-th power). In matrix notation we use the canonical basis, hence sequences vanishing at 0 are identified with infinite column vectors indexed by $\{1, 2, \cdots\}$. The diagonal matrix with terms given by the sequence a is denoted by $\operatorname{diag}(a)$. Sequence inequalities, etc., should be interpreted termwise. Empty sums are equal to 0 and empty products to 1. Constants such as K may vary from line to line. Let $g_\theta = (\theta^k)_{k\geq 1}$ be the geometric sequence with parameter θ.

For a sequence $w = (w(k))_{k\geq 1}$ such that $w(k) > 0$ we define the Hilbert spaces

$$L_2(w) = \left\{ x \in \mathbb{R}^{\mathbb{N}} : x(0) = 0\,,\ \|x\|_{L_2(w)}^2 = \sum_{k\geq 1} \left(\frac{x(k)}{w(k)} \right)^2 w(k) = \sum_{k\geq 1} x(k)^2 w(k)^{-1} < \infty \right\}$$

and in matrix notation $(x, y)_{L_2(w)} = x^* \operatorname{diag}(w^{-1}) y$. We use the notation $L_2(w)$ since its elements will often be considered as measures identified with their densities with respect to the reference measure w. In this perspective $L_1(w) = \ell_1^0$ and if w is summable then $\|x\|_1 \leq \|w\|_1^{1/2} \|x\|_{L_2(w)}$ and $L_2(w) \subset \ell_1^0$. Using $L_2(1) = \ell_2^0$ as a pivot space, for bounded w we have the Gelfand triplet $L_2(w) \subset \ell_2^0 \subset L_2(w)^* = L_2(w^{-1})$.

Another natural perspective on $L_2(w)$ is that it is an ℓ_2 space with weights, and we consider the ℓ_1 space with same weights (the notation being chosen for consistency)

$$\ell_1(w) = \left\{ x \in \mathbb{R}^{\mathbb{N}} : x(0) = 0\,,\ \|x\|_{\ell_1(w)} = \sum_{k\geq 1} |x(k)| w(k)^{-1} < \infty \right\}$$

and $x \in L_2(w) \Leftrightarrow x^2 \in \ell_1(w)$ with $\|x\|_{L_2(w)}^2 = \|x^2\|_{\ell_1(w)}$. The inclusion $\mathcal{V} \cap \ell_1(w) \hookrightarrow \mathcal{V} \cap L_2(w)$ is continuous since $x^2 \leq |x|$ for $|x| \leq 1$. The following result is trivial.

Lemma 2.1 *If $w = O(v)$ and $v = O(w)$ then the $L_2(v)$ and $L_2(w)$ norms are equivalent, and the $\ell_1(v)$ and $\ell_1(w)$ norms are equivalent.*

In the sequel we often assume that $w = (w_k)_{k\geq 1}$ satisfies the condition that

$$\exists\, c, d > 0\,,\ \forall k \geq 1\,,\ 0 < cw(k+1) \leq w(k) \leq dw(k+1)\,, \tag{2.1}$$

which is satisfied by $g_\theta = (\theta^k)_{k\geq 1}$ with $c = d = 1/\theta$ for $\theta > 0$. It implies that $w(1)d(1/d)^k \leq w(k) \leq w(1)c(1/c)^k$ which bounds w by geometric sequences. The norms have exponentially strong weights for $c > 1$. We give a refined existence result for (1.2). (Proofs are left for later.)

Theorem 2.2 *Let w satisfy (2.1). Then in $\mathcal{V}$ the mappings F, F_+ and F_- are Lipschitz for the $L_2(w)$ and the $\ell_1(w)$ norms. Existence and uniqueness holds for (1.2) in $\mathcal{V} \cap L_2(w)$ and in $\mathcal{V} \cap \ell_1(w)$.*

2.2 The functional CLT for converging initial data

The time-inhomogeneous Ornstein-Uhlenbeck process

In $\mathcal{V}$, the linearization of (1.2) around a particular solution u is the linearization of the recentered equation satisfied by $y = g - u$ where g is a generic solution for (1.2). It is given for $t \geq 0$ by

$$\dot{z}_t = \mathbf{K}(u_t) z_t \tag{2.2}$$

where for v in $\mathcal{V}$ the linear operator $\mathbf{K}(v) : x \mapsto \mathbf{K}(v)x$ on c_0^0 is given by

$$\mathbf{K}(v)x(k) = \alpha L v(k-1)^{L-1} x(k-1) - (\alpha L v(k)^{L-1} + \beta) x(k) + \beta x(k+1)\,, \quad k \geq 1\,, \tag{2.3}$$

and is identified with its infinite matrix in the canonical basis $(0, 1, 0, 0\ldots)$, $(0, 0, 1, 0\ldots)$, $\ldots$

$$\mathbf{K}(v) = \begin{pmatrix} -\big(\alpha L v(1)^{L-1} + \beta\big) & \beta & 0 & \cdots \\ \alpha L v(1)^{L-1} & -\big(\alpha L v(2)^{L-1} + \beta\big) & \beta & \cdots \\ 0 & \alpha L v(2)^{L-1} & -\big(\alpha L v(3)^{L-1} + \beta\big) & \cdots \\ 0 & 0 & \alpha L v(3)^{L-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $(M(k))_{k\in\mathbb{N}}$ be independent real continuous centered Gaussian martingales, determined in law by their deterministic Doob-Meyer brackets given for $t \geq 0$ by

$$\langle M(k) \rangle_t = \int_0^t \big\{ F_+(u_s)(k) + F_-(u_s)(k) \big\}\, ds\,. \tag{2.4}$$

The processes $M = (M(k))_{k\geq 0}$ and $\langle M \rangle = (\langle M(k) \rangle)_{k\in\mathbb{N}}$ have values in c_0^0.

Theorem 2.3 *Let w satisfy (2.1) and u_0 be in $\mathcal{V} \cap \ell_1(w)$. Then the Gaussian martingale M is square-integrable in $L_2(w)$.*

Proof. We have $\mathbf{E}\Big(\|M_t\|^2_{L_2(w)}\Big) = \|\langle M\rangle_t\|_{\ell_1(w)}$ and we conclude using (2.4), Theorem 2.2, and uniform bounds in $\ell_1(w)$ on $(u_s)_{0\le s\le t}$ in function of u_0 given by the Gronwall Lemma. $\square$

The limit equation for the fluctuations is a Gaussian perturbation of (2.2), the inhomogeneous affine SDE given for $t \ge 0$ by

$$Z_t = Z_0 + \int_0^t \mathbf{K}(u_s) Z_s \, ds + M_t \,. \tag{2.5}$$

A well-defined solution is called an Ornstein-Uhlenbeck process, in short OU process. We recall that strong (or pathwise) uniqueness implies weak uniqueness, and that $\ell_1(w) \subset L_2(w)$.

Theorem 2.4 *Let the sequence w satisfy (2.1).*
(a) For v in $\mathcal{V}$, the operator $\mathbf{K}(v)$ is bounded in $L_2(w)$ with operator norm uniformly bounded in v.
(b) Let u_o be in $\mathcal{V} \cap L_2(w)$. Then in $L_2(w)$ there is a unique solution $z_t = \mathrm{e}^{\int_0^t \mathbf{K}(u_s)\, ds} z_0$ for (2.2) and strong uniqueness of solutions holds for (2.5).
(c) Let u_o be in $\mathcal{V} \cap \ell_1(w)$. Then in $L_2(w)$ there is a unique strong solution $Z_t = \mathrm{e}^{\int_0^t \mathbf{K}(u_s)\, ds} Z_0 + \int_0^t \mathrm{e}^{\int_s^t \mathbf{K}(u_r)\, dr} dM_s$ for (2.5) and if $\mathbf{E}\Big(\|Z_0\|^2_{L_2(w)}\Big) < \infty$ then $\mathbf{E}\Big(\sup_{t\le T} \|Z_t\|^2_{L_2(w)}\Big) < \infty$.

Tightness bounds and the CLT

The finite-horizon bounds in the following lemma will yield tightness estimates for the processes Z^N used in the compactness-uniqueness proof for the subsequent theorem.

