

EMBEDDINGS OF GENERALIZED DANIELEWSKI SURFACES IN AFFINE SPACES

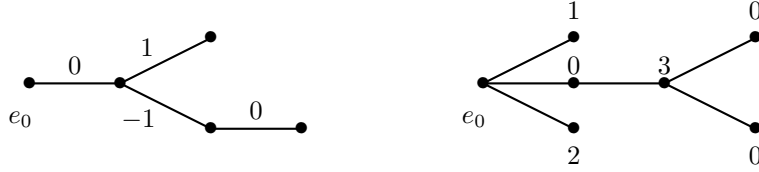
A. DUBOULOZ

Institut Fourier, Laboratoire de Mathématiques
UMR5582 (UJF-CNRS)
BP 74, 38402 ST MARTIN D'HERES CEDEX
FRANCE
adrien.dubouloz@ujf-grenoble.fr

ABSTRACT. We construct explicit embeddings of generalized Danielewski surfaces [5] in affine spaces. The equations defining these embeddings are obtained from the 2×2 minors of a matrix attached to a labelled-rooted tree \mathfrak{g} . The corresponding surfaces $V_{\mathfrak{g}}$ come equipped with canonical \mathbb{C}_+ -actions which appear as the restrictions of certain \mathbb{C}_+ -actions on the ambient affine space. We characterize those surfaces $V_{\mathfrak{g}}$ with a trivial Makar-Limanov invariant, and we complete the study of $\log \mathbb{Q}$ -homology planes with a trivial Makar-Limanov invariant initiated by Miyanishi and Masuda [11].

INTRODUCTION

In [5], the author introduced the notion of generalized Danielewski surface (*GDS* for short). This is a normal affine surface V which admits a faithfully flat morphism $q : V \rightarrow Z = \mathbb{A}_{\mathbb{C}}^1$ with reduced fibers and generic fiber $\mathbb{A}_{K(Z)}^1$, such that all but possibly one closed fiber $q^{-1}(z_0)$ are integral. For instance, a nonsingular ordinary Danielewski surface $V_{P,n} \subset \mathbb{C}[x, y, z]$ with equation $x^n z - P(y) = 0$, where P is a nonconstant polynomial with simple roots, is a *GDS* for the projection $pr_x : V_{P,n} \rightarrow Z = \text{Spec}(\mathbb{C}[x])$. Generalized Danielewski surfaces appear naturally as locally trivial fiber bundles $\rho : V \rightarrow X$ over an affine line with a multiple origin (see e.g. [6]). In [5], the author established that every such bundle is obtained from an invertible sheaf \mathcal{L} on X and a Čech 1-cocycle g with values in the dual \mathcal{L}^\vee of \mathcal{L} . In turn, this invertible sheaf \mathcal{L} and the cocycle g are encoded in a combinatorial data consisting of a rooted tree with weighted edges, which we call a *weighted rooted tree* (see [5, Theorem 4.1] and 3.3 below). Here we use rooted trees in a different way to construct embeddings of *GDS*'s into affine spaces. More precisely, starting with certain rooted trees with weights on their nodes, which we call *labelled rooted trees*, we construct explicit ideals of certain polynomial rings. In turn, these ideals define affine surfaces which are naturally *GDS*'s over the affine line $\mathbb{A}_{\mathbb{C}}^1$.

A weighted tree γ rooted in e_0 .A labelled tree \mathfrak{g} rooted in e_0 .

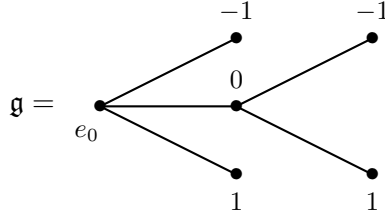
The paper is divided as follows. In section 1 we recall basic facts on rooted trees and we give a procedure to construct an affine scheme $V_{\mathfrak{g}}$ from the data consisting of a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$. These schemes $V_{\mathfrak{g}} = \text{Spec}(B_{\mathfrak{g}})$ are naturally schemes over an affine line $Z = \text{Spec}(\mathbb{C}[h])$, and they come embedded into affine spaces $\mathbb{A}_Z^{d(\Gamma)+1} = \text{Spec}(\mathbb{C}[h][\Gamma])$ depending on the tree Γ . In section 2, we prove the following result (theorem 2.1)

Theorem 0.1. *For every fine-labelled rooted tree \mathfrak{g} (see definition 1.3 below), the scheme $V_{\mathfrak{g}} = \text{Spec}(B_{\mathfrak{g}})$ is a GDS over $Z = \text{Spec}(\mathbb{C}[h])$.*

For instance, the Bandman and Makar-Limanov surface [1] $V \subset \mathbb{C}[x, y, z, u]$ with equation

$$xz = y(y^2 - 1), \quad yu = z(z^2 - 1), \quad xu = (y^2 - 1)(z^2 - 1)$$

is a GDS over $Z = \text{Spec}(\mathbb{C}[x])$ for the morphism $pr_x : V \rightarrow Z$. This data corresponds to the fine-labelled rooted tree



In section 2, we recall [5] how a weighted rooted tree γ defines a GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$. Then, starting with such a weighted rooted tree $\gamma = (\Gamma, w)$, we construct a fine-labelled rooted tree $\mathfrak{g}_w = (\Gamma, lb_w)$ with the same underlying tree Γ and a closed embedding $i_{\gamma} : W_{\gamma} \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1}$ inducing an isomorphism $W_{\gamma} \simeq V_{\mathfrak{g}_w}$. This leads to the following result (theorem 3.1)

Theorem 0.2. *Every GDS $q : V \rightarrow Z$ is Z -isomorphic to a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ for an appropriate fine-labelled rooted tree \mathfrak{g} .*

In section 3 we study $\mathbb{G}_{a,Z}$ -actions on the GDS's $q_{\mathfrak{g}} : V_{\mathfrak{g}} = \text{Spec}(B_{\mathfrak{g}}) \rightarrow Z$. We construct explicit locally nilpotent derivations of the algebras $B_{\mathfrak{g}}$. In [8, 9], Makar-Limanov proved that every \mathbb{C}_+ -action on an ordinary Danielewski surface $V_{P,n} \subset \text{Spec}(\mathbb{C}[x, y, z])$ is the restriction of a \mathbb{C}_+ -action on $\mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, y, z])$. We prove the following general result (theorem 4.9).

Theorem 0.3. *Every $\mathbb{G}_{a,Z}$ -action on a GDS $q : V_{\mathfrak{g}} \rightarrow Z$ defined by a fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$ is induced by a $\mathbb{G}_{a,Z}$ -action on the ambient space $\mathbb{A}_Z^{d(\Gamma)+1} = \text{Spec}(\mathbb{C}[h][\Gamma])$.*

In section 4, we consider GDS 's $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ with a trivial Makar-Limanov invariant. As a consequence of [5, Theorem 7.2], we obtain the following criterion (theorem 5.1).

Theorem 0.4. *A GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ has a trivial Makar-Limanov invariant if and only if \mathfrak{g} is a fine-labelled comb (see 5.1 below).*

This leads to the following description (see 5.2 below). Let $P_1, \dots, P_n \in \mathbb{C}[T]$ be a collection of nonconstant polynomials with simple roots, one of these root, say $\lambda_{i,1}$, $1 \leq i \leq n$, being distinguished. We let

$$P_i(T) = \prod_{j=1}^{r_i} (T - \lambda_{i,j}) = (T - \lambda_{i,1}) \tilde{P}_i(T) \in \mathbb{C}[T], \quad 1 \leq i \leq n.$$

Then every GDS with a trivial Makar-Limanov is isomorphic, for an appropriate choice of the polynomials P_1, \dots, P_n , to a surface

$$V_{P_1, \dots, P_n} \subset \mathbb{A}_{\mathbb{C}}^{n+2} = \text{Spec}(\mathbb{C}[x][y_1, \dots, y_{n+1}])$$

with equations

$$\begin{cases} xy_{i+1} & = \left(\prod_{k=1}^{i-1} \tilde{P}_k(y_k) \right) P_i(y_i) & 1 \leq i \leq n \\ (y_{j-1} - \lambda_{j-1,1}) y_{i+1} & = y_j \left(\prod_{k=j}^{i-1} \tilde{P}_k(y_k) \right) P_i(y_i) & 2 \leq j \leq i \leq n \end{cases},$$

where, by convention, $\prod_{k=j}^{i-1} \tilde{P}_k(y_k) = 1$ if $i = j$. From this description, we recover the following characterization of nonsingular ordinary Danielewski surfaces, due to Bandman and Makar-Limanov [1] (5.4).

Theorem 0.5. *For a GDS $q : V \rightarrow Z$ with a trivial Makar-Limanov invariant, the following are equivalent.*

- 1) V admits a free $\mathbb{G}_{a,Z}$ -action.
- 2) The canonical sheaf ω_V of V is trivial.
- 3) V is isomorphic to an ordinary Danielewski surface $V_{P,1} \subset \mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ with the equation $xz - P(y) = 0$ for a certain nonconstant polynomial P with simple roots.

