

NON ABELIAN TWISTED REIDEMEISTER TORSION FOR FIBERED KNOTS

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ABSTRACT. In this article, we give an explicit formula to compute the non-abelian twisted sign-determined Reidemeister torsion of the exterior of a fibered knot in terms of its monodromy. As an application, we give explicit formulae for the non-abelian Reidemeister torsion of torus knots and of the figure eight knot.

1. INTRODUCTION

A knot in the 3-sphere is called *fibered* if its exterior has the structure of a surface bundle over the circle. For example, each torus knot is fibered and the figure eight knot is also fibered. The aim of this paper is to compute the sign-determined non-abelian Reidemeister torsion defined by the author in [Dub03b] (see also [Dub03a]) for a fibered knot in S^3 in terms of the eigenvalues of the tangent map induced by its monodromy on the moduli representation of the fundamental group of its fibre (see Main Theorem). This non-abelian Reidemeister torsion is a combinatorial invariant of knots.

In [Fri88], D. Fried already computed the twisted Reidemeister torsion for bundles over the circle but in an *acyclic* case. In the situation of fibered knots describes in this article, we work in a *non-acyclic* case. The key idea of our computations is to look at the Wang sequence in cohomology associated to the fibration and to compute the twisted Reidemeister torsion of the exterior of the fibered knot in terms of the Reidemeister torsion of this Wang sequence. In [LST98], W. Lück, T. Schick and T. Thielmann study the behaviour of the analytic torsion under smooth fibrations. They obtain a general formula which involves several Reidemeister torsion's, namely the torsion's of the fibre, of the basis and of the Leray-Serre spectral sequence for deRham cohomology induced by the fibration. This last term is of course the most difficult to compute. In our situation it coincides with the Wang sequence (see [Ser51]) and is precisely the one we must focus on and explicitly compute in terms of the monodromy.

The paper is organised as follows. Section 2 reviews the sign-determined Reidemeister torsion. Section 3 deals with the construction of the non-abelian twisted Reidemeister torsion for knots. In Section 4, we prove the main theorem of the paper (see Main Theorem) about the twisted Reidemeister torsion associated to fibered knots. Finally Section 6 treats some examples.

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2. REVIEW ON THE SIGN-DETERMINED REIDEMEISTER TORSION

The Reidemeister torsion of a finite simplicial complex W is a more subtle invariant than the usual ones traditionally used in algebraic topology, because it uses an action of the fundamental group $\pi_1(W)$ on the universal cover of W . This section reviews the basic definitions and sets up the conventions which will be used. For more details, we refer to Milnor's survey [Mil66] and to Turaev's monographs [Tur01] & [Tur02].

Notation. In this paper, \mathbb{F} is one of the fields \mathbb{R} or \mathbb{C} ; \mathfrak{G} is one of the Lie groups $SU(2)$ or $SL_2(\mathbb{C})$, and \mathfrak{g} is the associated Lie algebra $\mathfrak{su}(2)$ or $\mathfrak{sl}_2(\mathbb{C})$. We denote by $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ the Killing form of \mathfrak{g} . It is well-known that $B_{\mathfrak{g}}$ is non-degenerated. The Lie algebra $\mathfrak{su}(2)$ is identified with the *pure quaternions*, i.e. with the quaternions of the form $q = ai + bj + ck$. In this case, $B_{\mathfrak{su}(2)}$ is equal to the usual scalar product $\langle \cdot, \cdot \rangle$ multiplied by -2 . The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is identified with the space of matrices with zero trace. As a consequence, the Killing form satisfies

$$B_{\mathfrak{sl}_2(\mathbb{C})} \left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \right) = 8aa' + 4(bc' + cb').$$

2.1. Basic definitions. Let E be an n -dimensional vector space over \mathbb{F} . For two ordered basis $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of E , we write $[\mathbf{a}/\mathbf{b}] = \det(p_{ij})_{i,j}$, where $a_i = \sum_{j=1}^n p_{ij}b_j$, for all i . The bases \mathbf{a} and \mathbf{b} are called *equivalent* if $[\mathbf{a}/\mathbf{b}] = +1$. In this case, we write $\mathbf{a} \sim \mathbf{b}$.

Let $C_* = (0 \rightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} C_0 \rightarrow 0)$ be a chain complex. For each i , consider $B_i = \text{im}(d_{i+1} : C_{i+1} \rightarrow C_i)$, $Z_i = \text{ker}(d_i : C_i \rightarrow C_{i-1})$ and the homology group $H_i = Z_i/B_i$. We say that C_* is *acyclic* if it has zero homology.

Suppose that for all $i = 1, \dots, n$ both C_i and H_i are endowed with *reference bases*. In this situation, C_* is said to be *based* and *homology based*; one defines the Reidemeister torsion of C_* as follows. Denote by \mathbf{c}^i the reference basis in C_i and by \mathbf{h}^i the one in H_i . Let \mathbf{b}^i be a sequence of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis in B_{i-1} and denote by $\tilde{\mathbf{h}}^i$ a lift of \mathbf{h}^i in Z_i . Then for every i , $d_{i+1}(\mathbf{b}^{i+1})$, $\tilde{\mathbf{h}}^i$ and \mathbf{b}^i combines to yield a new basis $d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i\mathbf{b}^i$ in C_i . The *Reidemeister torsion* of C_* (in the bases \mathbf{c}^* and \mathbf{h}^*) is the alternating product (see [Tur01, Definition 3.1]):

$$(1) \quad \text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = \prod_{i=0}^n [d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{F} \setminus \{0\}.$$

The torsion $\text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choice of \mathbf{b}^i and $\tilde{\mathbf{h}}^i$, and does only depend on the equivalence classes of \mathbf{c}^i and \mathbf{h}^i . More precisely, if \mathbf{c}'^i is another basis in C_i and \mathbf{h}'^i another one in H_i , then we have the so-called *basis change formula*

$$(2) \quad \frac{\text{tor}(C_*, \mathbf{c}'^*, \mathbf{h}'^*)}{\text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)} = \prod_{i=0}^n \left(\frac{[\mathbf{c}'^i/\mathbf{c}^i]}{[\mathbf{h}'^i/\mathbf{h}^i]} \right)^{(-1)^i}.$$

2.2. The Reidemeister torsion of a CW-complex. If formula (1) is used to define the Reidemeister torsion of a CW-complex, then we will fall into the well-known *up-to-sign ambiguity* of the Reidemeister torsion. To solve this problem, V. Turaev has introduced a *sign-determined* Reidemeister torsion.

The sign-determined torsion. Set

$$\alpha_i(C_*) = \sum_{k=0}^i \dim C_k \in \mathbb{Z}/2\mathbb{Z}, \quad \beta_i(C_*) = \sum_{k=0}^i \dim H_k \in \mathbb{Z}/2\mathbb{Z},$$

$$|C_*| = \sum_{k \geq 0} \alpha_k(C_*) \beta_k(C_*) \in \mathbb{Z}/2\mathbb{Z}.$$

The *sign-determined Reidemeister torsion* of C_* is the “sign-corrected” torsion

$$(3) \quad \text{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) \in \mathbb{F} \setminus \{0\},$$

see [Tur86, Section 3.1] or [Tur02, formula (1.a)].