Lemma 2.5 *Let w satisfy (2.1). Let u_0 be in $\mathcal{V} \cap \ell_1(w)$ and R_0^N be in $\mathcal{V}^N$. For any $T \ge 0$*

$$\limsup_{N\to\infty} \mathbf{E}\left(\left\|Z_0^N\right\|^2_{L_2(w)}\right) < \infty \Rightarrow \limsup_{N\to\infty} \mathbf{E}\left(\sup_{0\le t\le T} \left\|Z_t^N\right\|^2_{L_2(w)}\right) < \infty \,.$$

We refer to Jakubowski [8] for the Skorokhod topology for non-metrizable topologies. For the weak topology of a reflexive Banach space, the relatively compact sets are the bounded sets for the norm, see Rudin [13] Theorems 1.15 (b), 3.18, and 4.3. Hence, if $B(r)$ denotes the closed ball centered at 0 of radius r, a set $\mathcal{T}$ of probability measures is tight if and only if for all $\varepsilon > 0$ there exists $r_\varepsilon < \infty$ such that $p(B(r_\varepsilon)) > 1 - \varepsilon$ uniformly for p in $\mathcal{T}$. We state the functional CLT.

Theorem 2.6 *Let w satisfy (2.1). Consider $L_2(w)$ with its weak topology and $\mathbb{D}(\mathbb{R}_+, L_2(w))$ with the corresponding Skorokhod topology. Let u_0 be in $\mathcal{V} \cap \ell_1(w)$, R_0^N in $\mathcal{V}^N$, and Z^N be given by (1.4). If $(Z_0^N)_{N\ge L}$ converges in law to Z_0 and is tight, then $(Z^N)_{N\ge L}$ converges in law to the unique OU process solving (2.5) starting at Z_0 and is tight.*

2.3 The functional CLT in equilibrium

We assume the stability condition $\rho = \alpha/\beta < 1$ holds, and consider $u_0 = \tilde{u}$.

The Ornstein-Uhlenbeck process

We set $\mathcal{K} = \mathbf{K}(\tilde{u})$ and (2.3) yields that $\mathcal{K} : x \in c_0^0 \mapsto \mathcal{K}x \in c_0^0$ is given by

$$\mathcal{K}x(k) = \mathbf{K}(\tilde{u})x(k) = \beta L\rho^{L^{k-1}}x(k-1) - \left(\beta L\rho^{L^k} + \beta\right) x(k) + \beta x(k+1)\,, \quad k \geq 1\,, \tag{2.6}$$

identified with its infinite matrix in the canonical basis

$$\mathcal{K} = \begin{pmatrix} -\left(\beta L\rho^{L} + \beta\right) & \beta & 0 & 0 & \cdots \\ \beta L\rho^{L} & -\left(\beta L\rho^{L^2} + \beta\right) & \beta & 0 & \cdots \\ 0 & \beta L\rho^{L^2} & -\left(\beta L\rho^{L^3} + \beta\right) & \beta & \cdots \\ 0 & 0 & \beta L\rho^{L^3} & -\left(\beta L\rho^{L^4} + \beta\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.7}$$

Note that $\mathcal{K} = \mathcal{A}^*$ where $\mathcal{A}$ is the generator of a sub-Markovian birth and death process. We give the Karlin-McGregor spectral decomposition for $\mathcal{K}$ in Section 4.2, to which we make a few forward references (it is *not* a resolution of the identity, see Rudin [13]). The potential coefficients of $\mathcal{A}$ are

$$\pi = (\pi(k))_{k\geq 1}\,, \qquad \pi(k) = L^{k-1}\rho^{(L^k - L)/(L-1)} = \rho^{-1}L^{k-1}\tilde{u}(k)\,, \tag{2.8}$$

and solve the detailed balance equations $\pi(k+1) = L\rho^{L^k}\pi(k)$ with $\pi(1) = 1$. The linearization of (1.2) around its stable point $\tilde{u}$ is the forward Kolmogorov equation for $\mathcal{A}$ given for $t \geq 0$ by

$$\dot{z}_t = \mathcal{K}z_t \tag{2.9}$$

which is special case of (2.2). Considering (2.4) and $F(\tilde{u}) = F_+(\tilde{u}) - F_-(\tilde{u}) = 0$, the martingale $M = (M(k))_{k\in\mathbb{N}}$ has the same law as a c_0^0-valued sequence $B = (B(k))_{k\in\mathbb{N}}$ of independent centered Brownian motions such that $B(0) = 0$ and for $k \geq 1$

$$\tilde{v}(k) := \mathrm{var}(B_1(k)) = \mathbf{E}\big(B_1(k)^2\big) = 2\beta\left(\tilde{u}(k) - \tilde{u}(k+1)\right) = 2\beta\rho^{(L^k-1)/(L-1)}\left(1 - \rho^{L^k}\right),$$

and B has diagonal infinitesimal covariance matrix $\mathrm{diag}(\tilde{v})$. The following result is obvious.

Theorem 2.7 *The process B is an Hilbertian Brownian motion in $L_2(w)$ if and only if $\tilde{u}$ is in $\ell_1(w)$. This is true for $w = \pi$ and $w = g_\theta$ for $\theta > 0$ when $L \geq 2$ or for $w = g_\theta$ for $\theta > \rho$ when $L = 1$.*

The Ornstein-Uhlenbeck (OU) process $Z = (Z(k))_{k\in\mathbb{N}}$ solves the affine SDE given for $t \geq 0$ by

$$Z_t = Z_0 + \int_0^t \mathcal{K}Z_s\,ds + B_t \tag{2.10}$$

which is a Brownian perturbation of (2.9). For $L \geq 2$, existence and uniqueness results hold under much weaker assumptions than (2.1).

Theorem 2.8 *Let w be such that there exists $c > 0$ and $d > 0$ with*

$$0 < cw(k+1) \le w(k) \le d\rho^{-2L^k} w(k+1)\,, \qquad k \ge 1\,.$$

(a) In $L_2(w)$, the operator $\mathcal{K}$ is bounded, the equation (2.9) has a unique solution $z_t = \mathrm{e}^{\mathcal{K}t} z_0$ where $\mathrm{e}^{\mathcal{K}t}$ has a spectral representation given by (4.1), and there is uniqueness of solutions for the SDE (2.10). The assumptions and conclusions hold for $w = \pi$ and $w = g_\theta$ for $\theta > 0$.
(b) In addition let w be such that $\tilde{u}$ is in $\ell_1(w)$. The SDE (2.10) has a unique solution $Z_t = \mathrm{e}^{\mathcal{K}t} Z_0 + \int_0^t \mathrm{e}^{\mathcal{K}(t-s)}\, dB_s$ in $L_2(w)$ further made explicit in (4.2). This the case for $w = \pi$ and $w = g_\theta$ for $\theta > 0$ when $L \ge 2$ or for $w = g_\theta$ for $\theta > \rho$ when $L = 1$.

We use results in van Doorn [3] to prove the existence of a spectral gap, and use this fact for an exponential stability result inspired from Callaert and Keilson [2] Section 10.

Theorem 2.9 *(Spectral gap.) The operator $\mathcal{K}$ is bounded self-adjoint in $L_2(\pi)$. The least point γ of the spectrum of $\mathcal{K}$ is such that $0 < \gamma \le \beta$. The solution $z_t = \mathrm{e}^{\mathcal{K}t} z_0$ for (2.9) in $L_2(\pi)$ satisfies $\|z_t\|_{L_2(\pi)} \le \mathrm{e}^{-\gamma t} \|z_0\|_{L_2(\pi)}$.*

For $L \ge 2$ the sequence π decays hyper-exponentially, see (2.8), and (1.3) implies that the $L_2(\pi)$ norm is too strong for the CLT. Further, the mapping F_+ is not Lipschitz in $\mathcal{V} \cap L_2(\pi)$ for the $L_2(\pi)$ norm, see Theorems 2.2 and 2.8 and their contrasting assumptions and proofs. Hence we prove exponential stability and (exponential) ergodicity for the OU process in appropriate spaces.

Theorem 2.10 *Let $0 < \theta < 1$ when $L \ge 2$ or $\rho \le \theta < 1$ when $L = 1$. There exists $\gamma_\theta > 0$ and $C_\theta < \infty$ such that the solution $z_t = \mathrm{e}^{\mathcal{K}t} z_0$ for (2.9) in $L_2(g_\theta)$ satisfies $\|z_t\|_{L_2(g_\theta)} \le \mathrm{e}^{-\gamma_\theta t} C_\theta \|z_0\|_{L_2(g_\theta)}$.*

Theorem 2.11 *Let $w = \pi$ or $w = g_\theta$ with $0 < \theta < 1$ when $L \ge 2$ or let $w = g_\theta$ with $\rho < \theta < 1$ when $L = 1$. Any solution for the SDE (2.10) in $L_2(w)$ converges in law for large times to its unique invariant law (exponentially fast). This law is the law of $\int_0^\infty \mathrm{e}^{\mathcal{K}t} dB_t$ which is Gaussian centered with covariance matrix $\int_0^\infty \mathrm{e}^{\mathcal{K}t} \mathrm{diag}(\tilde{v}) \mathrm{e}^{\mathcal{K}^* t} dt$ made more explicit in (4.3) and (4.4). There is a unique stationary OU process solving the SDE (2.10) in $L_2(w)$.*

Global exponential stability for (1.2), infinite-horizon and invariant law bounds, and the CLT

We state an important global exponential stability result at $\tilde{u}$ for the non-linear dynamical system. This is essential in the proof of the subsequent infinite-horizon bounds for the marginals of the processes, which yield bounds on their long time limit, the invariant law. We need uniformity over the state space, and results for the linearized equation (2.9) are *not* enough.