Following a remark of the author, Miyanishi and Masuda [11] proved that every nonsingular \mathbb{Q} -homology plane with a trivial Makar-Limanov invariant is a cyclic quotient of an ordinary Danielewski surface $V \subset \text{Spec}(\mathbb{C}[x, t, z])$ with equation $xz = t^m - 1$ by a \mathbb{Z}_m -action $(x, t, z) \mapsto (\varepsilon x, \varepsilon^q y, \varepsilon^{-1} z)$, where ε is a primitive m -th root of unity and $\gcd(q, m) = 1$. More generally, the author proved in [5, theorem 7.7] that every normal affine surface S with a trivial Makar-Limanov invariant is a cyclic quotient of a GDS V . In case that S is a log \mathbb{Q} -homology plane, this GDS can be determined explicitly. This leads to the following result (theorem 5.6), see also [2].

Theorem 0.6. *Every log \mathbb{Q} -homology plane $S \not\cong \mathbb{A}_{\mathbb{C}}^2$ with a trivial Makar-Limanov invariant is isomorphic to a quotient of an ordinary Danielewski surface $V \subset \text{Spec}(\mathbb{C}[x, t, z])$ with equation $xz = t^n - 1$ by a \mathbb{Z}_m -action $(x, t, z) \mapsto (\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z)$, where ε is a primitive m -th root of unity, n divides m and $\gcd(q, m/n) = 1$.*

1. AFFINE SCHEMES DEFINED FROM LABELLED ROOTED TREES

In this section we explain how to construct affine schemes from labelled rooted trees.

Basics on rooted trees.

Let $G = (N, \leq)$ be a nonempty, finite, partially ordered set (a *poset*, for short). The elements of N are sometimes called the *nodes* of G . A totally ordered subset $N' \subset N$ is called a *chain of length* $l(N') = \text{Card}(N') - 1$. A chain which is maximal for the inclusion is called a *maximal chain*. For every $e \in N$, we let

$$(\uparrow e)_G = \{e' \in N, e' \geq e\} \quad \text{and} \quad (\downarrow e)_G = \{e' \in N, e' \leq e\}$$

For every $e < e'$, $e, e' \in N$, we let $[e', e]_G := (\uparrow e')_G \cap (\downarrow e)_G$. A pair $e < e'$ such that $[e', e]_G = \{e' < e\}$ is called an edge of G , and we denote the set of all edges in G by $E(G)$.

Definition 1.1. A *rooted tree* Γ is poset $\Gamma = (N(\Gamma), \leq)$ with a unique minimal element e_0 called the *root*, and such that $(\downarrow e)_\Gamma$ is a chain for every $e \in N(\Gamma)$.

1.2. The maximal elements of a rooted tree $\Gamma = (N(\Gamma), \leq)$ are called the *leaves* of Γ . We denote the set of those elements by $\text{Leaves}(\Gamma)$. An element of $N(\Gamma) \setminus \text{Leaves}(\Gamma)$ is called a *parent*, and we denote the set of those nodes by $\mathbf{P}(\Gamma)$. Given $e \in N(\Gamma) \setminus \{e_0\}$, an element of the chain $\text{Anc}(e) = (\downarrow e) \setminus \{e\}$ is called an *ancestor* of e . The *parent* of e is the maximal element $\text{Par}(e)$ of $\text{Anc}(e)$. More generally, the n -th *ancestor* of e is defined recursively by $\text{Par}^n(e) = \text{Par}(\text{Par}^{n-1}(e)) \in \text{Anc}(e)$. Given two different nodes $g, g' \in N(\Gamma)$, the *first common ancestor* of g and g' is the maximal element $\text{Anc}(g, g')$ of the chain $\text{Anc}(g) \cap \text{Anc}(g')$. Given $e \in \mathbf{P}(\Gamma)$, the minimal elements of $(\uparrow e)_\Gamma \setminus \{e\}$ are called the *children* of e , and we denote the set of those nodes by $\text{Child}_\Gamma(e)$. The *degree* $\text{deg}_\Gamma(e)$ of a node e is the number of its children. Given $e \in N(\Gamma)$, the *maximal rooted subtree of Γ rooted in e* is the tree $\Gamma(e) = ((\uparrow e)_\Gamma, \leq)$. A node $e \in N(\Gamma)$ such that $l((\downarrow e)_\Gamma) = n$ is said to be at *level n* , and we denote the set of those nodes by $N_n(\Gamma)$. The maximal chains of a rooted tree Γ are the chains

$$(1.1) \quad (\downarrow e)_\Gamma = \{e_0 < e_1 < \cdots < e_{n_f-1} < e_{n_f} = e\}, \quad e \in \text{Leaves}(\Gamma).$$

The *height* $h(\Gamma)$ of Γ is the maximum of the lengths $l((\downarrow e)_\Gamma) = n_e$, $e \in \text{Leaves}(\Gamma)$.

In 1.3 and 1.4 below, we introduce two different notions of weights on a rooted tree Γ .

Definition 1.3. A *labelling* (with values in \mathbb{C}) on a tree Γ rooted in e_0 is a function

$$lb : N(\Gamma) \setminus \{e_0\} \rightarrow \mathbb{C}$$

A labelling lb is called *fine* if $lb(f) \neq lb(f')$ whenever f and f' share the same parent $e \in \mathbf{P}(\Gamma)$. A rooted tree Γ equipped with a labelling lb will be referred to as a *labelled rooted tree* and will be denoted by $\mathfrak{g} = (\Gamma, lb)$. A rooted tree Γ equipped with a fine labelling lb will be referred to as a *fine-labelled rooted tree*. For every $e \in \mathbf{P}(\Gamma)$, we let $\mathfrak{g}(e) = (\Gamma(e), lb)$ be the maximal subtree of Γ rooted in e , equipped with the restriction of the labelling lb to $N(\Gamma(e)) \setminus \{e\}$.

Definition 1.4. A *weight function* (with values in \mathbb{C}) on a rooted tree Γ is a function

$$w : E(\Gamma) \rightarrow \mathbb{C},$$

i.e. a function which assign a complex number $w([e, f]_\Gamma)$ to every edge $[e, f]_\Gamma$ of Γ , such that $w([e, f]) \neq w([e, f'])$ whenever f and f' share the same parent $e \in \mathbf{P}(\Gamma)$. A rooted tree Γ

equipped with a weight function w will be referred to as a *weighted rooted tree* and will be denoted by $\gamma = (\Gamma, w)$. For every $e \in \mathbf{P}(\Gamma)$, we let $\gamma(e) = (\Gamma(e), w)$ be the maximal subtree of Γ rooted in e , equipped with the restriction of the weight function w to $E(\Gamma(e))$.

Affine scheme associated to a labelled rooted tree.

In this subsection we give a general procedure for constructing an affine scheme $V_{\mathfrak{g}}$ from the data consisting of a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$. To fix the notation, we let $A = \mathbb{C}[h]$ be a polynomial ring in one variable, $z_0 = (h) \in Z = \text{Spec}(A)$ and $Z_* = Z \setminus \{z_0\} \simeq \text{Spec}(A_h)$. We let $pr_Z : \mathbb{A}_Z^1 = \text{Spec}(A[X_0]) \rightarrow Z$ be the trivial line bundle over Z .

Definition 1.5. Given a rooted tree Γ , we let $\mathbf{S}(M_\Gamma)$ be the symmetric algebra of the free A -module M_Γ with basis $(X_e)_{e \in \mathbf{P}(\Gamma)}$. This is a polynomial ring over A in

$$(1.2) \quad d(\Gamma) := \sum_{i=1}^{h(\Gamma)} \text{Card}(N_{i-1}(\Gamma) \setminus \text{Leaves}(\Gamma))$$

variables. Then we let $A[\Gamma] = A[X_0] \otimes_A \mathbf{S}(M_\Gamma)$.

Notation 1.6. If $e' \in \mathbf{P}(\Gamma)$ is the parent of a given $e \in \mathbf{P}(\Gamma) \setminus \{e_0\}$ then we will sometimes denote $X_{e'} \in A[\Gamma]$ as $X_{\text{Par}(e)}$. We also extend this relationship between the variables X_e , $e \in \mathbf{P}(\Gamma)$, by letting $X_{\text{Par}(e_0)} = X_0 \in A[\Gamma]$.

For every node $e \in \mathbf{P}(\Gamma)$ of a given labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, we introduce, in 1.7-1.9 below, three polynomials $S_e(\mathfrak{g}), R_e(\mathfrak{g}), Q_e(\mathfrak{g}) \in A[\Gamma]$, defined recursively through the labelling lb .

Definition 1.7. For every $e \in \mathbf{P}(\Gamma)$ and every subset $J \subset \text{Child}(e)$ we let

$$S_e^J(\mathfrak{g}) = \prod_{e'' \in (\text{Child}(e) \setminus J)} (X_{\text{Par}(e)} - lb(e'')) \in \mathbb{C}[X_{\text{Par}(e)}] \subset A[\Gamma].$$

We call $S_e(\mathfrak{g}) := S_e^\emptyset(\mathfrak{g})$ the *sibling polynomial* of e . This is a polynomial of degree $\deg_\Gamma(e)$. The roots of $S_e(\mathfrak{g})$ are simple if and only if $lb : N(\Gamma) \setminus \{e_0\} \rightarrow \mathbb{C}$ restricts to a fine labelling on the subtree $(\{e\} \cup \text{Child}(e), \leq_\Gamma)$ of Γ .