For an acyclic based chain complex C_* , we have $\text{Tor}(C_*) = \text{tor}(C_*)$.

The Reidemeister torsion of a CW-complex. Let W be a finite CW-complex; consider a representation $\rho : \pi_1(W) \rightarrow \mathfrak{G}$. The universal covering \widetilde{W} of W is endowed with the induced CW-complex structure and the fundamental group¹ $\pi_1(W)$ acts on \widetilde{W} by the covering transformations. This action turns $C_*(\widetilde{W}; \mathbb{Z})$ into a chain complex of left $\mathbb{Z}[\pi_1(W)]$ -modules. The Lie algebra \mathfrak{g} can be viewed as a left $\mathbb{Z}[\pi_1(W)]$ -module via the composition $Ad \circ \rho$, where $Ad : \text{SU}(2) \rightarrow \text{Aut}(\mathfrak{su}(2))$, $A \mapsto Ad_A$ is the adjoint representation. This module will be denoted by \mathfrak{g}_ρ . The \mathfrak{g}_ρ -twisted cochain complex of W is

$$C^*(W; \mathfrak{g}_\rho) = \text{Hom}_{\pi_1(W)}(C_*(\widetilde{W}; \mathbb{Z}), \mathfrak{g}_\rho).$$

This cochain complex $C^*(W; \mathfrak{g}_\rho)$ computes the \mathfrak{g}_ρ -twisted cohomology of W , which will be denoted by $H^*(W; \mathfrak{g}_\rho)$. When $H^*(W; \mathfrak{g}_\rho) = 0$, we say that ρ is *acyclic*.

We choose a *cohomology orientation* of W , i.e. an orientation of the real vector space $H^*(W; \mathbb{R}) = \bigoplus_{i \geq 0} H^i(W; \mathbb{R})$; we denote by \mathfrak{o} such an orientation and by $\{e_1^{(i)}, \dots, e_{n_i}^{(i)}\}$ the set of i -dimensional cells of W .

Denote by $\tilde{e}_j^{(i)}$ a lift in \widetilde{W} of the cell $e_j^{(i)}$ and choose an arbitrary order and an arbitrary orientation for the cells $\tilde{e}_j^{(i)}$. Thus $\mathbf{c}^i = (\tilde{e}_1^{(i)}, \dots, \tilde{e}_{n_i}^{(i)})$ is a $\mathbb{Z}[\pi_1(W)]$ -basis in $C_i(\widetilde{W}; \mathbb{Z})$. If $\mathcal{B} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an orthonormal basis in \mathfrak{g} , we consider

$$\mathbf{c}_{\mathcal{B}}^i = \left(\tilde{e}_{1,\mathbf{a}}^{(i)}, \tilde{e}_{1,\mathbf{b}}^{(i)}, \tilde{e}_{1,\mathbf{c}}^{(i)}, \dots, \tilde{e}_{n_i,\mathbf{a}}^{(i)}, \tilde{e}_{n_i,\mathbf{b}}^{(i)}, \tilde{e}_{n_i,\mathbf{c}}^{(i)} \right),$$

the corresponding “dual” basis for $\text{Hom}_{\pi_1(W)}(C_*(\widetilde{W}; \mathbb{Z}), \mathfrak{g}_\rho)$. As a consequence, if \mathbf{h}^i denotes a basis in $H^i(W; \mathfrak{g}_\rho)$ then $\text{Tor}(C^*(W; \mathfrak{g}_\rho), \mathbf{c}_{\mathcal{B}}^*, \mathbf{h}^*) \in \mathbb{F} \setminus \{0\}$ is well-defined.

The cells $(\tilde{e}_j^{(i)})_{0 \leq i \leq \dim W, 1 \leq j \leq n_i}$ are in one-to-one correspondence with the cells of W and their order and orientation induce an order and an orientation for the cells $(e_j^{(i)})_{0 \leq i \leq \dim W, 1 \leq j \leq n_i}$. We thus produce a basis over \mathbb{R} in $C^*(W; \mathbb{R})$, which we denote by c^* . Provide each vector space $H^i(W; \mathbb{R})$ with a reference basis h^i such that the basis $(h^0, \dots, h^{\dim W})$ in $H^*(W; \mathbb{R})$ is *positively oriented* with respect to the cohomology orientation \mathfrak{o} . Compute the sign-determined Reidemeister torsion $\text{Tor}(C^*(W; \mathbb{R}), c^*, h^*) \in \mathbb{R} \setminus \{0\}$ of the resulting based and cohomology based chain complex and consider its sign $\tau_0 = \text{sgn}(\text{Tor}(C^*(W; \mathbb{R}), c^*, h^*)) \in \{\pm 1\}$. The *sign-determined Reidemeister torsion* of the cohomology oriented CW-complex W is

$$\text{TOR}(W; \mathfrak{g}_\rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \text{Tor}(C^*(W; \mathfrak{g}_\rho), \mathbf{c}_{\mathcal{B}}^*, \mathbf{h}^*) \in \mathbb{F} \setminus \{0\}.$$

¹We will not specify any base point, because all the constructions we do are invariant under conjugation.

The torsion $\text{TOR}(W; \mathfrak{g}_\rho, \mathbf{h}^*, \mathfrak{o})$ is well-defined. It is independent from the orthonormal basis \mathcal{B} of \mathfrak{g} , from the choice of the lifts $\tilde{e}_j^{(i)}$ and from the choice of the positively oriented basis in $H^*(W; \mathbb{R})$. Moreover, it is independent from order and orientation of the cells (because they appear twice). Finally, it just depends on the conjugacy class of ρ .

One can prove that TOR is invariant under cellular subdivision, homeomorphism and simple homotopy equivalence. In fact, it is precisely the sign $(-1)^{|C_*|}$ in (3) which ensures all these particularly important properties of invariance (see [Dub03b, Chapter 2] for detailed proofs).

2.3. The multiplicativity lemma. This lemma appears to be a very powerful tool for computing Reidemeister torsion's, and will be used all over this paper.