Theorem 2.12 *Let $\rho \le \theta < 1$ and u be the solution of (1.2) starting at u_0 in $\mathcal{V} \cap L_2(g_\theta)$. There exists $\gamma_\theta > 0$ and $C_\theta < \infty$ such that $\|u_t - \tilde{u}\|_{L_2(g_\theta)} \le \mathrm{e}^{-\gamma_\theta t} C_\theta \|u_0 - \tilde{u}\|_{L_2(g_\theta)}$.*

This does *not* hold in $L_2(\pi)$ for $L \ge 2$, else Lemma 2.13 below would also hold in $L_2(\pi)$, which would contradict (1.3). Theorem 3.6 in Mitzenmacher [12] states a related result for some weighted ℓ_1 norms obtained by potential function techniques.

Lemma 2.13 *Let $\rho \le \theta < 1$ when $L \ge 2$ or $\rho < \theta < 1$ when $L = 1$. Then*

$$\limsup_{N\to\infty} \mathbf{E}\left(\|Z_0^N\|^2_{L_2(g_\theta)}\right) < \infty \Rightarrow \limsup_{N\to\infty} \sup_{t\ge 0} \mathbf{E}\left(\|Z_t^N\|^2_{L_2(g_\theta)}\right) < \infty$$

and under the invariant laws $\limsup_{N\to\infty} \mathbf{E}\left(\|Z_0^N\|^2_{L_2(g_\theta)}\right) < \infty$.

Our main result is the functional CLT in equilibrium, obtained with a compactness-uniqueness method using tightness of the invariant laws (based on Lemma 2.13) and Theorems 2.6 and 2.11.

Theorem 2.14 *Let the networks of size N be in equilibrium. For $L \ge 2$ consider $L_2(g_\rho)$ with its weak topology and $\mathbb{D}(\mathbb{R}_+, L_2(g_\rho))$ with the corresponding Skorokhod topology. Then $(Z^N)_{N\ge L}$ converges in law to the unique stationary OU process solving the SDE (2.10), in particular $(Z_0^N)_{N\ge L}$ converges in law to the invariant law for this process (see Theorem 2.11). For $L = 1$ the same result holds in $L_2(g_\theta)$ for $\rho < \theta < 1$.*

3 The proofs for converging initial conditions

3.1 Existence and uniqueness results

Proof of Theorem 2.2 (refined existence result for (1.2))

We give the proof for $L_2(w)$, the proof for $\ell_1(w)$ being similar. The assumption (2.1) and the identity $x^L - y^L = (x-y)(x^{L-1} + x^{L-2}y + \cdots + y^{L-1})$ yield

$$\left(u(k-1)^L - v(k-1)^L\right)^2 w(k)^{-1} \le (u(k-1) - v(k-1))^2 L^2 d w(k-1)^{-1} ,$$

$$\left(u(k)^L - v(k)^L\right)^2 w(k)^{-1} \le (u(k) - v(k))^2 L^2 w(k)^{-1} ,$$

$$(u(k+1) - v(k+1))^2 w(k)^{-1} \le (u(k+1) - v(k+1))^2 c^{-1} w(k+1)^{-1} ,$$

hence we have the Lipschitz bounds $\|F_+(u) - F_+(v)\|^2_{L_2(w)} \le 2\alpha^2 L^2 (d+1) \|u - v\|^2_{L_2(w)}$ and $\|F_-(u) - F_-(v)\|^2_{L_2(w)} \le 2\beta^2 (c^{-1} + 1) \|u - v\|^2_{L_2(w)}$. Existence and uniqueness follows by a classical Cauchy-Lipschitz method.

The derivation of the Ornstein-Uhlenbeck process

Let $(x)_k = x(x-1)\cdots(x-k+1)$ for $x \in \mathbb{R}$ (the falling factorial of degree $k \in \mathbb{N}$). Considering (1.1), let the mappings F_+^N and F^N with values in c_0^0 be given for v in c_0 by

$$F_+^N(v)(k) = \alpha \, \frac{(Nv(k-1))_L - (Nv(k))_L}{(N)_L}\,, \quad k \geq 1\,; \qquad F^N(v) = F_+^N(v) - F_-(v)\,. \tag{3.1}$$

The process R^N is Markov on $\mathcal{V}^N$, and when in state r has jumps in its k-th coordinate, $k \geq 1$, of size $1/N$ at rate $NF_+^N(r)(k)$ and size $-1/N$ at rate $NF_-(r)(k)$.

Lemma 3.1 *Let R_0^N be in $\mathcal{V}^N$, u solve (1.2) starting at u_0 in $\mathcal{V}$, and Z^N be given by (1.4). Then*

$$Z_t^N = Z_0^N + \int_0^t \sqrt{N}\,\big(F^N(R_s^N) - F(u_s)\big)\,ds + M_t^N \tag{3.2}$$

defines an independent family of square-integrable martingales $M^N = (M^N(k))_{k\in\mathbb{N}}$ independent of Z_0^N with Doob-Meyer brackets given by

$$\left\langle M^N(k)\right\rangle_t = \int_0^t \left\{F_+^N(R_s^N)(k) + F_-(R_s^N)(k)\right\} ds\,. \tag{3.3}$$

Proof. This follows from a classical application of the Dynkin formula. □

The first lemma below shows that it is indifferent to choose the L queues with or without replacement at this level of precision. The second one is a linearization formula.

Lemma 3.2 *For $N \geq L \geq 1$ and a in $\mathbb{R}$ we have*

$$A^N(a) := \frac{(Na)_L}{(N)_L} - a^L = \sum_{j=1}^{L-1} (a-1)^j a^{L-j} \sum_{1\leq i_1<\cdots<i_j\leq L-1} \frac{i_1\cdots i_j}{(N-i_1)\cdots(N-i_j)}$$

and $A^N(a) = N^{-1}O(a)$, uniformly for $0 \leq a \leq 1$, and $A^N(k/N) \leq 0$ for $k = 0, 1, \ldots, N$.

Proof. We develop $\frac{(Na)_L}{(N)_L} = \prod_{i=0}^{L-1} \frac{Na-i}{N-i} = \prod_{i=0}^{L-1}\left(a + (a-1)\frac{i}{N-i}\right)$ to obtain the identity for $A^N(a)$ which is clearly $N^{-1}O(a)$, uniformly for $0 \leq a \leq 1$. For $a = k/N$, $\prod_{i=0}^{L-1}\frac{Na-i}{N-i}$ is composed of terms bounded by a or contains a term equal to 0 and cannot exceed a^L. □

Lemma 3.3 *For $L \geq 1$ and a and h in $\mathbb{R}$ we have*

$$B(a,h) := (a+h)^L - a^L - La^{L-1}h = \sum_{i=2}^{L} \binom{L}{i} a^{L-i} h^i$$

with $B(a,h) = 0$ for $L = 1$ and $B(a,h) = h^2$ for $L = 2$. For $L \geq 2$ we have $0 \leq B(a,h) \leq h^L + \left(2^L - L - 2\right) a h^2$ for a and $a+h$ in $[0,1]$.

Proof. The identity is Newton's binomial formula. A convexity argument yields $B(a,h)\geq 0$. For a and $a+h$ in $[0,1]$, $B(a,h)\leq h^L+\sum_{i=2}^{L-1}\binom{L}{i}ah^2 = h^L+\left(2^L-L-2\right)ah^2\,.$ $\Box$

Let v be in $\mathcal{V}$ and x in c_0^0. Considering (1.1), (3.1) and Lemma 3.2, let $G^N:\mathcal{V}\to c_0^0$ be given by

$$G^N(v)(k)=\alpha A^N(v(k-1))-\alpha A^N(v(k))\,,\qquad k\geq 1\,,\tag{3.4}$$

and considering (1.1), (2.3) and Lemma 3.3 let $H:\mathcal{V}\times c_0^0\to c_0^0$ be given by

$$H(v,x)(k)=\alpha B(v(k-1),x(k-1))-\alpha B(v(k),x(k))\,,\qquad k\geq 1\,,\tag{3.5}$$

so that for $v+x$ in $\mathcal{V}$

$$F^N=F+G^N\,,\qquad F(v+x)-F(v)=\mathbf{K}(v)x+H(v,x)\,,\tag{3.6}$$

and we derive the limit equation (2.5) and (2.4) for the fluctuations from (3.2) and (3.3).