Definition 1.8. The *root polynomial* of $e \in N(\Gamma)$ is the polynomial $R_e(\mathfrak{g})$ defined recursively by $R_{e_0}(\mathfrak{g}) = 1$ and

$$R_e(\mathfrak{g}) = S_{\text{Par}(e)}^{\{e\}}(\mathfrak{g}) R_{\text{Par}(e)}(\mathfrak{g}) \in \mathbb{C}[X_0, (X_{e'})_{e' \in \text{Anc}(\text{Par}(e))}] \subset A[\Gamma].$$

Definition 1.9. For every $e \in \mathbf{P}(\Gamma)$, we let

$$Q_e(\mathfrak{g}) = S_e(\mathfrak{g}) R_e(\mathfrak{g}) \in \mathbb{C}[X_0, (X_{e'})_{e' \in \text{Anc}(e)}] \subset A[\Gamma].$$

Definition 1.10. Given a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, we let

$$M(\mathfrak{g}) = \left(M_0, (M_e)_{e \in \mathbf{P}(\Gamma)} \right) \in \text{Mat}_{2, (\text{Card}(\mathbf{P}(\Gamma))+1)}(A[\Gamma])$$

be the matrix with the columns

$$M_0 = \begin{pmatrix} h \\ 1 \end{pmatrix} \in \text{Mat}_{2,1}(A[\Gamma]) \quad \text{and} \quad M_e = \begin{pmatrix} Q_e(\mathfrak{g}) \\ X_e \end{pmatrix} \in \text{Mat}_{2,1}(A[\Gamma]), \quad e \in \mathbf{P}(\Gamma).$$

Definition 1.11. Given a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, we let $I_{\mathfrak{g}} \subset A[\Gamma]$ be the ideal generated by the polynomials

$$(1.3) \quad \begin{cases} \Delta_{0,e}(\mathfrak{g}) = \det(M_0, M_e) & e \in \mathbf{P}(\Gamma) \\ \Delta_{e',e}(\mathfrak{g}) = R_{e''}^{-1}(\mathfrak{g}) \det(M_{e'}, M_e) & \begin{array}{l} (e, e') \in \mathbf{P}(\Gamma) \times \text{Anc}(e) \\ e'' = \text{Child}(e') \cap (\downarrow e) \end{array} \end{cases} .$$

The *affine scheme* $V_{\mathfrak{g}}$ associated to \mathfrak{g} is the closed subscheme

$$V_{\mathfrak{g}} = \text{Spec}(B_{\mathfrak{g}}) \subset \text{Spec}(A[\Gamma]), \text{ with } B_{\mathfrak{g}} = A[\Gamma]/I_{\mathfrak{g}}$$

Remark 1.12. For every pair $(e, e') \in \mathbf{P}(\Gamma) \times \text{Anc}(e)$,

$$(1.4) \quad \Delta_{e',e}(\mathfrak{g}) = (X_{\text{Par}(e')} - lb(e'')) X_e - X_{e'} Q_{e',e}(\mathfrak{g}),$$

where $e'' = \text{Child}(e') \cap (\downarrow e)$ and $Q_{e',e}(\mathfrak{g}) = R_{e''}^{-1}(\mathfrak{g}) Q_e(\mathfrak{g}) \in \mathbb{C} \left[(X_g)_{g \in [e', \text{Par}(e)]_{\Gamma}} \right]$.

Remark 1.13. For every pair $(e, e') \in \mathbf{P}(\Gamma) \times \text{Anc}(e)$ as above, the polynomials $\Delta_{0,e}(\mathfrak{g})$, $\Delta_{0,e'}(\mathfrak{g})$ and $\Delta_{e',e}(\mathfrak{g})$ satisfy the syzygy relation

$$(1.5) \quad h \Delta_{e',e}(\mathfrak{g}) = (X_{\text{Par}(e')} - lb(e'')) \Delta_{0,e}(\mathfrak{g}) - Q_{e',e}(\mathfrak{g}) \Delta_{0,e'}(\mathfrak{g}).$$

Remark 1.14. Given a pair $(g_1, g_2) \in \mathbf{P}(\Gamma) \times \mathbf{P}(\Gamma)$ such that $g_i \notin (\downarrow g_j)$, $i, j \in \{1, 2\}$, we let $e \in \mathbf{P}(\Gamma)$ be the first common ancestor of g_1 and g_2 by e , and we let $e_i = \text{Child}(e) \cap (\downarrow g_i)$, $i = 1, 2$. Then

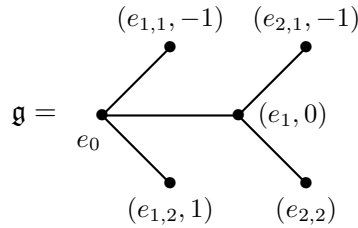
$$(1.6) \quad \Delta_{g_1, g_2} := \left(R_e(\mathfrak{g}) S_e^{\{e_1, e_2\}}(\mathfrak{g}) \right)^{-1} \det(M_{g_1}, M_{g_2})$$

$$(1.7) \quad = Q_{e, g_1}(\mathfrak{g}) \Delta_{e, g_2}(\mathfrak{g}) - Q_{e, g_2}(\mathfrak{g}) \Delta_{e, g_1}(\mathfrak{g})$$

This means that $I_{\mathfrak{g}}$ coincides with the ideal generated by the *simplified* 2×2 minors of $M(\mathfrak{g})$, i.e. the 2×2 minors of $M(\mathfrak{g})$ simplified by a common factor according to the rules (1.3) and (1.6).

Notation 1.15. Hereafter, we will usually write P instead of $P(\mathfrak{g})$, where $P(\mathfrak{g})$ stands for any of the polynomials $S_e(\mathfrak{g})$, $R_e(\mathfrak{g})$, $Q_e(\mathfrak{g})$, $\Delta_{0,e}(\mathfrak{g})$ and $\Delta_{e',e}(\mathfrak{g})$ attached to a labelled rooted tree \mathfrak{g} .

Example 1.16. Consider the following fine-labelled rooted tree



The corresponding algebra is $A[\Gamma] = \mathbb{C}[h][X_0, X_{e_0}, X_{e_1}]$, and the associated matrix is

$$M(\mathfrak{g}) = \begin{pmatrix} h & X_0(X_0^2 - 1) & (X_0^2 - 1)(X_{e_0}^2 - 1) \\ 1 & X_{e_0} & X_{e_1} \end{pmatrix}.$$

Therefore $V_{\mathfrak{g}}$ is the Bandman and Makar-Limanov surface [1] with the equations

$$hX_{e_0} = X_0(X_0^2 - 1), \quad X_0X_{e_1} = X_{e_0}(X_{e_0}^2 - 1), \quad hX_{e_1} = (X_0^2 - 1)(X_{e_0}^2 - 1).$$

This is a *GDS* over $Z = \text{Spec}(A)$ for the morphism $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ induced by the inclusion $A \hookrightarrow B_{\mathfrak{g}}$. This \mathbb{A}^1 -fibration $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ restricts to the trivial line bundle over $Z_* = Z \setminus \{z_0\}$, whereas $q_{\mathfrak{g}}^{-1}(z_0)$ is the disjoint union of 4 curves

$$C_{e_{1,1}}, C_{e_{1,2}} \simeq \text{Spec}(\mathbb{C}[X_{e_0}]) \quad \text{and} \quad C_{e_{2,1}}, C_{e_{2,2}} \simeq \text{Spec}(\mathbb{C}[X_{e_1}]).$$

2. *GDS*'S DEFINED BY A FINE-LABELLED ROOTED TREES

This section is devoted to the proof of the following result (see also theorem 3.1 below).

Theorem 2.1. *For every fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, the scheme $V_{\mathfrak{g}} = \text{Spec}(B_{\mathfrak{g}})$ is a *GDS* over $Z = \text{Spec}(A)$ for the morphism $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ induced by the inclusion $A \hookrightarrow B_{\mathfrak{g}}$.*

Example 2.2. Consider the ordinary Danielewski surface $V = V_{P,1} \subset \text{Spec}(\mathbb{C}[x, y, z])$ with equation $xz = P(y)$, where P is a polynomial with $r \geq 1$ simple roots y_1, \dots, y_r . It is a *GDS* via $pr_x : V \rightarrow Z = \text{Spec}(\mathbb{C}[x])$. Since $xz - P(y) \in \mathbb{C}[x, y, z]$ is an irreducible polynomial, V is an irreducible. By the Jacobian criterion, it is also nonsingular. Thus pr_x is a flat morphism. It restricts to a trivial line bundle over $Z_* = \text{Spec}(\mathbb{C}[x, x^{-1}])$. Since the roots of P are simple, the polynomials $y - y_i \in \mathbb{C}[y]$, $1 \leq i \leq r$ are relatively prime and so,

$$pr_x^{-1}(0) = \text{Spec}(\mathbb{C}[x, y, z] / (x, xz - P(y))) \simeq \bigsqcup_{i=1}^r \text{Spec}(\mathbb{C}[y, z] / (y - y_i))$$

is the disjoint union of curves $C_i \simeq \text{Spec}(\mathbb{C}[z])$, $1 \leq i \leq r$. In general, a scheme $V_{\mathfrak{g}}$ associated to a fine-labelled rooted tree \mathfrak{g} is given by more than one equation. Therefore, it is hard to check directly if it is irreducible or not. In particular, a necessary condition for $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ to be flat is that no irreducible or embedded component of $V_{\mathfrak{g}}$ is supported on the fiber $q_{\mathfrak{g}}^{-1}(z_0)$. Let us give a direct proof of the flatness of $pr_x : V \rightarrow Z = \text{Spec}(\mathbb{C}[x])$ which illustrates the constructions used below. Since pr_x restricts to a trivial line bundle over Z_* , it suffices to show that $\mathcal{O}_{V,v}$ is a torsion free \mathcal{O}_{Z,z_0} -module for every point $v \in V$ such that $pr_x(v) = z_0 = Z \setminus Z_*$. For every $i \in I$, we let

$$R_i(y) = \prod_{j \neq i} (y - y_j) = (y - y_i)^{-1} P(y) \in \mathbb{C}[x, y, z].$$