Multiplicativity Lemma. *Let*

$$(4) \quad 0 \longrightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \longrightarrow 0$$

be an exact sequence of chain complexes. Assume that C'_ , C_* and C''_* are based and homology based. For all i , denote by \mathbf{c}'^i , \mathbf{c}^i and \mathbf{c}''^i the reference bases in C'_i , C_i and C''_i respectively; and assume that these bases are compatible, in the sense that $\mathbf{c}^i \sim \mathbf{c}'^i \mathbf{c}''^i$. Associated to (4) is the long sequence in homology*

$$\cdots \longrightarrow H_i(C'_*) \longrightarrow H_i(C_*) \longrightarrow H_i(C''_*) \longrightarrow H_{i-1}(C'_*) \longrightarrow \cdots$$

We denote by \mathcal{H}_ this acyclic chain complex and base $\mathcal{H}_{3i+2} = H_i(C'_*)$, $\mathcal{H}_{3i+1} = H_i(C_*)$ and $\mathcal{H}_{3i} = H_i(C''_*)$ with the reference bases in $H_i(C'_*)$, $H_i(C_*)$ and $H_i(C''_*)$ respectively. Then*

$$\text{Tor}(C_*) = (-1)^{\alpha(C'_*, C''_*) + \varepsilon(C'_*, C_*, C''_*)} \text{Tor}(C'_*) \cdot \text{Tor}(C''_*) \cdot \text{tor}(\mathcal{H}_*),$$

where

$$\alpha(C'_*, C''_*) = \sum_{i \geq 0} \alpha_{i-1}(C'_*) \alpha_i(C''_*) \in \mathbb{Z}/2\mathbb{Z}$$

and

$$\varepsilon(C'_*, C_*, C''_*) = \sum_{i \geq 0} [(\beta_i(C_*) + 1)(\beta_i(C'_*) + \beta_i(C''_*)) + \beta_{i-1}(C'_*) \beta_i(C''_*)] \in \mathbb{Z}/2\mathbb{Z}.$$

The proof is a carefully computation based on linear algebra, see [Tur86, Lemma 3.4.2] and [Mil66, Theorem 3.2].

3. NON ABELIAN REIDEMEISTER TORSION FOR KNOT EXTERIORS

In this section, we assume that S^3 is oriented and that $K \subset S^3$ is an oriented knot. Let $M_K = S^3 \setminus N(K)$ be the exterior of K and $G_K = \pi_1(M_K)$ its group. Here, we construct a non abelian twisted Reidemeister torsion for K , which appears to be a combinatorial invariant of knots. To accomplish this, we must produce some reference bases for the twisted cohomology group $H^*(M_K; \mathfrak{g}_\rho)$.

3.1. Representations of knot groups. For a finitely generated group π , we denote by $R(\pi; \mathfrak{G}) = \text{Hom}(\pi; \mathfrak{G})$ the space of \mathfrak{G} -representation of π endowed with the *compact-open topology*, where π is assumed to have the discrete topology. A representation $\rho : \pi \rightarrow \mathfrak{G}$ is called *abelian* (resp. *metabelian*) if its image $\rho(\pi)$ is an abelian subgroup of \mathfrak{G} (resp. if $\rho([\pi, \pi])$ is an abelian subgroup of \mathfrak{G}). A representation $\rho : \pi \rightarrow \mathfrak{G}$ is called *reducible* if there exist a proper non-trivial subspace U of \mathbb{C}^2 such that $\rho(g)(U) \subset U$, for all $g \in \pi$. Observe that all abelian (resp. metabelian) \mathfrak{G} -representations of π are

reducible. A representation is called *irreducible* if it is not reducible. We denote by $\tilde{R}(\pi; \mathfrak{G})$ the subspace of irreducible representations.

The Lie group \mathfrak{G} acts on $R(\pi; \mathfrak{G})$ by conjugation and we denote by $\mathcal{R}(\pi; \mathfrak{G}) = R(\pi; \mathfrak{G})/\mathfrak{G}$ the moduli space. The action by conjugation of $SU(2)$ on $R(G; SU(2))$ factors through $SO(3) = SU(2)/\{\pm 1\}$ as a *free* action on the open subspace $\tilde{R}(\pi; SU(2))$ and we set $\hat{R}(\pi; SU(2)) = \tilde{R}(\pi; SU(2))/SO(3)$. If $\rho \in \tilde{R}(\pi; SU(2))$, then we denote by $[\rho]$ its conjugacy class. In this way, we can see $\hat{R}(\pi; SU(2))$ as the base space of a principal $SO(3)$ -bundle with total space $\tilde{R}(\pi; SU(2))$, see [GM92], Section 3.A. In the case of $SL_2(\mathbb{C})$, the quotient $\mathcal{R}(\pi; SL_2(\mathbb{C}))$ is not Hausdorff in general. Following [CS83] we will focus on the *representation variety* $X(\pi; SL_2(\mathbb{C}))$, which is the set of *characters* of π . Associated to $\rho \in R(\pi; SL_2(\mathbb{C}))$ is its character $\chi_\rho : \pi \rightarrow \mathbb{C}$, defined by $\chi_\rho(g) = \text{Tr}(\rho(g))$. In some sense $X(\pi; SL_2(\mathbb{C}))$ is the ‘‘algebraic quotient’’ of $R(\pi; SL_2(\mathbb{C}))$ by the action by conjugation of $PSL_2(\mathbb{C})$. We also set $\hat{R}(\pi; SL_2(\mathbb{C})) = \tilde{R}(\pi; SL_2(\mathbb{C}))/PSL_2(\mathbb{C})$ the image of $\tilde{R}(\pi; SL_2(\mathbb{C}))$ under $R(\pi; SL_2(\mathbb{C})) \rightarrow X(\pi; SL_2(\mathbb{C}))$.

3.2. Twisted cohomology of the torus. Let M be an n -dimensional compact manifold possibly with boundary ∂M . By the ‘‘inward pointing normal vector in last position’’ convention, ∂M is an oriented $(n - 1)$ -manifold. The Killing form $B_{\mathfrak{g}}$ induces a *cup-product*

$$(5) \quad \cup : H^p(M; \mathfrak{g}_\rho) \times H^{n-p}(M, \partial M; \mathfrak{g}_\rho) \rightarrow H^n(M, \partial M; \mathbb{F}),$$

which is (as $B_{\mathfrak{g}}$) non-degenerated.

Denote by T^2 the 2-dimensional torus. For any non trivial $\rho \in R(\pi_1(T^2); \mathfrak{G})$, we observe that $H^0(T^2; \mathfrak{g}_\rho) = \mathfrak{g}^{Ad \circ \rho(\pi_1(T^2))} \cong \mathbb{F}$. If P^ρ denotes a generator in $H^0(T^2; \mathfrak{g}_\rho)$, then the cup product (5) and Poincar duality combines to make the map $\phi_{P^\rho}^{(2)} : H^2(T^2; \mathfrak{g}_\rho) \rightarrow H^2(T^2; \mathbb{F})$, given by $\phi_{P^\rho}^2(z) = P^\rho \cup z$. This is a natural isomorphism.