Proof of Theorem 2.4 (existence and uniqueness for the OU process)

Considering (2.3), $v\leq 1$, convexity bounds, and (2.1), we have

$$\begin{aligned}\|\mathbf{K}(v)x\|_{L_2(w)}^2 &\leq 2(\alpha L+\beta)\sum_{k\geq 1}\big(\alpha Lx(k-1)^2dw(k-1)^{-1}+(\alpha L+\beta)x(k)^2w(k)^{-1}\\ &\qquad\qquad +\beta x(k+1)^2c^{-1}w(k+1)^{-1}\big)\\ &\leq 2(\alpha L+\beta)(\alpha L(d+1)+\beta(c^{-1}+1))\|x\|_{L_2(w)}^2\end{aligned}$$

and (a) and (b) follow. For u_o in $\mathcal{V}\cap\ell_1(w)$ the martingale M is square-integrable in $L_2(w)$. If $\mathbf{E}\Big(\|Z_0\|_{L_2(w)}^2\Big)<\infty$ then the formula for Z is well-defined, solves the SDE, and the Gronwall Lemma yields $\mathbf{E}\Big(\sup_{t\leq T}\|Z_t\|_{L_2(w)}^2\Big)<\infty$. Else for any $\varepsilon>0$ there is $r_\varepsilon<\infty$ such that $\mathbf{P}\big(\|Z_0\|_{L_2(w)}>r_\varepsilon\big)<\varepsilon$ and a localization procedure using pathwise uniqueness yields existence.

3.2 The proof of the CLT

Proof for Lemma 2.5 (finite-horizon bounds)

Using (3.2) and (3.6)

$$Z_t^N=Z_0^N+M_t^N+\sqrt{N}\int_0^t G^N(R_s^N)\,ds+\int_0^t\sqrt{N}\left(F(R_s^N)-F(u_s)\right)ds\tag{3.7}$$

where Lemma 3.2 yields $G^N(R_s^N)(k)=N^{-1}O\big(R_s^N(k-1)+R_s^N(k)\big)$ and considering (2.1)

$$\left\|G^N(R_s^N)\right\|_{L_2(w)}=N^{-1}O\big(\left\|R_s^N\right\|_{L_2(w)}\big)\,.\tag{3.8}$$

We have

$$\left\|R_s^N\right\|_{L_2(w)}\leq\|u_s\|_{L_2(w)}+N^{-1/2}\left\|Z_s^N\right\|_{L_2(w)}\tag{3.9}$$

and since F_+, F_- and F are Lipschitz (Theorem 2.2) the Gronwall Lemma yields that for some $K_T < \infty$ we have $\|u_s\|_{L_2(w)} \le K_T \|u_0\|_{L_2(w)}$ and

$$\sup_{0\le t\le T} \left\|Z_t^N\right\|_{L_2(w)} \le K_T\left(\left\|Z_0^N\right\|_{L_2(w)} + N^{-1/2}K_T \|u_0\|_{L_2(w)} + \sup_{0\le t\le T}\left\|M_t^N\right\|_{L_2(w)}\right).$$

We conclude using the Doob inequality, (3.3), (3.6), (3.8), (3.9), and

$$\left\|F_+(R_s^N) + F_-(R_s^N)\right\|_{L_2(w)} \le K \left\|R_s^N\right\|_{L_2(w)}. \tag{3.10}$$

Tightness for the process

Lemma 3.4 *Let w satisfy (2.1), and consider $L_2(w)$ with its weak topology and $\mathbb{D}(\mathbb{R}_+, L_2(w))$ with the corresponding Skorokhod topology. Let u_0 be in $\mathcal{V} \cap \ell_1(w)$ and R_0^N in $\mathcal{V}^N$, and Z^N be given by (1.4). If $(Z_0^N)_{N\ge L}$ is tight then $(Z^N)_{N\ge L}$ is tight and its limit points are continuous.*

Proof. For $\varepsilon > 0$ let $r_\varepsilon < \infty$ be such that $\mathbf{P}(Z_0^N \in B(r_\varepsilon)) > 1-\varepsilon$ for $N \ge 1$ (see the discussion prior to Theorem 2.6). Let $R_0^{N,\varepsilon}$ be equal to R_0^N on $\{Z_0^N \in B(r_\varepsilon)\}$ and such that $Z_0^{N,\varepsilon}$ is uniformly bounded in $L_2(w)$ on $\{Z_0^N \notin B(r_\varepsilon)\}$. Then $Z_0^{N,\varepsilon}$ is uniformly bounded in $L_2(w)$ and we may use a coupling argument to construct $Z^{N,\varepsilon}$ and Z^N coinciding on $\{Z_0^N \in B(r_\varepsilon)\}$. Hence to prove tightness of $(Z^N)_{N\ge L}$ we may restrict our attention to $(Z_0^N)_{N\ge L}$ uniformly bounded in $L_2(w)$, for which we may use Lemma 2.5.

The compact subsets of $L_2(w)$ are Polish, a fact yielding tightness criteria. We deduce from Theorems 4.6 and 3.1 in Jakubowski [8], which considers completely regular Hausdorff spaces (Tychonoff spaces) of which $L_2(w)$ with its weak topology is an example, that $(Z^N)_{N\ge L}$ is tight if

1. For each $T \ge 0$ and $\varepsilon > 0$ there is a bounded subset $K_{T,\varepsilon}$ of $L_2(w)$ such that for $N \ge L$ we have $\mathbf{P}\big(Z^N \in \mathbb{D}([0,T], K_{T,\varepsilon})\big) > 1-\varepsilon$.
2. For each $d \ge 1$, the d-dimensional processes $(Z^N(1), \dots, Z^N(d))_{N\ge L}$ are tight.

Lemma 2.5 and the Markov inequality yield condition 1. We use (3.7) (see (3.2) and (3.6)), and (3.3) and (3.6), and the bounds (3.8), (3.9) and (3.10). The bounds in Lemma 2.5 and the fact that $Z^N(k)$ has jumps of size $1/\sqrt{N} = o(N)$ classically imply that the above finite-dimensional processes are tight and have continuous limit points, see for instance Ethier-Kurtz [4] Theorem 4.1 p. 354 or Joffe-Métivier [9] Proposition 3.2.3 and their proofs. □

Proof of Theorem 2.6 (the functional CLT)

Lemma 3.4 implies that from any subsequence of Z^N we may extract a further subsequence which converges to some Z^∞ with continuous sample paths. Necessarily Z_0^∞ has same law as Z_0. In (3.7) we have considering (3.6) that

$$\sqrt{N}\big(F(R_s^N)(k)-F(u_s)(k)\big)=\mathbf{K}(u_s)Z_s^N+\sqrt{N}H\big(u_s,N^{-1/2}Z_s^N\big)\,. \tag{3.11}$$

We use the bounds (3.8), (3.9) and (3.10), the uniform bounds in Lemma 2.5, and additionally (3.5) and Lemma 3.3. We deduce by a martingale characterization that Z^∞ has the law of the OU process unique solution for (2.5) in $L_2(w)$ starting at Z_0^∞, see Theorem 2.4; the drift vector is given by the limit for (3.2) and (3.7) considering (3.11), and the martingale bracket by the limit for (3.3). See for instance Ethier-Kurtz [4] Theorem 4.1 p. 354 or Joffe-Métivier [9] Theorem 3.3.1 and their proofs for details. Thus, this law is the unique accumulation point for the relatively compact sequence of laws of $(Z^N)_{N\geq 1}$, which must then converge to it, proving Theorem 2.6.