Since the roots of $P(y)$ are simple, the principal open subsets

$$\begin{aligned} U_i = V \cap D(R_i) &\simeq \text{Spec}(\mathbb{C}[x, y, z][t] / (xz - P(y), R_i(y)t - 1)), \quad 1 \leq i \leq r. \\ &\simeq \text{Spec}(\mathbb{C}[x, z, t] / (tR_i(y_i + xzt) - 1)) \end{aligned}$$

cover V . Since x does not divide $tR_i(y_i + xzt) - 1$, U_i is flat over Z . Thus V is covered by open subsets U_i , $1 \leq i \leq r$, flat over Z , whence is flat over Z too. We also deduce directly from this description that no irreducible component of V is supported on the fiber $pr_x^{-1}(0)$. Indeed, the subscheme $V(R_i) \cap V$ is the union of the closures of the curves

$$S_j = \{y = y_j, z = 0, x \neq 0\} \cap V \subset V \setminus pr_x^{-1}(0), \quad j \neq i,$$

and of the irreducible components C_j , $j \neq i$, of the fiber $pr_x^{-1}(0)$. This shows that for every $1 \leq i \leq r$ the generic point of $(C_i)_{red}$ is contained in the closure in V of the open subset $V \setminus pr_x^{-1}(0)$.

The proof divides as follows. In subsection 1 we show that $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ is a flat morphism. Then in subsection 2 we describe the fiber $q_{\mathfrak{g}}^{-1}(z_0)$.

1. Flatness of $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$.

In this subsection we prove the following result.

Proposition 2.3. *If \mathfrak{g} is a fine-labelled rooted tree then $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ is a flat morphism restricting over Z_* to a trivial line bundle.*

2.4. If $N(\Gamma) = \{e_0\}$ then $B_{\mathfrak{g}} = A[X_0]$ and $q_{\mathfrak{g}} = pr_Z$ is a flat morphism. Otherwise, (1.5) implies that the image of $I_{\mathfrak{g}}$ in $A_h[\Gamma] = A[\Gamma] \otimes_A A_h$ is generated by the polynomials

$$(2.1) \quad \delta_{0,e} = h^{-1} \Delta_{0,e} = X_e - h^{-1} Q_e, \quad e \in \mathbf{P}(\Gamma).$$

Since Q_e only involves the variables X_0 and $X_{e'}$, $e' \in \text{Anc}(e)$, we recursively arrive at an A_h -algebras isomorphism $A_h[\Gamma]/I_0 \simeq A_h[X_0]$. Thus $q_{\mathfrak{g}}$ restricts to a trivial line bundle over $Z_* \simeq \text{Spec}(A_h)$ and so, $\mathcal{O}_{V_{\mathfrak{g}},x}$ is a flat $\mathcal{O}_{Z,z}$ -module for every $x \in V_{\mathfrak{g}}$ such that $q_{\mathfrak{g}}(x) = z \in Z_*$. Given $x \in V_{\mathfrak{g}}$ such that $q_{\mathfrak{g}}(x) = z_0$, $\mathcal{O}_{V_{\mathfrak{g}},x}$ is a flat \mathcal{O}_{Z,z_0} -module provided that h is not a zero divisor in $\mathcal{O}_{V_{\mathfrak{g}},x}$. So it suffices to show that h is not a zero divisor in $B_{\mathfrak{g}}$. In turn, it suffices to find $g_1, \dots, g_r \in B_{\mathfrak{g}}$ which generate the unit ideal, such that h is not a zero divisor in $(B_{\mathfrak{g}})_{g_i}$ for every $1 \leq i \leq r$. The following lemma says that $V_{\mathfrak{g}}$ admits a natural covering by principal open subsets.

Lemma 2.5. *The scheme $V_{\mathfrak{g}}$ is covered by the principal open subsets*

$$U_e = D(R_e) \cap V_{\mathfrak{g}} \simeq \text{Spec}(A[\Gamma][T]/(I_{\mathfrak{g}}, R_e T - 1)), \quad e \in \text{Leaves}(\Gamma).$$

Proof. Let us show that the ideal $J \subset A[\Gamma]$ generated by the polynomials R_e , $e \in \text{Leaves}(\Gamma)$, is the unit ideal of $A[\Gamma]$. Given $e' \in \mathbf{P}(\Gamma)$ such that $\text{Child}(e') \subset \text{Leaves}(\Gamma)$, the polynomials $S_{e'}^{\{e\}} \in \mathbb{C}[X_{\text{Par}(e')}]$, $e \in \text{Child}(e')$, are relatively prime as \mathfrak{g} is fine-labelled. Thus there exist polynomials $\Lambda_e \in A[\Gamma]$, $e \in \text{Child}(e')$, such that

$$R_{e'} = R_{e'} \sum_{e \in \text{Child}(e')} \Lambda_e S_{e'}^{\{e\}} = \sum_{e \in \text{Child}(e')} \Lambda_e R_e$$

and so, $R_{e'} \in J$. By repeatedly applying the same argument, we conclude that $1 = R_{e_0} \in J$. Therefore the open subsets $D(R_e)$, $e \in \text{Leaves}(\Gamma)$, cover $\text{Spec}(A[\Gamma])$ and so, the open subsets $(U_e)_{e \in \text{Leaves}(\Gamma)}$ cover $V_{\mathfrak{g}}$. \square

Given $e \in \text{Leaves}(\Gamma)$, we have an isomorphism of \mathbb{C} -algebras

$$\Gamma(U_e, \mathcal{O}_{V_{\mathfrak{g}}}) \simeq (B_{\mathfrak{g}})_{\overline{R_e}} \simeq A[\Gamma][T]/(I_{\mathfrak{g}}, G_e),$$

where $G_e = TR_e - 1$ and $\overline{R_e}$ denotes the image of $R_e \in A[\Gamma]$ in $B_{\mathfrak{g}}$. In view of 2.4 above, the following proposition completes the proof of proposition 2.3.

Proposition 2.6. *There exists a nonzero polynomial $P_e \in A[X_{\text{Par}(e)}, T]$ such that*

$$(B_{\mathfrak{g}})_{\overline{R_e}} \simeq A[X_{\text{Par}(e)}, T]/(TP_e - 1).$$

In particular, h is not a zero divisor in $(B_{\mathfrak{g}})_{\overline{R_e}}$ as it does not divide $(TP_e - 1)$.

Proof. With the notation of (1.1), we let

$$L_i = R_{e_{i+1}}^{-1} R_e \in \mathbb{C}[X_{e_i}, \dots, X_{e_{n_e-1}}] \subset A[\Gamma], \quad 0 \leq i \leq n_e - 2.$$

Given $e' \in \mathbf{P}(\Gamma) \setminus (\downarrow e)$, the first common ancestor $\text{Anc}(e', e)$ of e and e' is a node e_i for a certain indice $i \leq n_e - 1$, and $e'' = \text{Child}(e_i) \cap (\downarrow e) \neq e_{i+1}$. We let

$$K_{e'} = (X_{e_{i-1}} - lb(e''))^{-1} R_e \in \mathbb{C}[X_0, X_{e_0}, \dots, X_{e_{n_e-1}}] \subset A[\Gamma].$$

Modulo the ideal I_0 generated by the polynomials

$$\left\{ \begin{array}{l} \delta_{0,i} = -(X_{e_{i-1}} - lb(e_{i+1})) + hX_{e_i}I_iT \\ \quad = TL_i\Delta_{0,e_i} + (X_{e_{i-1}} - lb(e_{i+1}))G_e \end{array} \right. \quad 0 \leq i \leq n_e - 2$$

$$\left\{ \begin{array}{l} \delta_{e'} = X_{e'} - TK_{e'}X_{e_i}Q_{e_i,e'} \\ \quad = TK_{e'}\Delta_{e_i,e'} - X_{e'}G_e \end{array} \right. \quad \begin{array}{l} e' \in \mathbf{P}(\Gamma) \setminus (\downarrow e) \\ \text{Anc}(e', e) = e_i \end{array}$$

we can recursively eliminate all the variables in $A[\Gamma][T]$ but $X_{\text{Par}(e)}$ and T . Therefore, there exists a nonzero polynomial $P_e \in A[X_{\text{Par}(e)}, T]$ such that $R_e \equiv P_e \pmod{I_0}$ and so,

$$A[\Gamma][T]/(I_0, G_e) \simeq A[X_{\text{Par}(e)}, T]/(TP_e - 1)$$

as $G_e = R_eT - 1$. In particular h is not a zero divisor modulo (I_0, G_e) .