It is well-known that the non trivial \mathfrak{G} -representations of $\pi_1(T^2)$ are of two kinds: the *hyperbolic* ones and the *parabolic* ones. Here we say that $\rho \in R(\pi_1(T^2); \mathfrak{G})$ is hyperbolic if each element in $\rho(\pi_1(T^2))$ is diagonalizable; ρ is called parabolic if each element in $\rho(\pi_1(\partial M_K))$ is non-diagonalizable or is $\mathbf{1}$. Further notice that each element in $R(\pi_1(T^2); SU(2))$ is hyperbolic. Assume that ρ is hyperbolic. One can prove that the map $\phi_{P^\rho}^{(i)} : H^i(T^2; \mathfrak{g}_\rho) \rightarrow H^i(T^2; \mathbb{F})$, given by $\phi_{P^\rho}^*(z) = P^\rho \cup z$, is a natural isomorphism, for each $i = 0, 1, 2$, because $B_{\mathfrak{g}}(P^\rho, P^\rho) \neq 0$ (see [Por97, Proposition 3.18]). If ρ is assumed to be parabolic, then we can notice that $\phi_{P^\rho}^{(1)} = P^\rho \cup \cdot$ is not an isomorphism.

3.3. μ -regular representations. We turn now to the case of knot groups. Let μ be a simple closed unoriented curve in ∂M_K . Among irreducible representations we focus on the μ -regular ones. We say that $\rho \in \tilde{R}(G_K; \mathfrak{G})$ is μ -regular, if (see [Por97, Definition 3.21]):

- (1) the map $\alpha^* : H^1(M_K; \mathfrak{g}_\rho) \rightarrow H^1(\mu; \mathfrak{g}_\rho)$, induced by the inclusion $\alpha : \mu \hookrightarrow M_K$, is *injective*,
- (2) if $\text{Tr}(\rho(\pi_1(\partial M_K))) \subset \{\pm 2\}$, then $\rho(\mu) \neq \pm \mathbf{1}$.

It is easy to see that this notion is invariant by conjugation.

Lemma 1. *If ρ is μ -regular, then $\dim_{\mathbb{F}} H^1(M_K; \mathfrak{g}_\rho) = \dim_{\mathbb{F}} H^2(M_K; \mathfrak{g}_\rho) = 1$.*

Proof. If ρ is μ -regular, then the map $i^* : H^1(M_K; \mathfrak{g}_\rho) \rightarrow H^1(\partial M_K; \mathfrak{g}_\rho)$, induced by the inclusion $i : \partial M_K \hookrightarrow M_K$, is injective. We have $\dim_{\mathbb{F}} H^1(\partial M_K; \mathfrak{g}_\rho) = 2$ and Poincar duality implies $\text{rk}_{\mathbb{F}} i^* = 1$. As a consequence $\dim_{\mathbb{F}} H^1(M_K; \mathfrak{g}_\rho) = \text{rk}_{\mathbb{F}} i^* = 1$, and thus $\dim_{\mathbb{F}} H^2(M_K; \mathfrak{g}_\rho) = 1$ (because $H^0(M_K; \mathfrak{g}_\rho) = 0$ and $\sum_i (-1)^i \dim_{\mathbb{F}} H^i(M_K; \mathfrak{g}_\rho) = 3\chi(M_K) = 0$). \square

Here is an alternative formulation of μ -regularity which will be more useful for us. The notion of μ -regularity is defined for an unoriented curve μ but to avoid any ambiguity in what follows we must endow μ with a “coherent” orientation.

Let $\text{int}(\cdot, \cdot)$ be the intersection form associated to the orientation of ∂M_K induced by the one of M_K . The peripheral subgroup $\pi_1(\partial M_K)$ is generated by the meridian-longitude system m, l of K , where m is oriented by the convention $\ell k(K, m) = +1$ and l is oriented by using the requirement that $\text{int}(m, l) = +1$. We orient the curve μ as follows. If μ is parallel to l , then μ and l are endowed with the same orientation, if not μ is endowed with the orientation such that $\text{int}(\mu, l) \geq 0$. The resulting oriented curve will be denoted by $\tilde{\mu}$.

Fix a generator P^ρ in $H^0(\partial M_K; \mathfrak{g}_\rho)$ and consider the linear form $f_\mu^\rho : H^1(M_K; \mathfrak{g}_\rho) \rightarrow \mathbb{F}$ induced by the inclusion $\alpha : \mu \hookrightarrow M_K$ and by the cup product. We have

$$f_\mu^\rho(v) = B_{\mathfrak{g}}(P^\rho, v(\tilde{\mu})), \text{ for all } v \in H^1(M_K; \mathfrak{g}_\rho).$$

And we observe $f_{\tilde{\mu}^{-1}}^\rho = -f_\mu^\rho$.

Proposition 2. *The representation $\rho \in \tilde{R}(G_K; \mathfrak{G})$ is μ -regular if and only if the linear form $f_\mu^\rho : H^1(M_K; \mathfrak{g}_\rho) \rightarrow \mathbb{F}$ is an isomorphism.*

Proof. Observe first that if ρ is μ -regular, then $\rho|_{\pi_1(\partial M_K)}$ is non-trivial, and we have $\dim_{\mathbb{F}} H^1(M_K; \mathfrak{g}_\rho) = 1$. We split the first part of the proof into two steps to be clearer.

- (1) If $\rho|_{\pi_1(\partial M_K)}$ is hyperbolic, then $H^*(\partial M_K; \mathfrak{g}_\rho) \cong H^*(\partial M_K; \mathbb{F})$, thus the linear form f_μ^ρ is non-trivial.
- (2) If $\rho|_{\pi_1(\partial M_K)}$ is parabolic then $\rho(\mu) \neq \pm \mathbf{1}$, thus $\mathfrak{g}^{Ad \circ \rho(\mu)} = \mathfrak{g}^{Ad \circ \rho(\pi_1(\partial M_K))}$. As a consequence $P^\rho \cup \cdot : H^1(\mu; \mathfrak{g}_\rho) \rightarrow H^1(\mu; \mathbb{F})$ is an isomorphism, and thus f_μ^ρ is non-trivial.

Assume now that f_μ^ρ is an isomorphism. We have $f_\mu^\rho = F \circ \alpha^*$, where $F : H^1(\mu; \mathfrak{g}_\rho) \rightarrow \mathbb{F}$ is the linear form induced by the cup product. Thus α^* is injective which proved the first assumption. Next, if $\rho(\mu) = \pm \mathbf{1}$, then $H^1(\mu; \mathfrak{g}_\rho) \cong \mathfrak{g}$ and thus $\rho|_{\pi_1(\partial M_K)}$ must be non-parabolic (see [Por97, Proposition 3.18]). \square

Example 1. Let K be a torus knot. All the irreducible representations of G_K into \mathfrak{G} are m -regular and also l -regular (see [Dub03b, Example 1.43]).

3.4. Reference bases of the twisted cohomology group of M_K . Here we suppose that ρ is μ -regular.

Firstly, the reference generator of $H^1(M_K; \mathfrak{g}_\rho)$ is defined by

$$(6) \quad h_\rho^{(1)}(\tilde{\mu}) = (f_\mu^\rho)^{-1}(1).$$

Remark 1. The generator depends on the orientation of $\tilde{\mu}$, more precisely we have $h_\rho^{(1)}(\tilde{\mu}^{-1}) = -h_\rho^{(1)}(\tilde{\mu})$.