4 The properties of $\mathcal{K}=\mathbf{K}(\tilde u)$

4.1 Proof of Theorem 2.8 (existence and uniqueness results)

Considering (2.6) and convexity bounds we have

$$\begin{aligned}
\|\mathcal{K}z\|_{L_2(w)}^2 &= \beta^2\sum_{k\geq 1}\Big(L\rho^{L^{k-1}}z(k-1)-(L\rho^{L^k}+1)z(k)+z(k+1)\Big)^2 w(k)^{-1}\\
&\leq \beta^2(2L+2)\Bigg(L\sum_{k\geq 1}\rho^{2L^{k-1}}z(k-1)^2w(k)^{-1}+L\sum_{k\geq 1}\rho^{2L^k}z(k)^2w(k)^{-1}\\
&\qquad\qquad +\sum_{k\geq 1}z(k)^2w(k)^{-1}+\sum_{k\geq 1}z(k+1)^2w(k)^{-1}\Bigg)\\
&\leq \beta^2(2L+2)\Bigg(Ld\sum_{k\geq 2}z(k-1)^2w(k-1)^{-1}+(L\rho^{2L}+1)\sum_{k\geq 1}z(k)^2w(k)^{-1}\\
&\qquad\qquad +c^{-1}\sum_{k\geq 1}z(k+1)^2w(k+1)^{-1}\Bigg)\\
&\leq \beta^2(2L+2)\left(L\rho^{2L}+Ld+c^{-1}+1\right)\|z\|_{L_2(w)}^2\,.
\end{aligned}$$

The Gronwall Lemma yields uniqueness. For $k\geq 1$ we have

$$\left(L\rho^L\right)^{-1}\pi(k+1)\leq\pi(k)=\left(L\rho^{L^k}\right)^{-1}\pi(k+1)\leq\left(L^{-1}\rho^L\rho^{-2L^k}\right)\pi(k+1)\,,$$

$$\theta^{-1}\theta^{k+1}\leq\theta^k\leq\left(\theta^{-1}\rho^L\rho^{-2L^k}\right)\theta^{k+1}\,.$$

When B is an Hilbertian Brownian motion, the formula for Z yields a well-defined solution.

4.2 A related birth and death process, and the spectral decomposition

Considering (2.7), $\mathcal{A}=\mathcal{K}^*$ is the infinitesimal generator of the sub-Markovian birth and death process on the irreducible class $(1,2,\dots)$ with birth rates $\lambda_k=\beta L\rho^{L^k}$ and death rates $\mu_k=\beta$ for $k\geq 1$ (killed at rate $\mu_1=\beta$ at state 1). The process is well-defined since the rates are bounded.

Karlin and McGregor [10, 11] give a spectral decomposition for such processes, used by Callaert and Keilson [1, 2] and van Doorn [3] to study exponential ergodicity properties. The state space in these works is $(0, 1, 2, \dots)$, possibly extended by an absorbing barrier or graveyard state at -1. We consider $(1, 2, \dots)$ and adapt their notations to this simple shift.

The potential coefficients ([10] eq. (2.2), [3] eq. (2.10)) are given by

$$\pi(k) = \frac{\lambda_1 \cdots \lambda_{k-1}}{\mu_2 \cdots \mu_k} = L\rho^{L^1} \cdots L\rho^{L^{k-1}} = L^{k-1}\rho^{(L^k - L)/(L-1)}, \qquad k \geq 1\,,$$

and solve the detailed balance equations $\mu_{k+1}\pi(k+1) = \lambda_k \pi(k)$ with $\pi(1) = 1$, see (2.8).

The equation $\mathcal{A}Q(x) = -xQ(x)$ for an eigenvector $Q(x) = (Q_n(x))_{n\geq 1}$ of eigenvalue $-x$ yields $\lambda_1 Q_2(x) = (\lambda_1 + \mu_1 - x)Q_1(x)$ and $\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x)Q_n(x) - \mu_n Q_{n-1}(x)$ for $n \geq 2$. With the natural convention $Q_0 = 0$ and normalizing choice $Q_1 = 1$, we obtain inductively Q_n as the polynomial of degree $n-1$ satisfying the recurrence relation

$$-xQ_n(x) = \beta Q_{n-1}(x) - \left(\beta L\rho^{L^n} + \beta\right) Q_n(x) + \beta L\rho^{L^n} Q_{n+1}(x)\,, \qquad n \geq 1\,,$$

corresponding to [10] eq. (2.1) and [3] eq. (2.15). Such a sequence of polynomials is orthogonal with respect to a probability measure ψ on $\mathbb{R}_+$ and, for $i, j \geq 1$ with $i \neq j$, $\int_0^\infty Q_i(x)^2\, \psi(dx) = \pi(i)^{-1}$ and $\int_0^\infty Q_i(x)Q_j(x)\, \psi(dx) = 0$ or in matrix notation $\int_0^\infty Q(x)Q(x)^*\, \psi(dx) = \mathrm{diag}(\pi^{-1})$.

Let $P_t = (p_t(i,j))_{i,j\geq 1}$ denote the sub-stochastic transition matrix for $\mathcal{A}$. The adjoint matrix P_t^* is the fundamental solution for the forward equation $\dot{z}_t = \mathcal{A}^* z_t = \mathcal{K} z_t$ given in (2.9). The representation formula of Karlin and McGregor [10, 11], see (1.2) and (2.18) in [3], yields

$$\mathrm{e}^{\mathcal{K}t} = P_t^* = (p_t^*(i,j))_{i,j\geq 1}\,, \qquad p_t^*(i,j) = p_t(j,i) = \pi(i) \int_0^\infty \mathrm{e}^{-xt} Q_i(x) Q_j(x)\, \psi(dx)\,, \quad (4.1)$$

or in matrix notation $\mathrm{e}^{\mathcal{K}t} = \mathrm{diag}(\pi) \int_0^\infty \mathrm{e}^{-xt} Q(x)Q(x)^*\, \psi(dx)$.

The probability measure ψ is called the spectral measure, its support S is called the spectrum, and we set $\gamma = \min S$. The OU process in Theorem 2.8 (b) and its invariant law and its covariance matrix in Theorems 2.11 and 2.14 can be written

$$Z_t = \mathrm{diag}(\pi) \int_S \mathrm{e}^{-xt} Q(x)^* \left(Z_0 + \int_0^t \mathrm{e}^{xs}\, dB_s \right) Q(x)\, \psi(dx)\,, \quad (4.2)$$

$$\int_0^\infty \mathrm{e}^{\mathcal{K}t}\, dB_t = \mathrm{diag}(\pi) \int_S \left(Q(x)^* \int_0^\infty \mathrm{e}^{-xt}\, dB_t \right) Q(x)\, \psi(dx)\,, \quad (4.3)$$

$$\int_0^\infty \mathrm{e}^{\mathcal{K}t} \mathrm{diag}(\tilde{v}) \mathrm{e}^{\mathcal{K}^* t}\, dt = \mathrm{diag}(\pi) \int_{S^2} \frac{Q(x)^* \mathrm{diag}(\tilde{v}) Q(y)}{x+y} Q(x)Q(y)^*\, \psi(dx)\psi(dy)\, \mathrm{diag}(\pi). \quad (4.4)$$

4.3 The spectral gap, exponential stability, and ergodicity

Proof of Theorem 2.9 (spectral gap and exponential stability in the self-adjoint case)

The potential coefficients $(\pi(k))_{k\geq 1}$ solve the detailed balance equations for $\mathcal{A}$ and hence $\mathcal{K} = \mathcal{A}^*$ is self-adjoint in $L_2(\pi)$. For the spectral gap, we follow Van Doorn [3], Section 2.3. The orthogonality properties imply that Q_n has $n-1$ distinct zeros $0 < x_{n,1} < \ldots < x_{n,n-1}$ such that $x_{n+1,i} < x_{n,i} < x_{n+1,i+1}$ for $1 \le i \le n-1$. Hence $\xi_i = \lim_{n\to\infty} x_{n,i} \ge 0$ exists, $\xi_i \le \xi_{i+1}$, and $\sigma = \lim_{i\to\infty} \xi_i$ exists in $[0,\infty]$. Theorem 5.1 in [3] establishes that $\gamma > 0$ if and only if $\sigma > 0$ and Theorem 5.3 (i) in [3] that $\sigma = \left(\sqrt{\lim_k \lambda_k} - \sqrt{\lim_k \mu_k}\right)^2 = \beta > 0$. (Theorem 3.3 in [3] states that $\gamma = \xi_1 \le \sigma$, but estimating ξ_1 is impractical.)

For the exponential stability, we have $\|z_t\|^2_{L_2(\pi)} = \left(\mathrm{e}^{\mathcal{K}t} z_0, \mathrm{e}^{\mathcal{K}t} z_0\right)_{L_2(\pi)}$ and the fact that $\mathrm{e}^{\mathcal{K}t}$ is self-adjoint in $L_2(\pi)$ and the spectral representation (4.1) yield

$$\begin{aligned}\left(\mathrm{e}^{\mathcal{K}t} z_0, \mathrm{e}^{\mathcal{K}t} z_0\right)_{L_2(\pi)} &= \left(z_0, \mathrm{e}^{2\mathcal{K}t} z_0\right)_{L_2(\pi)} = \int_S \mathrm{e}^{-2xt} z_0^* Q(x) Q(x)^* z_0 \, \psi(dx) \\ &\le \mathrm{e}^{-2\gamma t} \int_S z_0^* Q(x) Q(x)^* z_0 \, \psi(dx) = \mathrm{e}^{-2\gamma t} \left(z_0, z_0\right)_{L_2(\pi)} .\end{aligned}$$

Proof of Theorem 2.10 (exponential stability, non self-adjoint case)

It is similar to and simpler than the proof for Theorem 2.12 to which Section 5 is devoted, and we postpone the proof until the end of that section.