Now it suffices to show that the ideals (I_0, G_e) and $(I_{\mathbf{g}}, G_e)$ of $A[\Gamma][T]$ coincide. By construction $I_0 \subset (I_{\mathbf{g}}, G_e)$. Conversely, the identities

$$\left\{ \begin{array}{l} \Delta_{0,e_i} = R_{e_{i+1}}\delta_{0,i} - hX_{e_i}G_e \\ \Delta_{0,e'} = h\delta_{e'} + (X_{e_{i-1}} - lb(e_{i+1}))^{-1}Q_{e'}\delta_{0,i} \end{array} \right. \quad \begin{array}{l} 0 \leq i \leq n_e - 2 \\ e' \in \mathbf{P}(\Gamma) \setminus (\downarrow e) \\ \text{Anc}(e', e) = e_i \end{array},$$

guarantee that $\Delta_{0,e'} \in (I_0, G_e)$ for every $e' \in \mathbf{P}(\Gamma)$. In turn, this implies that $h\Delta_{e'',e'} \in (I_0, G_e)$ for every pair $(e', e'') \in \mathbf{P}(\Gamma) \times \text{Anc}(e')$ by virtue of (1.5). Since h is not a zero divisor modulo (I_0, G_e) we conclude that $\Delta_{e'',e'}(\mathbf{g}) \in (I_0, G_e)$ for every such pair (e', e'') . This completes the proof. \square

Remark 2.7. Since $q_{\mathbf{g}}$ is flat, (1.5) and (2.1) implies that $B_{\mathbf{g}}$ is a sub- \mathbb{C} -algebra of $B_{\mathbf{g}} \otimes_A A_h \simeq A_h[X_0]$. On the other hand, (1.5) and proposition 2.3 assert that it suffices to add the polynomials $\Delta_{e',e}(\mathbf{g})$, $(e, e') \in \mathbf{P}(\Gamma) \times \text{Anc}_{\Gamma}(e)$, to the obvious ones $\Delta_{0,e}(\mathbf{g})$, $e \in \mathbf{P}(\Gamma)$, to guarantee that the fiber $q_{\mathbf{g}}^{-1}(z_0)$ is the flat limit of the fibers $q_{\mathbf{g}}^{-1}(z)$, $z \in Z_*$.

2. The fiber $q_{\mathbf{g}}^{-1}(z_0)$.

Since $q_{\mathbf{g}} : V_{\mathbf{g}} \rightarrow Z$ is a flat morphism restricting to a trivial line bundle over Z_* , $V_{\mathbf{g}}$ is a GDS over Z provided that the fiber $q_{\mathbf{g}}^{-1}(z_0)$ is nonempty and reduced. To describe this fiber, we need the following auxiliary result.

Lemma 2.8. *If $\text{Child}(e_0) \neq \emptyset$ then there exists an isomorphism of \mathbb{C} -algebras*

$$A[\Gamma]/(h, I_{\mathbf{g}}) \simeq \prod_{e \in \text{Child}(e_0)} (A[\Gamma(f)]/(h, I_{\mathbf{g}(e)})).$$

2.9. The polynomial $\Delta_{0,e_0} \in I_{\mathbf{g}}$ reduces to Q_{e_0} modulo h . Since the roots $lb(e) \in \mathbb{C}$, $e \in \text{Child}(e_0)$, of Q_{e_0} are simple, we deduce that

$$(2.2) \quad A[\Gamma]/(h, I_{\mathbf{g}}) \simeq \prod_{e \in \text{Child}(e_0)} (A[\Gamma]/(J_e, I_{\mathbf{g}})),$$

where $J_e = (h, X_0 - lb(e))$. Thus lemma 2.8 is a consequence of the following result.

Lemma 2.10. *For every $e \in \text{Child}(e_0)$, the \mathbb{C} -algebras $A[\Gamma]/(J_e, I_{\mathbf{g}})$ and $A[\Gamma(e)]/(h, I_{\mathbf{g}(e)})$ are isomorphic.*

Proof. Since $X_{e_0} \notin N(\Gamma(e))$, we have

$$A[\Gamma(e)] = A[X_0] \otimes_A \mathbf{S}(M_{\Gamma(e)}) \simeq A[X_{e_0}] \otimes_A \mathbf{S}(M_{\Gamma(e)}).$$

With this choice of coordinates,

$$\begin{cases} Q_{e'}(\mathfrak{g}(e)) &= Q_{e_0, e'}(\mathfrak{g}) & e' \in \mathbf{P}(\Gamma(e)) \\ Q_{e'', e'}(\mathfrak{g}(e)) &= Q_{e'', e'}(\mathfrak{g}) & (e', e'') \in \mathbf{P}(\Gamma(e)) \times \text{Anc}_{\Gamma(e)}(e') \end{cases}$$

We let $I_0 \subset A[\Gamma]$ be the ideal generated by the polynomials

$$\begin{cases} Q_{e_0, e'}(\mathfrak{g}) & e' \in \mathbf{P}(\Gamma(e)) \\ \Delta_{e'', e'}(\mathfrak{g}) & (e', e'') \in \mathbf{P}(\Gamma(e)) \times (\text{Anc}_{\Gamma(e)}(e')) \\ \delta_{e_0, e'} = (lb(e) - lb(g)) X_{e'} - X_{e_0} Q_{e_0, e'}(\mathfrak{g}) & \begin{cases} e' \in \mathbf{P}(\Gamma) \setminus (\{e_0\} \cup \mathbf{P}(\Gamma(e))) \\ g = \text{Child}(e_0) \cap (\downarrow e') \neq e \end{cases} \end{cases}.$$

Since \mathfrak{g} is fine-labelled, $lb(e) - lb(g) \in \mathbb{C}^*$ for every $g \in \text{Child}(e_0) \setminus \{e\}$ and so, we can eliminate modulo I_0 all the variables $X_{e'}$ corresponding to nodes $e' \in \mathbf{P}(\Gamma) \setminus (\mathbf{P}(\Gamma(e)) \cup \{e_0\})$. We conclude that

$$A[\Gamma] / (J_e, I_0) = A[\Gamma(e)][X_0] / (h, X_0 - lb(e), I_0) \simeq A[\Gamma(e)] / (h, I_{\mathfrak{g}(e)}).$$

Now it suffices to show that the ideals (J_e, I_0) and $(J_e, I_{\mathfrak{g}})$ of $A[\Gamma]$ coincide.

If $e' \in \mathbf{P}(\Gamma) \setminus (\mathbf{P}(\Gamma(e)))$ then $\Delta_{0, e'}(\mathfrak{g}) \in J_e$ as $(X_0 - lb(e))$ divides $Q_{e'}(\mathfrak{g})$. If $e'' \in \text{Anc}_{\Gamma}(e')$ then, letting $g' = \text{Child}(e'') \cap (\downarrow e')$, we obtain that

$$\Delta_{e'', e'}(\mathfrak{g}) \bmod J_e = \begin{cases} \delta_{e_0, e'} \in I_0 & \text{if } e'' = e_0 \\ (X_{\text{Par}(e'')} - lb(g')) \delta_{e_0, e'} - \delta_{e_0, e''} Q_{e'', e'}(\mathfrak{g}) \in I_0 & \text{otherwise} \end{cases}.$$

If $e' \in \mathfrak{P}(\Gamma(e))$ then $\Delta_{0, e'}(\mathfrak{g})$ reduces to $S_{e_0}^{\{e\}}(lb(e)) Q_{e_0, e'}(\mathfrak{g}) \in I_0$ modulo J_e . If $e'' \in \text{Anc}_{\Gamma}(e')$ then either $e'' = e_0$ and $\Delta_{e_0, e'}(\mathfrak{g})$ reduces to $Q_{e_0, e'}(\mathfrak{g}) \in I_0$ modulo J_e , or $e'' \neq e_0$ and $\Delta_{e'', e'}(\mathfrak{g}) \in I_0$. This proves that $(J_e, I_{\mathfrak{g}}) = (J_e, I_0)$. \square

The following proposition completes the proof of theorem 2.1.

Proposition 2.11. *The fiber $q_{\mathfrak{g}}^{-1}(z_0)$ of $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ is nonempty and reduced, consisting of the disjoint union of curves $C_e \simeq \text{Spec}(\mathbb{C}[X_{\text{Par}(e)}])$, $e \in \text{Leaves}(\Gamma)$, with defining ideals*

$$I_{\mathfrak{g}}(e) = \left(I_{\mathfrak{g}}, h, \left(X_{\text{Par}^{k+1}(e)} - lb(\text{Par}^{k-1}(e)) \right)_{1 \leq k \leq n_e} \right) \subset A[\Gamma].$$

Proof. We proceed by induction on the height $h(\Gamma)$ of Γ . If $h(\Gamma) = 0$ then $V_{\mathfrak{g}} = \text{Spec}(A[X_0])$ and $q_{\mathfrak{g}}^{-1}(z_0) \simeq \text{Spec}(\mathbb{C}[X_0])$ is reduced. Otherwise, if $\text{Child}(e_0) \neq \emptyset$ then lemma 2.8 implies that

$$q_{\mathfrak{g}}^{-1}(z_0) \simeq \text{Spec}(A[\Gamma] / (h, I_{\mathfrak{g}})) \simeq \bigsqcup_{e' \in \text{Child}(e_0)} \text{Spec}(A[\Gamma(e')] / (h, I_{\mathfrak{g}(e')}))$$

decomposes as the disjoint union of curves $D_{e'}$ isomorphic to the fibers $q_{\mathfrak{g}(e')}^{-1}(z_0)$ of $q_{\mathfrak{g}(e')} : V_{\mathfrak{g}(e')} \rightarrow Z$, $e' \in \text{Child}(e_0)$. Since $\Gamma(e')$ has height $h(\Gamma) - 1$, these fibers are nonempty and reduced, whence $q_{\mathfrak{g}}^{-1}(z_0)$ is. In view of the isomorphism (2.2), the description of the irreducible components of $q_{\mathfrak{g}}^{-1}(z_0)$ follows easily by induction. \square

Corollary 2.12. *For every fine-labelled rooted tree \mathfrak{g} , the surface $V_{\mathfrak{g}}$ is nonsingular.*

Proof. Indeed $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ is a flat morphism with regular geometric fibers, hence a smooth morphism. Thus $V_{\mathfrak{g}}$ is regular as $Z \simeq \mathbb{A}_{\mathbb{C}}^1$ is. \square

3. EMBEDDINGS OF GDS 'S IN AFFINE SPACES

In this section, we consider GDS 's $q : V \rightarrow Z = \text{Spec}(A)$ such that $q^{-1}(z)$ is integral for every $z \in Z_*$. We prove the following result.