Secondly, the construction of the reference generator of $H^2(M_K; \mathfrak{g}_\rho)$ works as follows (see [Por97, Corollary 3.23]). The long exact sequence in \mathfrak{g}_ρ -twisted cohomology associated to the pair $(M_K, \partial M_K)$ implies that the homomorphism $i^* : H^2(M_K; \mathfrak{g}_\rho) \rightarrow$

$H^2(\partial M_K; \mathfrak{g}_\rho)$, induced by the inclusion $\partial M_K \hookrightarrow M_K$ is an isomorphism (because $\dim_{\mathbb{F}} H^2(M_K; \mathfrak{g}_\rho) = \dim_{\mathbb{F}} H^2(\partial M_K; \mathfrak{g}_\rho) = 1$). As a consequence, the composition

$$\phi_{P^\rho}^{(2)} \circ i^* : H^2(M_K; \mathfrak{g}_\rho) \rightarrow H^2(\partial M_K; \mathfrak{g}_\rho) \rightarrow H^2(\partial M_K; \mathbb{F}) = H^2(\partial M_K; \mathbb{Z}) \otimes \mathbb{F}$$

is an *isomorphism*. Let c be the generator in $H^2(\partial M_K; \mathbb{Z}) = \text{Hom}(H_2(\partial M_K; \mathbb{Z}), \mathbb{Z})$ corresponding to the fundamental class $[[\partial M_K]] \in H_2(\partial M_K; \mathbb{Z})$ induced by the orientation of ∂M_K . The *reference generator* of $H^2(M_K; \mathfrak{g}_\rho)$ is defined by

$$(7) \quad h_\rho^{(2)} = (\phi_{P^\rho}^{(2)} \circ i^*)^{-1}(c).$$

3.5. The Reidemeister torsion \mathbb{T}_μ^K . We equip the exterior of K with its *canonical cohomology orientation* defines as follows (see [Tur02, Section V.3]). We have

$$H^*(M_K; \mathbb{R}) = H^0(M_K; \mathbb{R}) \oplus H^1(M_K; \mathbb{R})$$

and we base this \mathbb{R} -vector space with $([[pt]], m^*)$. Here $[[pt]]$ is the cohomology class of a point, and $m^* : m \mapsto 1$ is the dual of the meridian m of K . This reference basis in $H^*(M_K; \mathbb{R})$ induces the so-called canonical cohomology orientation of M_K , which will be denoted by \mathfrak{o} in the sequel.

Let $\rho : G_K \rightarrow \mathfrak{G}$ be a μ -regular representation; the *Reidemeister torsion* \mathbb{T}_μ^K at ρ is defined by

$$\mathbb{T}_\mu^K(\rho) = \text{TOR} \left(M_K; \mathfrak{g}_\rho, (h_\rho^{(1)}(\tilde{\mu}), h_\rho^{(2)}), \mathfrak{o} \right).$$

Observe that the μ -torsion $\mathbb{T}_\mu^K(\rho)$ does not depend on the choice of the generator P^ρ in $H^0(\partial M_K; \mathfrak{g}_\rho)$. This property is a consequence of formula (2), because $h_\rho^{(1)}(\tilde{\mu})$ and $h_\rho^{(2)}$ change in the same way at the same time.

Proposition 3. *The torsion $\mathbb{T}_\mu^K(\rho)$ does not depend on the orientation of K .*

Proof. If we change the orientation of K , then the orientations of m and l change simultaneously; thus the orientation of $\tilde{\mu}$ is reversed. As a consequence, the reference generator of $H^2(M_K; \mathfrak{g}_\rho)$ is unchanged, but the one of $H^1(M_K; \mathfrak{g}_\rho)$ and the cohomology orientation are reversed simultaneously. Thus $\mathbb{T}_\mu^K(\rho)$ does not change. \square

Remark 2. For an hyperbolic knot K , the torsion $\mathbb{T}_\mu^K(\rho)$ is a sign-refined version of the inverse of Porti's torsion function (see [Por97]).

4. THE REIDEMEISTER TORSION FOR FIBERED KNOTS

In this section, we give an explicit formula to compute the sign-determined non abelian Reidemeister torsion associated to fibered knots using the monodromy of the knot.

Here $K \subset S^3$ is a fibered knot, we denote by F the fiber of K and by γ the boundary of F . Recall that γ corresponds to the longitude of K and thus is an oriented curve in ∂M_K (see Subsection 3.3). If g denotes the genus of the surface F , then $\widehat{R}(\pi_1(F); \mathfrak{G})$ is smooth and $\dim_{\mathbb{F}} \widehat{R}(\pi_1(F); \mathfrak{G}) = 6g - 3$. Let $\varphi = \rho|_{\pi_1(F)}$ be the restriction of ρ to $\pi_1(F)$. The monodromy $\phi : F \rightarrow F$ induces a diffeomorphism $R(\phi) : \widehat{R}(F; \mathfrak{G}) \rightarrow \widehat{R}(F; \mathfrak{G})$. Let $I_\gamma : \widehat{R}(F; \mathfrak{G}) \rightarrow \mathbb{F}$, be the function defined by $I_\gamma : \varrho \mapsto \text{Tr}(\varrho(\gamma))$. As ϕ preserves the boundary $\gamma = \partial F$ and the trace function is invariant by conjugation, we see that $I_\gamma \circ R(\phi) = I_\gamma$.

Main Theorem. *Let $K \subset S^3$ be a fibered knot. Let F be its fiber, a surface of genus g and boundary γ . Assume that ε_0 is the sign of the determinant of the isomorphism $\text{Id} - \phi^* : H^1(F; \mathbb{R}) \rightarrow H^1(F; \mathbb{R})$, where ϕ^* is induced by the monodromy. If $\rho : G_K \rightarrow \mathfrak{G}$ is a non-metabelian γ -regular representation, then the tangent map at φ to $R(\phi) : \widehat{R}(F; \mathfrak{G}) \rightarrow \widehat{R}(F; \mathfrak{G})$ admits 1 as simple eigenvalue. If we denote by λ_i , $1 \leq i \leq 6g - 4$, its other eigenvalues then*

$$(8) \quad \mathbb{T}_\gamma^K(\rho) = -\varepsilon_0 \cdot \prod_{i=1}^{6g-4} \frac{1}{1 - \lambda_i} \in \mathbb{F} \setminus \{0\}.$$

In the Main Theorem, we restrict our attention to the irreducible non-metabelian representations of G_K into \mathfrak{G} . This restriction is just technical; moreover, there exists only a finite number of irreducible metabelian representations of G_K into $\text{SU}(2)$ (see [Lin01, Proposition 4.2]).

Let $\phi_\varphi^* : H^1(F; \mathfrak{g}_\varphi) \rightarrow H^1(F; \mathfrak{g}_\varphi)$ be the tangent map at φ to $R(\phi)$. The equality $I_\gamma \circ R(\phi) = I_\gamma$ implies that 1 is always an eigenvalue of ϕ_φ^* . Further observe that 1 is a simple eigenvalue because ρ is γ -regular and $\text{rk}_{\mathbb{F}}(\text{Id} - \phi_\varphi^*) = 6g - 4$.