Proof of Theorem 2.11 (ergodicity for the OU process)

We use the uniqueness result and explicit formula in Theorem 2.8, and Theorem 2.9 or 2.10.

5 Exponential stability for the nonlinear system

5.1 Some comparison results

Considering (3.6) with $\mathcal{K} = \mathbf{K}(\tilde{u})$ and $F(\tilde{u}) = 0$, if u solves (1.2) in $\mathcal{V}$ then $y = u - \tilde{u}$ solves the recentered equation given by $\dot{y}_t(k) = F(\tilde{u} + y) = \mathcal{K} y_t(k) + H(\tilde{u}, y_t)(k)$ or

$$\begin{aligned}\dot{y}_t(k) &= \beta L \rho^{L^{k-1}} y_t(k-1) + \alpha B(\tilde{u}(k-1), y_t(k-1)) \\ &\quad - \left(\beta L \rho^{L^k} y_t(k) + \alpha B(\tilde{u}(k), y_t(k)) + \beta y_t(k)\right) + \beta y_t(k+1) , \qquad k \ge 1 . \end{aligned} \tag{5.1}$$

If u_0 is in $\mathcal{V} \cap \ell_1$ then u is in $\mathcal{V} \cap \ell_1$ and hence y is in ℓ_1^0 and for $k \ge 1$ we have

$$\dot{y}_t(k) + \dot{y}_t(k+1) + \cdots = \beta L \rho^{L^{k-1}} y_t(k-1) + \alpha B(\tilde{u}(k-1), y_t(k-1)) - \beta y_t(k) . \tag{5.2}$$

If y solves (5.1) starting at y_0 such that $y_0 + \tilde{u}$ is in $\mathcal{V}$, then $u = y + \tilde{u}$ solves (1.2) in $\mathcal{V}$ starting at $u_0 = y_0 + \tilde{u}$. Then $-\tilde{u} \le y \le 1 - \tilde{u}$ and $-1 < y < 1$. For $y_0 + \tilde{u}$ in $\mathcal{V} \cap \ell_1$ we have y in ℓ_1^0.

Lemma 5.1 *Let u and v be two solutions for (1.2) in $\mathcal{V}$ such that $u_0 \le v_0$. Then $u_t \le v_t$ for $t \ge 0$. Let $y_0 + \tilde{u}$ be in $\mathcal{V}$ and y solve (5.1). If $y_0 \ge 0$ then $y_t \ge 0$ and if $y_0 \le 0$ then $y_t \le 0$ for $t \ge 0$.*

Proof. Lemma 6 in [15] yields the result for (1.2) (the proof written for $L = 2$ is valid for $L \geq 1$). The result for (5.1) follows by considering $u = y + \tilde{u}$ and $\tilde{u}$ which solve (1.2). $\square$

We compare solutions of the nonlinear equation (5.1) and of certain linear equations.

Lemma 5.2 *Let $\hat{\mathcal{A}}$ be the generator of the sub-Markovian birth and death process with birth rate $\hat{\lambda}_k \geq 0$ and death rate β at $k \geq 1$. Let $\sup_k \hat{\lambda}_k < \infty$. The linear operator $x \mapsto \hat{\mathcal{A}}^* x$ given by*

$$\hat{\mathcal{A}}^* x(k) = \hat{\lambda}_{k-1} x(k-1) - (\hat{\lambda}_k + \beta) x(k) + \beta x(k+1) \,, \qquad k \geq 1 \,,$$

is bounded in ℓ_1^0. There exists a unique $z = (z_t)_{t\geq 0}$ given by $z_t = \mathrm{e}^{\hat{\mathcal{A}}^* t} z_0$ solving the forward Kolmogorov equation $\dot{z} = \hat{\mathcal{A}}^* z$ in ℓ_1^0. It is such that if $z_0 \geq 0$ then $z_t \geq 0$ and if $z_0 \leq 0$ then $z_t \leq 0$, and $\dot{z}_t(k) + \dot{z}_t(k+1) + \cdots = \hat{\lambda}_{k-1} z_t(k-1) - \beta z_t(k)$ for $k \geq 1$.

Proof. The operator norm in ℓ_1^0 of $\hat{\mathcal{A}}^*$ is bounded by $2(\sup_k \hat{\lambda}_k + \beta)$, hence existence and uniqueness. Uniqueness and linearity imply that if $z_0 = 0$ then $z_t = 0$ and else if $z_0 \geq 0$ then $z_t \|z_0\|_1^{-1}$ is the instantaneous law of the process starting at $z_0 \|z_0\|_1^{-1}$ and hence $z_t \geq 0$. If $z_0 \leq 0$ then $-z$ solves the equation starting at $-z_0 \geq 0$ and hence $-z_t \geq 0$. The last result is obtained by summation. $\square$

Lemma 5.3 *Let $L \geq 2$ and $y = (y_t)_{t\geq 0}$ solve (5.1) with $y_0 + \tilde{u}$ in $\mathcal{V} \cap \ell_1$. Under the assumptions of Lemma 5.2, let $z = (z_t)_{t\geq 0}$ solve $\dot{z} = \hat{\mathcal{A}}^* z$ in ℓ_1^0. Let $h = (h_t)_{t\geq 0}$ be given in ℓ_1^0 by*

$$h(k) = z(k) + z(k+1) + \cdots - (y(k) + y(k+1) + \cdots) \,, \qquad k \geq 1 \,.$$

(a) Let $\hat{\lambda}_k \geq \beta L \rho^{L^k} + \alpha\big(1 + \big(2^L - L - 2\big)\, \tilde{u}(k)\big)$ for $k \geq 1$, $y_0 \geq 0$, and $h_0 \geq 0$. Then $h_t \geq 0$ for $t \geq 0$.

(b) Let $\hat{\lambda}_k \geq \beta L \rho^{L^k}$ for $k \geq 1$, $y_0 \leq 0$, and $h_0 \leq 0$. Then $h_t \leq 0$ for $t \geq 0$.

Proof. We prove (a). For $\varepsilon > 0$ let $\hat{\mathcal{A}}_\varepsilon^*$ correspond to $\hat{\lambda}_k^\varepsilon = \hat{\lambda}_k + \varepsilon$. The operator norm in ℓ_1^0 of $\hat{\mathcal{A}}_\varepsilon^* - \hat{\mathcal{A}}^*$ is bounded by 2ε, hence $\lim_{\varepsilon\to 0} \mathrm{e}^{\hat{\mathcal{A}}_\varepsilon^* t} z_0 = z_t$ in ℓ_1^0 and we may assume that $\hat{\lambda}_k > \beta L \rho^{L^k} + \alpha\big(1 + \big(2^L - L - 2\big)\, \tilde{u}(k)\big)$ for $k \geq 1$. Since $z_t = \mathrm{e}^{\hat{\mathcal{A}}^* t} z_0$ depends continuously on z_0 in ℓ_1^0 we may assume $h_0 > 0$. Let $\tau = \inf\{t \geq 0 : \{k \geq 1 : h_t(k) = 0\} \neq \emptyset\}$ be the first time when $h(k) = 0$ for some $k \geq 1$. We have $\tau > 0$.

The result (a) holds if $\tau = \infty$. If $\tau \neq \infty$, Lemma 5.2 and (5.2) yield

$$\begin{aligned} \dot{h}_\tau(k) &= \hat{\lambda}_{k-1} y_\tau(k-1) - \beta L \rho^{L^{k-1}} y_\tau(k-1) - \alpha B(\tilde{u}(k-1), y_\tau(k-1)) \\ &\qquad + \hat{\lambda}_{k-1}(z_\tau(k-1) - y_\tau(k-1)) - \beta(z_\tau(k) - y_\tau(k)) \,. \end{aligned}$$

Lemma 5.1 yields $y \geq 0$ and Lemma 3.3 and $y \leq 1$ yield

$$B(\tilde{u}(k-1), y(k-1)) \leq y(k-1)^L + \left(2^L - L - 2\right) \tilde{u}(k-1) y(k-1)^2$$
$$\leq \left(1 + \left(2^L - L - 2\right) \tilde{u}(k-1)\right) y(k-1)\,,$$

hence $\hat{\lambda}_{k-1} y(k-1) - \beta L \rho^{L^{k-1}} y(k-1) - \alpha B(\tilde{u}(k-1), y(k-1)) \geq 0$ with equality only when $y(k-1) = 0$. For k in $\mathcal{Z} = \{k \geq 1 : h_\tau(k) = 0\} \neq \emptyset$ we have

$$z_\tau(k-1) - y_\tau(k-1) = h_\tau(k-1) \geq 0\,, \quad z_\tau(k) - y_\tau(k) = -h_\tau(k+1) \leq 0\,,$$

hence $\dot{h}_\tau(k) \geq 0$ with equality if only if $k-1$ is in $\mathcal{Z} \cup \{0\}$ and $k+1$ is in $\mathcal{Z}$. Moreover $h_t(k) > 0$ for $t < \tau$ and $h_\tau(k) = 0$ imply $\dot{h}_\tau(k) \leq 0$. Hence $\dot{h}_\tau(k) = 0$, and the above signs and equality cases yield that $z_\tau(k-1) = y_\tau(k-1) = 0$ and $k-1$ is in $\mathcal{Z} \cup \{0\}$ and $k+1$ is in $\mathcal{Z}$. By induction $z_\tau(i) = y_\tau(i) = 0$ for $i \geq 1$ which implies $z_t = y_t = 0$ for $t \geq \tau$, and the proof of (a) is complete.