Theorem 3.1. *Every GDS $q : V \rightarrow Z$ is Z -isomorphic to a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ for a certain fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$.*

The proof divides as follows. In the first subsection we recall [5] how to attach a weighted rooted tree $\gamma = (\Gamma, w)$ to a pair $(q : V \rightarrow Z, \phi : V \rightarrow \mathbb{A}_Z^1)$ consisting of a GDS $q : V \rightarrow Z$ and a Z -morphism $\phi : V \rightarrow \mathbb{A}_Z^1$ restricting to an isomorphism over Z_* . In turn, this weighted rooted tree γ defines a GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$ which is Z -isomorphic to V . In the second subsection, given a weighted rooted tree $\gamma = (\Gamma, w)$ we construct a fine-labelled tree $\mathfrak{g}_w = (\Gamma, lb_w)$ with the same underlying tree Γ and a closed embedding $i_{\gamma} : W_{\gamma} \hookrightarrow \text{Spec}(A[\Gamma])$ inducing a Z -isomorphism $\psi_{\gamma} : W_{\gamma} \xrightarrow{\sim} V_{\mathfrak{g}_w}$.

From GDS 's to weighted rooted trees.

In this subsection, we freely use the description of GDS 's given in [5]. We recall without proof how to associate a weighted rooted tree $\gamma = (\Gamma, w)$ to a GDS $q : V \rightarrow Z$. We also recall how such a weighted rooted tree γ defines a GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$ which comes equipped with a canonical Z -morphism $\phi_{\gamma,0} : W_{\gamma} \rightarrow \mathbb{A}_Z^1$ restricting to an isomorphism over Z_* .

3.2. Given a GDS $q : V = \text{Spec}(B) \rightarrow Z$, q restricts to the trivial line bundle over $Z_* = \text{Spec}(A_h)$. Since q is flat, we can find an isomorphism $A_h[T] \simeq B \otimes_A A_h$ which extends to an injection $A[T] \hookrightarrow B$. The corresponding Z -morphism $\phi_0 : V \rightarrow \mathbb{A}_Z^1 = \text{Spec}(A[T])$ restricts to an isomorphism over Z_* . This data determines a weighted rooted tree $\gamma = (\Gamma, w)$ as follows.

3.3. If $\phi_0 : V \rightarrow \mathbb{A}_Z^1$ is an isomorphism then we let γ be the trivial tree with one element e_0 . Otherwise, ϕ_0 is constant on the irreducible components C_1, \dots, C_r of the fiber $q^{-1}(z_0)$, and the open subsets

$$V_i = (V \setminus q^{-1}(z_0)) \cup C_i \quad 1 \leq i \leq r$$

are Z -isomorphic to \mathbb{A}_Z^1 . Therefore, there exist $n_i \geq 1$, a polynomial $\sigma_i \in A = \mathbb{C}[h]$ of degree $\deg_h(\sigma_i) < n_i$ and an isomorphism $\tau_i : H^0(Z, q_* \mathcal{O}_{V_i}) \xrightarrow{\sim} A[T_i]$ such that $\phi_0|_{V_i}$ is induced by the polynomial $h^{n_i} T_i + \sigma_i \in A[T_i]$, $1 \leq i \leq r$. Note that over Z_* , the transition isomorphism $\tau_j \circ \tau_i^{-1}$ is given by the A_h -algebras isomorphism

$$A_h[T_i] \xrightarrow{\sim} A_h[T_j], T_i \mapsto h^{n_j - n_i} T_j + h^{-n_i} (\sigma_j - \sigma_i)$$

(compare with [3] and [6]). Letting

$$\sigma_i = \sum_{k=0}^{n_i-1} w_{i,k} h^k \in \mathbb{C}[h] = A, \quad 1 \leq i \leq r,$$

we consider the chains $(C_i, w_i) = \{e_{i,0} < e_{i,1} < \dots < e_{i,n_i}\}$ with the weights

$$w_i([e_{i,k}, e_{i,k+1}]) = w_{i,k} \quad 1 \leq i \leq r, 0 \leq k \leq n_i - 1.$$

Since $q : V \rightarrow Z$ is affine, proposition 2.6 in [5] implies that $n_{ij} = \text{ord}_{z_0}(\sigma_j - \sigma_i) < \min(n_i, n_j)$. Consequently, there exist isomorphisms of weighted subchain

$$\theta_{ij} : C_{ij} = (\downarrow e_{i,n_{ij}})_{C_i} \xrightarrow{\sim} C_{ji} = (\downarrow e_{j,n_{ij}})_{C_j} \quad i \neq j, 1 \leq i, j \leq r,$$

and a unique weighted rooted tree $\gamma = (\Gamma, w)$ with the leaves f_i , together with isomorphisms of weighted chains $\theta_i : (C_i, w_i) \simeq (\downarrow f_i)_\Gamma$, $1 \leq i \leq r$, such that $\theta_i = \theta_j \circ \theta_{ij}$ on C_{ij} (see [5, Proposition 1.12]).

3.4. Starting with a weighted rooted tree $\gamma = (\Gamma, w)$, we construct a *GDS* $q_\gamma : W_\gamma \rightarrow Z$ as follows. If Γ is the trivial tree with one element e_0 then we let $W_\gamma = \text{Spec}(A[X_0]) \simeq \mathbb{A}_Z^1$ and $q_\gamma = pr_Z$. Otherwise, we let $r(\gamma) = \text{Card}(\text{Leaves}(\Gamma)) \geq 1$, and we consider the maximal chains

$$(3.1) \quad (\downarrow f)_\Gamma = \{e_{f,0} = e_0 < e_{f,1} < \cdots < e_{f,n_f-1} < e_{f,n_f} = f\}, \quad f \in \text{Leaves}(\Gamma).$$

The prescheme $\pi_\gamma : X_\gamma \rightarrow Z$ obtained from Z by replacing the closed point z_0 by $r(\gamma)$ points x_f , $f \in \text{Leaves}(\Gamma)$, admits a covering \mathcal{U}_γ by the open subsets $X_f = \pi_\gamma^{-1}(Z_*) \cup \{x_f\} \simeq Z$, $f \in \text{Leaves}(\Gamma)$. Moreover $X_{ff'} = X_f \cap X_{f'} \simeq Z_*$ for every two different leaves f and f' . We let \mathcal{L}_γ be the sub- \mathcal{O}_X -module of $\mathcal{K}(X_\gamma)$ generated by $(h \circ \pi_\gamma)^{-n_f}$ on X_f , $f \in \text{Leaves}(\Gamma)$, and we denote by $s_\gamma \in \Gamma(X, \mathcal{L}_\gamma)$ the canonical section of \mathcal{L}_γ corresponding to the constant section 1 of $\mathcal{K}(X_\gamma)$. We let $\sigma(\gamma) = \{\sigma_f(\gamma)\}_{f \in \text{Leaves}(\Gamma)} \in C^0(\mathcal{U}_\gamma, \mathcal{O}_{X_\gamma})$ be the 0-cochain defined by

$$\sigma_f(\gamma) = \sum_{j=0}^{n_f-1} w([e_{f,j}, e_{f,j+1}]) h^j \in A \simeq H^0(X_f, \mathcal{O}_{X_\gamma}), \quad f \in \text{Leaves}(\Gamma).$$

Since s_γ does not vanish on $X_{ff'} \simeq Z_*$, $f \neq f'$, $f, f' \in \text{Leaves}(\Gamma)$, there exists a unique Čech 1-cocycle $\{g_{ff'}\}_{f, f' \in \text{Leaves}(\Gamma)} \in C^1(\mathcal{U}_\gamma, \mathcal{L}_\gamma^\vee)$ such that

$$g_{ff'} \circ s_\gamma = \sigma_{f'}(\gamma)|_{X_{ff'}} - \sigma_f(\gamma)|_{X_{ff'}}, \quad f \neq f', f, f' \in \text{Leaves}(\Gamma).$$