The main tool to compute the γ -torsion associated to the fibered knot K with coefficients in \mathfrak{g}_ρ is the Wang sequence in twisted cohomology associated to the fibration $F \hookrightarrow M_K \rightarrow S^1$. This idea is in some sense the same as the one used by D. Fried in [Fri88] in an acyclic case but is technically different, because of the non triviality of $H^*(M_K; \mathfrak{g}_\rho)$.

Denote by $a_1, b_1, \dots, a_g, b_g$ generators of $\pi_1(F)$. If $\phi_* : \pi_1(F) \rightarrow \pi_1(F)$ is the homomorphism induced by the monodromy ϕ , then G_K admits the presentation

$$(9) \quad G_K = \langle a_1, b_1, \dots, a_g, b_g, t \mid t^{-1}a_it = \phi_*(a_i), t^{-1}b_it = \phi_*(b_i), 1 \leq i \leq g \rangle$$

where t corresponds to the meridian of K . We know that M_K collapses to a 2-dimensional cell complex. More precisely, the presentation (9) of G_K allows us to define a 2-dimensional cell complex X_K as follows. The 0-skeleton of X_K consists of one point, the 1-skeleton X_K^1 is a wedge of $2g + 1$ oriented circles corresponding to the generators of (9); finally X_K is obtained from X_K^1 by gluing $2g$ closed 2-cells attached using the relations of (9). A result of Waldhausen [Wal78] implies that the cell complexes X_K and M_K have the same simple homotopy type. As a consequence X_K will be used to explicitly compute $\mathbb{T}_\gamma^K(\rho)$. Remark that $H^*(X_K; \mathfrak{g}_\rho) = H^*(M_K; \mathfrak{g}_\rho)$.

5. PROOF OF THE MAIN THEOREM

With the notation of the previous section, we have

$$\mathbb{T}_\gamma^K(\rho) = \tau_0 \cdot \text{Tor}(X_K; \mathfrak{g}_\rho, (h_\rho^{(1)}(\gamma), h_\rho^{(2)})),$$

where τ_0 is the sign of $\text{Tor}(X_K; \mathbb{R})$. The proof of the Main Theorem consists in the computation of the torsion's $\text{Tor}(X_K; \mathfrak{g}_\rho)$ and $\text{Tor}(X_K; \mathbb{R})$ in terms of the torsion's of the Wang sequences with twisted and with real coefficients respectively. It is divided in several steps.

5.1. Preliminaries: The Wang sequence with twisted coefficients. The monodromy $\phi : F \rightarrow F$ induces an action $\phi_\varphi^* : C^*(F; \mathfrak{g}_\varphi) \rightarrow C^*(F; \mathfrak{g}_\varphi)$ at the level of the twisted chain complex of F . Thus

$$(10) \quad 0 \longrightarrow C^*(M_K; \mathfrak{g}_\rho) \xrightarrow{i^*} C^*(F; \mathfrak{g}_\varphi) \xrightarrow{\text{Id} - \phi_\varphi^*} C^*(F; \mathfrak{g}_\varphi) \longrightarrow 0,$$

is an exact sequence of chain complexes.

Observe that the representation φ is non-abelian, because $[G_K, G_K] = \pi_1(F)$ and ρ is supposed to be non-metabelian. Thus, $H^0(F; \mathfrak{g}_\varphi) = \mathfrak{g}^{Ad\circ\varphi(\pi_1(F))} = 0$ (see [Por97, Lemma 0.7]). As a consequence the sequence (10) induces the long exact sequence in twisted cohomology

$$(11) \quad \mathcal{W}_*^\rho = 0 \longrightarrow H^1(M_K; \mathfrak{g}_\rho) \xrightarrow{i^*} H^1(F; \mathfrak{g}_\varphi) \xrightarrow{\text{Id} - \phi_\varphi^*} H^1(F; \mathfrak{g}_\varphi) \xrightarrow{\delta} H^2(M_K; \mathfrak{g}_\rho) \longrightarrow 0.$$

It is called the *Wang sequence* (with twisted coefficients) associated to the fibration $F \hookrightarrow M_K \rightarrow S^1$.

First, $H^1(M_K; \mathfrak{g}_\rho) \cong \mathbb{F}$ is based with the generator $h_\rho^{(1)}(\gamma)$ (cf. equation (6)) and $H^2(M_K; \mathfrak{g}_\rho) \cong \mathbb{F}$ with the generator $h_\rho^{(2)}$ (cf. equation (7)). Next, fix a basis (over \mathbb{F}) in $H^1(F; \mathfrak{g}_\varphi)$ and observe that $\text{Tor}(\mathcal{W}_*^\rho)$ is independent from it (see equation (2)). Further notice that it is precisely this indetermination which will be used to compute $\text{Tor}(X_K; \mathfrak{g}_\rho)$ in terms of $\text{Tor}(\mathcal{W}_*^\rho)$. We have

$$\alpha = \alpha(C^*(X_K; \mathfrak{g}_\rho), C^*(F; \mathfrak{g}_\varphi)) = 1 \in \mathbb{Z}/2\mathbb{Z}$$

and

$$\varepsilon = \varepsilon(C^*(X_K; \mathfrak{g}_\rho), C^*(F; \mathfrak{g}_\varphi), C^*(F; \mathfrak{g}_\varphi)) = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

As a consequence, the multiplicativity lemma provides

$$(12) \quad \text{Tor}(X_K; \mathfrak{g}_\rho, (h_\rho^{(1)}(\gamma), h_\rho^{(2)})) = -(\text{Tor}(\mathcal{W}_*^\rho))^{-1}.$$

5.2. Torsion of the Wang sequence with twisted coefficients. The aim of this subsection is to show

Claim 4. *Under the hypothesis of the Main Theorem, we have*

$$(13) \quad \text{Tor}(\mathcal{W}_*^\rho) = \prod_{i=1}^{6g-4} (1 - \lambda_i) \in \mathbb{F} \setminus \{0\}.$$

Proof. Let $\mathbf{c} = (v_1, v_2, \dots, v_{6g-3})$ be a trigonalization-basis for ϕ_φ^* in $H_\varphi^1(F)$ such that $i^*(h_\rho^{(1)}(\gamma)) = v_{6g-3}$. In the sequel, we assume that $H^1(F; \mathfrak{g}_\varphi)$ is based with \mathbf{c} . Denote by $\lambda_1, \dots, \lambda_{6g-4}$ the eigenvalues of ϕ_φ^* different from 1. With this notation, we have:

- $\phi_\varphi^*(v_{6g-3}) = v_{6g-3}$ and $v_{6g-3} \notin \text{im}(\text{Id} - \phi_\varphi^*)$,
- $((\text{Id} - \phi_\varphi^*)(v_1), \dots, (\text{Id} - \phi_\varphi^*)(v_{6g-4}))$ is a basis in $\text{im}(\text{Id} - \phi_\varphi^*)$,
- $\delta(v_{6g-3}) = h_\rho^{(2)}$.