The proof for (b) is similar and involves obvious changes of sign. The assumption $\hat{\lambda}_k > \beta L \rho^{L^k}$ suffices to conclude since $B(\tilde{u}(k-1), y(k-1)) \geq 0$ (Lemma 3.3) and the non-linearity "pushes" in the right direction. $\square$

Lemma 5.4 *For any* $0 < \theta < 1$ *there exists* $K_\theta < \infty$ *such that for* x *in* $L_2(g_\theta) \subset \ell_1^0$

$$\|(x(k) + x(k+1) + \cdots)_{k \geq 1}\|_{L_2(g_\theta)} \leq K_\theta \|x\|_{L_2(g_\theta)}\,.$$

Proof. Using a classical convexity inequality

$$\sum_{k \geq 1} (x(k) + x(k+1) + \cdots)^2 \theta^{-k}$$
$$\leq \sum_{k \geq 1} n\big(x(k)^2 + x(k+1)^2 + \cdots + x(k+n-2)^2 + (x(k+n-1) + x(k+n) + \cdots)^2\big)\, \theta^{-k}$$
$$\leq n\big(1 + \theta + \cdots + \theta^{n-2}\big) \sum_{k \geq 1} x(k)^2 \theta^{-k} + n\, \theta^{n-1} \sum_{k \geq 1} (x(k) + x(k+1) + \cdots)^2 \theta^{-k}$$

and we take n large enough that $n\theta^{n-1} < 1$ and $K_\theta^2 = n\big(1 + \theta + \cdots + \theta^{n-2}\big)\,(1 - n\theta^{n-1})^{-1}$. $\square$

5.2 Proofs of the exponential stability results

Proof of Theorem 2.12 for $L \geq 2$

If u_0 is in $\mathcal{V} \cap L_2(g_\theta)$, then so are $u_0^- = \min\{u_0, \tilde{u}\}$ and $u_0^+ = \max\{u_0, \tilde{u}\}$ and hence the corresponding solutions u^- and u^+ for (1.2), see Theorem 2.2. Lemma 5.1 yields that $u_t^- \leq u_t \leq u_t^+$ and $u_t^- \leq \tilde{u} \leq u_t^+$ for $t \geq 0$. Then

$$y = u - \tilde{u}\,, \qquad y^+ = u^+ - \tilde{u} \geq 0\,, \qquad y^- = u^- - \tilde{u} \leq 0\,,$$

solve the recentered equation (5.1), and termwise

$$|y_0| = \max\{y_0^+, -y_0^-\}\,, \qquad |y_t| \le \max\{y_t^+, -y_t^-\}\,, \quad t \ge 0\,. \tag{5.3}$$

We consider the birth and death process with generator $\hat{\mathcal{A}}$ defined in Lemma 5.2 with

$$\hat\lambda_k = \max\left\{\beta L\rho^{L^k} + \alpha\big(1 + \big(2^L - L - 2\big)\,\tilde u(k)\big)\,, \beta\theta\right\}\,, \qquad k \ge 1\,,$$

which satisfies the assumptions of Lemma 5.3 (a) and (b). We reproduce the spectral study in Section 4.2 and the proof of Theorem 2.9 in Section 4.3 for $\hat{\mathcal{A}}$, corresponding objects being denoted with a hat. For $\rho \le \theta < 1$ we have $\alpha \le \beta\theta$ and hence $\hat\lambda_k$ is equivalent to $\beta\theta$ for large k, Theorems 5.1 and 5.3 (i) in [3] yield that $0 < \hat\gamma \le \hat\sigma = \left(\sqrt{\beta\theta} - \sqrt{\beta}\right)^2 = \beta\left(1 - \sqrt{\theta}\right)^2$, and if z solves $\dot z = \hat{\mathcal{A}}^* z$ then $\|z_t\|_{L_2(\hat\pi)} \le \mathrm{e}^{-\hat\gamma t}\|z_0\|_{L_2(\hat\pi)}$ for $t \ge 0$. Moreover

$$\theta^{k-1} \le \hat\pi(k) = \theta^{k-1}\prod_{i=1}^{k-1}\max\left\{\theta^{-1}L\rho^{L^k} + \theta^{-1}\rho\big(1 + \big(2^L - L - 2\big)\,\tilde u(k)\big)\,, 1\right\}$$

and the product converges, hence $\hat\pi(k) = O(\theta^k)$ and $\theta^k = O(\hat\pi(k))$ and Lemma 2.1 yields that there exists $c > 0$ and $d > 0$ such that $c^{-1}\|\cdot\|_{L_2(\hat\pi)} \le \|\cdot\|_{L_2(g_\theta)} \le d\|\cdot\|_{L_2(\hat\pi)}$. Hence for $t \ge 0$

$$\|z_t\|_{L_2(g_\theta)} \le d\|z_t\|_{L_2(\hat\pi)} \le \mathrm{e}^{-\hat\gamma t} d\|z_0\|_{L_2(\hat\pi)} \le \mathrm{e}^{-\hat\gamma t} cd\|z_0\|_{L_2(g_\theta)}\,.$$

Hence if z^+ solves $z^+ = \hat{\mathcal{A}}^* z^+$ starting at $z_0^+ = y_0^+ \ge 0$ then Lemmas 5.3 (a) and 5.4 yield

$$\begin{aligned}
\|y_t^+\|_{L_2(g_\theta)} &\le \|(y_t^+(k) + y_t^+(k+1) + \cdots)_{k\ge1}\|_{L_2(g_\theta)}\\
&\le \|(z_t^+(k) + z_t^+(k+1) + \cdots)_{k\ge1}\|_{L_2(g_\theta)}\\
&\le K_\theta\|z_t^+\|_{L_2(g_\theta)} \le \mathrm{e}^{-\hat\gamma t} cdK_\theta\|y_0^+\|_{L_2(g_\theta)}
\end{aligned}$$

and similarly if z^- solves $z^- = \hat{\mathcal{A}}^* z^-$ starting at $z_0^- = y_0^- \le 0$ then Lemmas 5.3 (b) and 5.4 yield $\|y_t^-\|_{L_2(g_\theta)} \le \mathrm{e}^{-\hat\gamma t} cdK_\theta\|y_0^-\|_{L_2(g_\theta)}$. We set $\gamma_\theta = \hat\gamma$ and $C_\theta = cdK_\theta$. Considering (5.3),

$$\|y_t\|_{L_2(g_\theta)}^2 \le \|y_t^+\|_{L_2(g_\theta)}^2 + \|y_t^-\|_{L_2(g_\theta)}^2 \le \mathrm{e}^{-2\gamma_\theta t} C_\theta^2\left(\|y_0^+\|_{L_2(g_\theta)}^2 + \|y_0^-\|_{L_2(g_\theta)}^2\right)$$

and we complete the proof by remarking that for $k \ge 1$, either $y_0^+(k) = y_0(k)$ and $y_0^-(k) = 0$ or $y_0^-(k) = y_0(k)$ and $y_0^+(k) = 0$, and hence $\|y_0^+\|_{L_2(g_\theta)}^2 + \|y_0^-\|_{L_2(g_\theta)}^2 = \|y_0\|_{L_2(g_\theta)}^2$.

Proof of Theorem 2.10 and of Theorem 2.12 for $L = 1$

The linearization (2.9) of Equation (1.2) is obtained by replacing B and H in Equation (5.1) by 0 and coincides with Equation (5.1) for $L = 1$. Likewise, the equation for (2.9) corresponding to (5.2) is obtained by omitting the terms $\alpha B(\tilde u(k-1), y_t(k-1))$. We obtain a result for Equation (2.9) corresponding to Lemma 5.3 (a) and (b) under the sole assumption $\hat\lambda_k \ge \beta L\rho^{L^k}$ for $k \ge 1$. The proof proceeds as for Theorem 2.12 for $L \ge 2$ with the difference that $\hat\lambda_k = \max\left\{\beta L\rho^{L^k}, \beta\theta\right\}$. We have $\hat\lambda_k$ equal to $\beta\theta$ for large k for $0 < \theta < 1$ when $L \ge 2$ and for $\rho \le \theta < 1$ when $L = 1$.