3.5. Now there exists a unique quasicohherent \mathcal{O}_{X_γ} -algebra \mathcal{A}_γ , together with isomorphisms $\tau_f : \mathcal{A}_\gamma|_{X_f} \xrightarrow{\sim} \mathbf{S}(\mathcal{L}_\gamma|_{X_f})$, $f \in \text{Leaves}(\Gamma)$, such that over the overlaps $X_{ff'}$, the $\mathcal{O}_{X_{ff'}}$ -algebras isomorphisms $\tau_{f'} \circ \tau_f^{-1} : \mathbf{S}(\mathcal{L}_\gamma|_{X_{ff'}}) \xrightarrow{\sim} \mathbf{S}(\mathcal{L}_\gamma|_{X_{ff'}})$ are induced by the $\mathcal{O}_{X_{ff'}}$ -modules homomorphisms

$$(3.2) \quad (g_{ff'}, \text{Id}) : \mathcal{L}_\gamma|_{X_{ff'}} \rightarrow \mathcal{O}_{X_{ff'}} \oplus \mathcal{L}_\gamma|_{X_{ff'}} \subset \mathbf{S}(\mathcal{L}_\gamma|_{X_{ff'}}).$$

By theorem 3.3 in [5], the morphism

$$q_\gamma = \pi_\gamma \circ \rho_\gamma : W_\gamma = \mathbf{Spec}(\mathcal{A}_\gamma) \xrightarrow{\rho_\gamma} X_\gamma \xrightarrow{\pi_\gamma} Z$$

is a *GDS* over Z . It restricts to the trivial line bundle over Z_* , whereas $q_\gamma^{-1}(z_0)$ is the disjoint union of the curves $C_f = \rho_\gamma^{-1}(x_f) \simeq \mathbb{A}_{\mathbb{C}}^1$, $f \in \text{Leaves}(\Gamma)$.

3.6. This *GDS* $q_\gamma : W_\gamma \rightarrow Z$ comes equipped with the canonical section $\phi_{\gamma,0} \in H^0(Z, (q_\gamma)_* \mathcal{O}_{W_\gamma})$ whose restriction to $W_\gamma|_{X_f}$, $f \in \text{Leaves}(\Gamma)$, is simply given by

$$\phi_{\gamma,0}|_{X_f} = (\sigma_f(\gamma), s_\gamma|_{X_f}) \in H^0(X_f, \mathcal{A}_\gamma) \simeq H^0(X_f, \mathbf{S}(\mathcal{L}_\gamma)).$$

Since s_γ does not vanish on $\pi_\gamma^{-1}(Z_*) \subset X_\gamma$, the corresponding Z -morphism $\phi_{\gamma,0} : W_\gamma \rightarrow \mathbb{A}_Z^1$ restricts to an isomorphism over Z_* . If γ is obtained from a pair $(q : V \rightarrow Z, \phi : V \rightarrow \mathbb{A}_Z^1)$ by the procedure described in 3.3 then, by construction, there exists a Z -isomorphism $\psi : V \xrightarrow{\sim} W_\gamma$ such that $\phi_0 = \phi_{\gamma,0} \circ \psi$. This leads to the following result.

Theorem 3.7. [5, Theorem 5.1] *For every GDS $q : V \rightarrow Z$ there exists a weighted rooted tree $\gamma = (\Gamma, w)$ and a Z -isomorphism $\psi : V \xrightarrow{\sim} W_\gamma$.*

Example 3.8. The nonsingular surface $V \subset \text{Spec}(\mathbb{C}[x, y, z])$ with equation

$$x^2z = y^2 - 2xy - 1$$

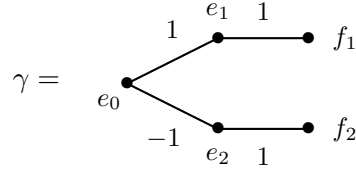
is a GDS over $Z = \text{Spec}(\mathbb{C}[x])$ for the morphism $pr_x : V \rightarrow \text{Spec}(\mathbb{C}[x])$. Letting $z_0 = (x) \in Z$, we get $V \times_Z Z_* \simeq \text{Spec}(\mathbb{C}[x, x^{-1}][y])$, whereas $pr_x^{-1}(z_0)$ is the disjoint union of curves

$$C_1 = \{x = 0, y = 1\} \cap V \quad \text{and} \quad C_2 = \{x = 0, y = -1\} \cap V.$$

isomorphic to $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[z])$. The projection $pr_{x,y} : V \rightarrow \mathbb{A}_Z^1 = \text{Spec}(\mathbb{C}[x][y])$ is a birational Z -morphism restricting to an isomorphism over Z_* . The rational functions $T_1 = x^{-2}(y-1) - x^{-1}$ and $T_2 = x^{-2}(y+1) - x^{-1}$ on V induce Z -isomorphisms

$$(V \setminus pr_x^{-1}(z_0)) \cup C_1 \simeq \text{Spec}(\mathbb{C}[x][T_1]) \quad \text{and} \quad (V \setminus pr_x^{-1}(z_0)) \cup C_2 \simeq \text{Spec}(\mathbb{C}[x][T_2])$$

Therefore, the pair $(pr_x : V \rightarrow Z, pr_{x,y} : V \rightarrow \mathbb{A}_Z^1)$ corresponds to the following weighted rooted tree



Embedding of a GDS W_γ in an affine space.

Given a GDS $q_\gamma : W_\gamma \rightarrow Z$ defined by a weighted rooted tree $\gamma = (\Gamma, w)$, we construct in 3.9-3.17 below a fine-labelled tree $\mathfrak{g} = (\Gamma, lb_w)$ with the same underlying rooted tree Γ , and a collection of regular functions

$$\left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e \in \mathbf{P}(\Gamma)} \right) \in B_\gamma = H^0(Z, (q_\gamma)_* \mathcal{O}_{W_\gamma})$$

defining a closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1} = \text{Spec}(A[\Gamma])$ with image $V_{\mathfrak{g}_w}$.

3.9. Consider the canonical section $\phi_{\gamma,0} \in B_\gamma$ defined in 3.6. If Γ is the trivial tree with one element e_0 then

$$W_\gamma \simeq \text{Spec}(A[\phi_{\gamma,0}]) \simeq \text{Spec}(A[\Gamma]).$$

Otherwise, for every $f \in \text{Leaves}(\Gamma)$, the multiplication by h^{n_f} gives rise to an \mathcal{O}_{X_f} -modules isomorphism $\mathcal{L}_\gamma|_{X_f} \xrightarrow{\sim} \mathcal{O}_{X_f}$, whence to an A -algebras isomorphism $\tau_f : H^0(X_f, \mathcal{A}_\gamma) \xrightarrow{\sim} A[T]$ such that

$$(3.3) \quad \tau_f(\phi_{\gamma,0}) = h^{n_f}T + \sigma_f(\gamma) = \sum_{k=0}^{n_f} w_{f,k}h^k \in A[T],$$

where $w_{f,n_f} = T$. Since $n_f \geq 1$, we deduce that $\phi_{\gamma,0}$ is locally constant on $q_\gamma^{-1}(z_0)$, with the value $w_{f,0} \in \mathbb{C}$ on $C_f = \rho_\gamma^{-1}(x_f) \simeq \text{Spec}(\mathbb{C}[T])$. For every $e \in N(\Gamma)$, we let

$$F(e) = \bigcup_{f \in \text{Leaves}(\Gamma(e))} C_f \subset q_\gamma^{-1}(z_0).$$

In particular, $F(e_0) = q_\gamma^{-1}(z_0)$ and $F(f) = C_f$ for every $f \in \text{Leaves}(\Gamma)$. We have the following result.

Proposition 3.10. *If $h(\Gamma) \geq 1$ then there exists a fine-labelled rooted tree $\mathfrak{g}_w = (\Gamma, lb_w)$ with underlying tree Γ , and a collection of regular functions $\phi_{\gamma, \text{Par}(e_0)} = \phi_{\gamma, 0}, \phi_{\gamma, e} \in B_\gamma$, $e \in \mathbf{P}(\Gamma)$, such that the following hold.*

1) For every $e \in \mathbf{P}(\Gamma)$ with $(\downarrow e)_\Gamma = \{e_0 < e_1 < \dots < e_n = e\}$,

$$\phi_{\gamma, e} = h^{-1}Q_e(\mathfrak{g}_w)(\phi_{\gamma, 0}, \phi_{\gamma, e_0}, \dots, \phi_{\gamma, e_{n-1}}).$$

2) Letting $\text{Child}(e) = \{g_1, \dots, g_r\}$, $\phi_{\gamma, \text{Par}(e)}$ is constant on $F(g_i)$ with the value $lb_w(g_i) \in \mathbb{C}$ whereas $\phi_{\gamma, e}$ restricts to a coordinate function on $C_g \subset q_\gamma^{-1}(z_0)$ for every $g \in \text{Child}(e) \cap \text{Leaves}(\Gamma)$.

3.11. Starting with $\phi_{\gamma, 0}$, we construct by induction the labelling $lb_w : N(\Gamma) \setminus \{e_0\} \rightarrow \mathbb{C}$ and the regular functions $\phi_{\gamma, e}$, $e \in \mathbf{P}(\Gamma)$. More precisely, for every $n \leq h(\Gamma)$ in $N(\Gamma)$, we define the labelling function lb_w on $N_n(\Gamma)$ in terms of the functions $\phi_{\gamma, e'}$ corresponding to nodes $e' \in N_k(\Gamma)$, $k \leq n-2$. Then we define the functions $\phi_{\gamma, e}$ for nodes $e \in N_{n-1}(\Gamma)$ in terms of those functions $\phi_{\gamma, e'}$ and the values of lb_w on the nodes $e'' \in N_k(\Gamma)$, $k \leq n$.