This last equality is obtained as follows. Combine the Wang sequences (with twisted coefficients) associated to the fibration $F \hookrightarrow M_K \rightarrow S^1$ and with the one associated to the fibration $\gamma \hookrightarrow \partial M_K \rightarrow S^1$ to obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(M_K; \mathfrak{g}_\rho) & \longrightarrow & H^1(F; \mathfrak{g}_\varphi) & \longrightarrow & H^1(F; \mathfrak{g}_\varphi) & \xrightarrow{\delta} & H^2(M_K; \mathfrak{g}_\rho) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \\ \cdots & \longrightarrow & H^1(\partial M_K; \mathfrak{g}_\rho) & \longrightarrow & H^1(\gamma; \mathfrak{g}_\varphi) & \longrightarrow & H^1(\gamma; \mathfrak{g}_\varphi) & \xrightarrow[\cong]{\Delta} & H^2(\partial M_K; \mathfrak{g}_\rho) & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are induced by inclusions.

If $\mathbf{b}^1 = (v_{6g-3})$ and $\mathbf{b}^2 = (v_1, \dots, v_{6g-4})$, then

$$\mathrm{Tor}(\mathcal{W}_*) = [(\mathrm{Id} - \phi_\varphi^*)(\mathbf{b}^2)\mathbf{b}^1/\mathbf{c}] = \prod_{i=1}^{6g-4} (1 - \lambda_i).$$

□

5.3. Torsion of the Wang sequence with real coefficients. For the same reason as in Subsection 5.1, the fibration $F \hookrightarrow M_K \rightarrow S^1$ induces a Wang sequence \mathfrak{W}_* with real coefficients which splits into three isomorphism's:

(14)

$$H^0(M_K; \mathbb{R}) \xrightarrow{\cong} H^0(F; \mathbb{R}), \quad H^0(F; \mathbb{R}) \xrightarrow{\cong} H^1(M_K; \mathbb{R}) \quad \text{and} \quad H^1(F; \mathbb{R}) \xrightarrow{\cong} H^1(F; \mathbb{R}).$$

Observe that $\mathrm{Tor}(\mathfrak{W}_*)$ does not depend on the choice of the bases in $H^*(F; \mathbb{R})$ used for the computation. As a consequence, we will choose appropriate bases to compute the torsion. Recall that $H^0(M_K; \mathbb{R})$ is based with the generator $\llbracket pt \rrbracket$ and $H^1(M_K; \mathbb{R})$ is based with m^* (see Subsection 3.5). Suppose that $H^0(F; \mathbb{R})$ is endowed with the generator $i^*(\llbracket pt \rrbracket)$. We fix an arbitrary basis in $H^1(F; \mathbb{R})$.

We have

$$\alpha = \alpha(C^*(X_K; \mathbb{R}), C^*(F; \mathbb{R})) = 1 \in \mathbb{Z}/2\mathbb{Z}$$

and

$$\varepsilon = \varepsilon(C^*(X_K; \mathbb{R}), C^*(F; \mathbb{R}), C^*(F; \mathbb{R})) = 1 \in \mathbb{Z}/2\mathbb{Z}.$$

The multiplicativity lemma thus provides

$$\mathrm{Tor}(X_K; \mathbb{R}) = (\mathrm{Tor}(\mathfrak{W}_*))^{-1},$$

so that

$$(15) \quad \tau_0 = \mathrm{sgn}(\mathrm{Tor}(\mathfrak{W}_*)).$$

We turn now to the computation of $\mathrm{Tor}(\mathfrak{W}_*)$. More precisely we show

Claim 5. *We have*

$$(16) \quad \mathrm{sgn}(\mathrm{Tor}(\mathfrak{W}_*)) = \varepsilon_0.$$

Proof. Notice that $\mathrm{Tor}(\mathfrak{W}_*)$ is the product of the determinants of the three isomorphism's (14) in the bases chosen above.

First, in the chosen bases, the isomorphism $i^* : H^0(X_K; \mathbb{R}) \rightarrow H^0(F; \mathbb{R})$ has determinant 1.

Second, we prove that

$$\mathrm{sgn}([\delta \circ i^*(\llbracket pt \rrbracket)]/m^*) = 1.$$

In fact, we show that $\delta(i^*(\llbracket pt \rrbracket)) = m^*$. Combine the Wang sequences in cohomology (with coefficients in \mathbb{Z}) associated to the fibration's $F \hookrightarrow M_K \rightarrow S^1$ and $\gamma \hookrightarrow \partial M_K \rightarrow S^1$ to obtain the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(M_K; \mathbb{Z}) & \xrightarrow{i^*} & H^0(F; \mathbb{Z}) & \xrightarrow{0} & H^0(F; \mathbb{Z}) & \xrightarrow{\delta} & H^1(M_K; \mathbb{Z}) & \xrightarrow{i^*} & H^1(F; \mathbb{Z}) & \longrightarrow & \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow j^* & & \downarrow & & \\ 0 & \longrightarrow & H^0(\partial M_K; \mathbb{Z}) & \xrightarrow{\cong} & H^0(\gamma; \mathbb{Z}) & \xrightarrow{0} & H^0(\gamma; \mathbb{Z}) & \xrightarrow{\Delta} & H^1(\partial M_K; \mathbb{Z}) & \xrightarrow{t^*} & H^1(\gamma; \mathbb{Z}) & \xrightarrow{0} & \cdots \end{array}$$

in which the vertical arrows are induced by inclusion's.

The group $H^0(\partial M_K; \mathbb{Z})$ is generated by $[[pt]]$, the group $H^0(\gamma; \mathbb{Z})$ by $i^*([pt])$ and $H^1(\partial M_K; \mathbb{Z})$ is based with (m^*, γ^*) ; here m denotes the meridian of K and $\gamma = \partial F$ its longitude. We know that $j^*(m^*) = m^*$.

Using the previous commutative diagram, to prove $\delta(i^*([pt])) = m^*$ it is enough to show $\Delta(i^*([pt])) = m^*$. This last equality is obtained by a careful examination of the short exact sequence

$$0 \longrightarrow H^0(\gamma; \mathbb{Z}) \xrightarrow{\Delta} H^1(\partial M_K; \mathbb{Z}) \xrightarrow{\iota^*} H^1(\gamma; \mathbb{Z}) \longrightarrow 0$$

in which $H^0(\gamma; \mathbb{Z})$ is generated by $i^*([pt])$, $H^1(\gamma; \mathbb{Z}) = \text{Hom}(H_1(\gamma; \mathbb{Z}), \mathbb{Z})$ by $\gamma^* : \gamma \mapsto 1$ and $H^1(\partial M_K; \mathbb{Z}) = \text{Hom}(H_1(\partial M_K; \mathbb{Z}), \mathbb{Z})$ is based with (m^*, γ^*) , where $\iota^*(\gamma^*) = \gamma^*$. \square

5.4. End of the proof. It remains to bring together all the computations we have done before.

Proof of the Main Theorem. Recall that

$$\mathbb{T}_\gamma^K(\rho) = \tau_0 \cdot \text{Tor}(X_K; \mathfrak{g}_\rho, (h_\rho^{(1)}(\gamma), h_\rho^{(2)})).$$

Equations (12) and (13) imply

$$\text{Tor}(X_K; \mathfrak{g}_\rho, (h_\rho^{(1)}(\gamma), h_\rho^{(2)})) = - \prod_{i=1}^{6g-4} \frac{1}{1 - \lambda_i}.$$

Next equations (15) and (16) imply $\tau_0 = \varepsilon_0$, and this completes the proof. \square

6. EXAMPLES

This last section is devoted to concrete computations. The Main Theorem can be applied to provide explicit formulae. We focus our attention on the $\text{SU}(2)$ -representation space of the group of torus knots and next on the $\text{SL}_2(\mathbb{C})$ -representation space of the group of the figure eight knot.