6 Tightness estimates and the functional CLT in equilibrium

6.1 Proof of Lemma 2.13 (infinite horizon and invariant law bounds)

Let $U_h(v)$ be the solution of (1.2) at time $h \geq 0$ with initial value v in $\mathcal{V}$. For $t_0 \geq 0$ let $Z^N_{t_0,h} = \sqrt{N}\left(R^N_{t_0+h} - U_h(R^N_{t_0})\right)$. Then $Z^N_{t_0+h} = Z^N_{t_0,h} + \sqrt{N}\left(U_h(R^N_{t_0}) - \tilde{u}\right)$ and Theorem 2.12 yields

$$\left\|Z^N_{t_0+h}\right\|_{L_2(g_\theta)} \leq \left\|Z^N_{t_0,h}\right\|_{L_2(g_\theta)} + \mathrm{e}^{-\gamma_\theta h} C_\theta \left\|Z^N_{t_0}\right\|_{L_2(g_\theta)} . \tag{6.1}$$

The conditional law of $(Z^N_{t_0,h})_{h\geq 0}$ given $R^N_{t_0} = r$ is the law of Z^N started with $R^N_0 = u_0 = r$, in particular with $Z^N_0 = Z^N_{t_0,0} = 0$. We reason as in (3.7)–(3.10) except that the bound (3.9) becomes

$$\left\|R^N_{t_0+s}\right\|_{L_2(g_\theta)} \leq \|\tilde{u}\|_{L_2(g_\theta)} + N^{-1/2}\left\|Z^N_{t_0+s}\right\|_{L_2(g_\theta)}$$

and we use (6.1) and obtain that for some $K_T < \infty$

$$\sup_{0\leq h\leq T}\left\|Z^N_{t_0,h}\right\|_{L_2(g_\theta)} \leq K_T\left(N^{-1/2}\|\tilde{u}\|_{L_2(g_\theta)} + N^{-1}C_\theta\left\|Z^N_{t_0}\right\|_{L_2(g_\theta)} + \sup_{0\leq h\leq T}\left\|M^N_{t_0+h} - M^N_{t_0}\right\|_{L_2(g_\theta)}\right)$$

which combined with (6.1) yields that for some $L_T < \infty$ we have for $0 \leq h \leq T$

$$\mathbf{E}\left(\left\|Z^N_{t_0+h}\right\|^2_{L_2(g_\theta)}\right) \leq L_T + 2(K_T N^{-1} + \mathrm{e}^{-\gamma_\theta h})^2 C^2_\theta\,\mathbf{E}\left(\left\|Z^N_{t_0}\right\|^2_{L_2(g_\theta)}\right). \tag{6.2}$$

We fix T large enough for $8\mathrm{e}^{-2\gamma_\theta T}C^2_\theta \leq \varepsilon < 1$. Uniformly for $N \geq K_T \mathrm{e}^{\gamma_\theta T}$, for $m \in \mathbb{N}$

$$\mathbf{E}\left(\left\|Z^N_{(m+1)T}\right\|^2_{L_2(g_\theta)}\right) \leq L_T + \varepsilon\,\mathbf{E}\left(\left\|Z^N_{mT}\right\|^2_{L_2(g_\theta)}\right)$$

and by induction

$$\mathbf{E}\left(\left\|Z^N_{mT}\right\|^2_{L_2(g_\theta)}\right) \leq L_T\sum_{j=1}^m \varepsilon^{j-1} + \varepsilon^m\,\mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right) \leq \frac{L_T}{1-\varepsilon} + \mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right),$$

and (6.2) also yields

$$\sup_{0\leq h\leq T}\mathbf{E}\left(\left\|Z^N_{mT+h}\right\|^2_{L_2(g_\theta)}\right) \leq L_T + 8C^2_\theta\,\mathbf{E}\left(\left\|Z^N_{mT}\right\|^2_{L_2(g_\theta)}\right),$$

hence the infinite horizon bound

$$\sup_{t\geq 0}\mathbf{E}\left(\left\|Z^N_t\right\|^2_{L_2(g_\theta)}\right) \leq L_T + 8C^2_\theta\left(\frac{L_T}{1-\varepsilon} + \mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right)\right).$$

Ergodicity and the Fatou Lemma yield that for Z^N_∞ distributed according to the invariant law

$$\mathbf{E}\left(\left\|Z^N_\infty\right\|^2_{L_2(g_\theta)}\right) \leq \liminf_{t\geq 0}\mathbf{E}\left(\left\|Z^N_t\right\|^2_{L_2(g_\theta)}\right) \leq \sup_{t\geq 0}\mathbf{E}\left(\left\|Z^N_t\right\|^2_{L_2(g_\theta)}\right)$$

and the invariant law bound follows if we show that we can choose R^N_0 in $\mathcal{V}^N$ such that

$$\limsup_{N\to\infty}\mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right) < \infty . \tag{6.3}$$

For this we consider $L \geq 2$, the case $L = 1$ being similar, and R_0^N given for $k \geq 0$ by $R_0^N(k) = iN^{-1}$ with $1 \leq i \leq N$ such that $-2^{-1}N^{-1} < \tilde{u}(k) - iN^{-1} \leq 2^{-1}N^{-1}$. For $x \geq 0$ and $0 < y \leq 1$

$$
\begin{aligned}
y = \rho^{(L^x-1)/(L-1)} &\Leftrightarrow x = \log\left(1 + (L-1)\log y/\log\rho\right)/\log L \\
&\Leftrightarrow \theta^{-x} = \left(1 + (L-1)\log y/\log\rho\right)^{-\log\theta/\log L}
\end{aligned}
$$

hence for $z(N) = \inf\{k \geq 1 : R_0^N(k) = 0\}$ we have $z(N) = \inf\{k \geq 1 : \tilde{u}(k) \leq 2^{-1}N^{-1}\} = \inf\left\{k \in \mathbb{N} : k \geq \log\left(1 + (L-1)\log\left(2^{-1}N^{-1}\right)/\log\rho\right)/\log L\right\}$. Then

$$
\left\|Z_0^N\right\|_{L_2(g_\theta)}^2 = N \sum_{k=1}^{z(N)-1} \left(R_0^N(k) - \tilde{u}(k)\right)^2 \theta^{-k} + N \sum_{k \geq z(N)} \tilde{u}(k)^2 \theta^{-k}
$$

with

$$
N \sum_{k=1}^{z(N)-1} \left(R_0^N(k) - \tilde{u}(k)\right)^2 \theta^{-k} \leq 2^{-2}N^{-1} \frac{\theta^{-z(N)} - \theta^{-1}}{\theta^{-1} - 1} = O\left(N^{-1}(\log N)^{-\log\theta/\log L}\right)
$$

and for large enough N (and hence $z(N)$)

$$
\begin{aligned}
N \sum_{k \geq z(N)} \tilde{u}(k)^2 \theta^{-k} &= N\tilde{u}(z(N))^2 \sum_{j \geq 0} \rho^{2L^{z(N)}(L^j-1)/(L-1)} \theta^{-(j+z(N))} \\
&\leq 2^{-2}N^{-1} \sum_{j \geq 0} \rho^{L^{z(N)}(L^j-1)/(L-1)} = o(N^{-1}),
\end{aligned}
$$

hence (6.3) holds and the proof is complete.

6.2 The functional CLT: Proof of Theorem 2.14

Lemma 2.13 and the Markov inequality imply that in equilibrium $(Z_0^N)_{N \geq L}$ is tight for the weak topology of $L_2(g_\rho)$, for which all bounded sets are relatively compact. Consider a subsequence. We can extract a further subsequence along which $(Z_0^N)_{N \geq L}$ converges in law to some square-integrable Z_0^∞ in $L_2(g_\rho)$, and Theorem 2.6 yields that along the further subsequence $(Z^N)_{N \geq L}$ converges in law to the OU process Z^∞ unique solution for (2.10) in $L_2(g_\rho)$ starting at Z_0^∞.

The limit in law of a sequence of stationary processes is stationary (Ethier-Kurtz [4] p. 131, Lemma 7.7 and Theorem 7.8). Hence the law of Z^∞ is determined as the unique law of the stationary OU process given by (2.10), see Theorem 2.11. From every subsequence we can extract a further subsequence converging in law to Z^∞, hence $\lim_{N\to\infty} Z^N = Z^\infty$ in law.

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