6.1. The trefoil knot. It is well-known that the (right hand) trefoil knot K is a fibered knot of genus 1. Denote by a, b the generators of the free group $\pi_1(F)$; here F is the fibre of K and $\gamma = \partial F$ is its longitude. The group G_K of K admits as a fibered knot the presentation

$$G_K = \langle a, b, t \mid t^{-1}at = ab^{-1}a^{-1}, t^{-1}bt = ab \rangle,$$

in which t represents the meridian of K . Remember that $\widehat{R}(G_K; \text{SU}(2))$ is the set of the $\text{SO}(3)$ -conjugacy classes of the irreducible representations $\rho : G_K \rightarrow \text{SU}(2)$ such that $\text{Tr}(\rho(t)) = \sqrt{3} \cos(\theta)$, for $\theta \in (0, \pi)$, see [Kla91, Theorem 1]. Observe that each irreducible representations of G_K into $\text{SU}(2)$ is γ -regular (see Example 1).

Recall that $H_1(F; \mathbb{Z}) = [\pi_1(F), \pi_1(F)]$; $H^1(F; \mathbb{Z}) = \text{Hom}(H_1(F; \mathbb{Z}), \mathbb{Z})$ is endowed with the basis (a^*, b^*) , where $a^* : a \mapsto 1$, $b^* : b \mapsto 1$. The monodromy $\phi : F \rightarrow F$ induces an endomorphism $\phi^* : H^1(F; \mathbb{Z}) \rightarrow H^1(F; \mathbb{Z})$ such that the matrix of ϕ^* in the basis (a^*, b^*) is $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. As a consequence, $\varepsilon_0 = +1$.

Let ρ be an irreducible $\text{SU}(2)$ -representation of G_K . Set $x_1 = I_a$, $x_2 = I_b$ and $x_3 = I_{ab}$. We know that the moduli space $\widehat{R}(F; \text{SU}(2))$ is parametrized by x_1, x_2, x_3 .

Observe that, with respect to the coordinates (x_1, x_2, x_3) , the action of ϕ_φ^* is given by $P = (x_2, x_3, x_1) \in \mathbb{Z}[x_1, x_2, x_3]^3$. Thus the tangent map at ρ to $R(\phi)$ is

$$\left(\frac{\partial P_i}{\partial x_j}(\varphi) \right)_{i,j} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which admits 1 , $e^{2i\pi/3}$ and $e^{-2i\pi/3}$ as eigenvalues. As a consequence

$$\mathbb{T}_\gamma^K(\rho) = -\frac{1}{3}.$$

6.2. The torus knots. More generally, let $p, q \in \mathbb{N}^*$ be coprime and denote by $K_{p,q}$ the (right hand) torus knot of type (p, q) . Its group admits the presentation $G_{p,q} = \langle x, y \mid x^p = y^q \rangle$. It is known that $K_{p,q}$ is fibered (see [BZ96]).

Let $\rho : G_{p,q} \rightarrow \mathrm{SU}(2)$ be the irreducible representation such that $\mathrm{Tr}(\rho(x)) = 2 \cos\left(\frac{\pi a}{p}\right)$, $\mathrm{Tr}(\rho(y)) = 2 \cos\left(\frac{\pi b}{q}\right)$, where $0 < a < p$, $0 < b < q$ and $a \equiv b \pmod{2}$. It is well-known that $\widehat{R}(G_{p,q}; \mathrm{SU}(2))$ is the conjugacy classes of all the previous representations (see [Kla91, Theorem 1]). In a similar way as for the trefoil knot, ρ is γ -regular and we obtain

$$\mathbb{T}_\gamma^{K_{p,q}}(\rho) = -\frac{16}{p^2 q^2} \sin^2\left(\frac{\pi a}{p}\right) \sin^2\left(\frac{\pi b}{q}\right).$$

6.3. The figure eight knot. We turn now to the case of an hyperbolic knot: the figure eight knot denoted by K . We study the moduli space of the $\mathrm{SL}_2(\mathbb{C})$ -representations of the group of K which admits as a fibered knot the presentation:

$$G_K = \langle a, b, t \mid t^{-1}at = ab, t^{-1}bt = bab \rangle.$$

The monodromy $\phi : F \rightarrow F$ induces an endomorphism $\phi^* : H^1(F; \mathbb{Z}) \rightarrow H^1(F; \mathbb{Z})$ such that the matrix of ϕ^* in the basis (a^*, b^*) is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. As a consequence, $\varepsilon_0 = -1$. With the same notation as in Subsection 6.1, the action of ϕ_φ^* is given by $P = (x_3, x_2x_3 - x_1, x_2x_3^2 - x_1x_3 - x_2) \in \mathbb{Z}[x_1, x_2, x_3]^3$ and we have (cf. [Por97, p. 113])

$$\mathbb{T}_\gamma^K(\rho) = \frac{1}{3 - 2(I_a(\rho) + I_b(\rho))}.$$

The well-known identity

$$\mathrm{Tr}(ABA^{-1}B^{-1}) = -2 - \mathrm{Tr}(A)\mathrm{Tr}(B)\mathrm{Tr}(AB) + (\mathrm{Tr}(A))^2 + (\mathrm{Tr}(B))^2 + (\mathrm{Tr}(AB))^2$$

implies $I_\gamma = x_1^2 + x_2^2 - x_1 - x_2 - 2$. As a consequence

$$\mathbb{T}_\gamma^K(\rho)^2 = \frac{1}{17 + 4I_\gamma(\rho)}.$$

The hyperbolic structure of the exterior of K determines an unique (up to complex-conjugation) *holonomy representation* into $\mathrm{PSL}_2(\mathbb{C})$ which lifts to two representations ϱ_\pm into $\mathrm{SL}_2(\mathbb{C})$ satisfying

$$\varrho_\pm(m) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \text{ and } \varrho_\pm(l) = \begin{pmatrix} 1 & \pm 2i\sqrt{3} \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\mathbb{T}_\gamma^K(\varrho_\pm) = \frac{1}{5}.$$

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