3.12. Step 1. Given $g \in \text{Child}(e_0)$, we let

$$lb_w(g) := \phi_{\gamma, 0} |_{F(g)} = w_{f, 0},$$

where f is any leaf of $\Gamma(g)$. This is well-defined as $g = e_{f, 1}$ for every such leaf f (see (3.1)). If g' is another child of e_0 then there exists $f' \in \text{Leaves}(\Gamma(g'))$ such that $g' = e_{f', 1}$, and hence

$$lb_w(g) = w_{f, 0} \neq w_{f', 0} = lb_w(g')$$

by definition 1.4. This defines the labelling function lb_w on the nodes at level 1. Clearly,

$$\phi_{\gamma, e_0} : = h^{-1}Q_{e_0}(\mathfrak{g}_w)(\phi_{\gamma, 0}) = h^{-1} \prod_{g \in \text{Child}(e_0)} (\phi_{\gamma, 0} - lb_w(g)) \in B_\gamma \otimes_A A_h$$

is a regular function on W_γ . Letting

$$s_g = S_{e_0}^{\{g\}}(\mathfrak{g}_w)(lb_w(g)) = \prod_{g' \in \text{Child}(e_0) \setminus \{g\}} (lb_w(g) - lb_w(g')) \in \mathbb{C}^*, \quad g \in \text{Child}(e_0),$$

we deduce from Taylor's Formula that for every $f \in \text{Leaves}(\Gamma(g))$ there exists $P_0(g, f) \in A[T]$ such that

$$\tau_f(\phi_{\gamma, e_0}) = s_g w_{f, 1} + h P_0(g, f) \in A[T].$$

If g is a leaf of Γ then $w_{g, 1} = T$ (see (3.3)) and so, ϕ_{γ, e_0} restricts to a coordinate function on $C_g \subset q_\gamma^{-1}(z_0)$. Thus (1)-(2) hold for $e_0 \in N(\Gamma)$.

3.13. Step 2. To define lb_w on the nodes at level 2, we proceed as follows. Given $g' \in \text{Child}(g)$, ϕ_{γ, e_0} is constant on $F(g') \subset q_\gamma^{-1}(z_0)$ with the value

$$lb_w(g') := s_g w_{f, 1} \in \mathbb{C},$$

where f is any leaf of $\Gamma(g')$. Given another child g'' of g and a leaf f' of $\Gamma(g'')$, $g = e_{f, 1} = e_{f', 1}$ is the first common ancestor of f and f' . Thus $e_{f, 2} \neq e_{f', 2}$, whence $w_{f, 1} \neq w_{f', 1}$ as γ is a weighted rooted tree. Since $s_g \in \mathbb{C}^*$ we deduce that $lb_w(g') \neq lb_w(g'')$. This defines the labelling function lb_w on the nodes at level 2.

Suppose that the labelling lb_w has been defined for every node e at level $k \leq n+1$ and that for every $e \in \mathbf{P}(\Gamma) \cap N_n(\Gamma)$ with $(\downarrow e) = \{e_0 < e_1 < \dots < e_n = e\}$, the regular functions $\phi_{e_i} \in B_\gamma$, $0 \leq i \leq n-1$ have been constructed, satisfying (1) and (2) of proposition 3.10. Since lb_w is defined on $N_k(\Gamma)$, $k \leq n+1$, the polynomial $Q_e(\mathbf{g}_w) \in \mathbb{C}[X_0, X_{e_0}, \dots, X_{e_{n-1}}]$ is well-defined (see 1.9). We have the following result.

Lemma 3.14. *If e is not a leaf of Γ then*

$$\phi_{\gamma, e_n} = h^{-1} Q_e(\mathbf{g}_w) (\phi_{\gamma, 0}, \phi_{\gamma, e_0}, \dots, \phi_{\gamma, e_{n-1}}) \in B_\gamma \otimes_A A_h$$

is a regular function on W_γ satisfying (1)-(2) in proposition 3.10.

Proof. In view of the definition of the functions ϕ_{γ, e_i} , $0 \leq i \leq n-1$, we deduce from Taylor's Formula that for every $g \in \text{Child}(e)$ and every $f \in \text{Leaves}(\Gamma(g))$, there exists $P_{n-1}(g, f) \in A[T]$ such that

$$\tau_f(\phi_{\gamma, e_{n-1}}) = lb_w(g) + \left(\prod_{i=0}^{n-1} s_{e_i}^{n-1-i} w_{f, n+1} + Q_{n-1, g}(w_{f, 0}, \dots, w_{f, n}) \right) h + h^2 P_{n-1}(g, f) \in A[T],$$

where $Q_{n-1, g} \in \mathbb{C}[Y_0, \dots, Y_n]$ is independent on the choice of a leaf f of $\Gamma(g)$ and

$$s_{e_i} = S_{e_i}^{\{e_{i+1}\}}(\mathbf{g}_w)(lb_w(e_{i+1})) \in \mathbb{C}^*, \quad 1 \leq i \leq n-1.$$

By definition,

$$\phi_{\gamma, e_n} = h^{-1} S_{e_n}(\mathbf{g}_w) (\phi_{\gamma, e_{n-1}}) \prod_{i=0}^{n-1} S_{e_i}^{\{e_{i+1}\}}(\mathbf{g}_w) (\phi_{\gamma, e_{i-1}}) \in B_\gamma \otimes_A A_h$$

and $\phi_{\gamma, e_{i-1}}|_{C(f)} = lb_w(e_{i+1})$, $0 \leq i \leq n$. Again, Taylor's Formula implies that there exists $P_n(g, f) \in A[T]$ such that

$$(3.4) \quad \tau_f(\phi_{\gamma, e_n}) = (\lambda_g w_{f, n+1} + \mu_g) + h P_n(g, f) \in A[T],$$

where

$$\begin{cases} \lambda_g = S_{e_n}^{\{g\}}(\mathbf{g}_w)(lb_w(g)) \left(\prod_{i=0}^{n-1} s_{e_i}^{n-i} \right) \in \mathbb{C}^*, \\ \mu_g = S_{e_n}^{\{g\}}(\mathbf{g}_w)(lb_w(g)) \left(\prod_{i=0}^{n-1} s_{e_i} \right) Q_{n-1, g}(w_{f, 0}, \dots, w_{f, n}) \in \mathbb{C}. \end{cases}$$

Thus ϕ_{γ, e_n} is regular on $W_\gamma|_{X_f}$ for every $f \in \text{Leaves}(\Gamma(e_n))$. Given $f' \in \text{Leaves}(\Gamma) \setminus \text{Leaves}(\Gamma(e_n))$, there exists an indice $k \leq n-1$ such that e_k is the first common ancestor of f' and e_n . Letting $e' = \text{Child}(e_k) \cap (\downarrow f')$, $(X_{e_{k-1}} - lb_w(e'))$ divides $Q_{e_n}(\mathbf{g}_w)$ and so, $\tau_{f'}(\phi_{\gamma, e_{k-1}} - lb_w(e')) \in hA[T]$. In turn, this implies that $\tau_{f'}(\phi_{\gamma, e_n}) \in A[T]$. Thus ϕ_{γ, e_n} is a regular function on W_γ . If $g \in \text{Child}(e_n)$ is a leaf of Γ then $w_{g, n+1} = T$ (see (3.3)) and so, ϕ_{γ, e_n} restricts to a coordinate function on $C_g \subset q_\gamma^{-1}(z_0)$. \square

3.15. Consider the regular function ϕ_{γ, e_n} constructed above. Given $g \in \text{Child}(e_n) \setminus \text{Leaves}(\Gamma)$ and $g' \in \text{Child}(g)$, we deduce from (3.4) that ϕ_{γ, e_n} is constant on $F(g') \subset q_\gamma^{-1}(z_0)$ with the value

$$(3.5) \quad lb_w(g') := \lambda_g w_{f, n+1} + \mu_g,$$

where f is any leaf of $\Gamma(g')$. Since neither λ_g nor μ_g depend on the choice of $f \in \text{Leaves}(\Gamma(g))$, the same argument as in 3.12 shows that $lb_w(g') \neq lb_w(g'')$ for every two distinct children g'

and g'' of g . In this way, we define the labelling function lb_w on the nodes of Γ at level $n + 2$. Now the proof of proposition 3.10 can be completed by induction.

Remark 3.16. By construction, the labelling $lb_w : N(\Gamma) \rightarrow \mathbb{C}$ is uniquely determined by the weight function $w : E(\Gamma) \rightarrow \mathbb{C}$. Conversely, since $lb_w(e) = w([e_0, e]_\Gamma)$ for every node $e \in N_1(\Gamma)$ (see 3.12), we deduce from (3.5) that the weight function $w : E(\Gamma) \rightarrow \mathbb{C}$ can be recursively recovered from the associated labelling lb_w .

The following result completes the proof of theorem 3.1.

Theorem 3.17. *For every GDS $q_\gamma : W_\gamma \rightarrow Z = \text{Spec}(A)$ defined from a weighted rooted tree $\gamma = (\Gamma, w)$, the A -algebras homomorphism*

$$A[\Gamma] \rightarrow B_\gamma, \begin{cases} X_0 \mapsto \phi_{\gamma,0} \\ X_e \mapsto \phi_{\gamma,e} \quad e \in \mathbf{P}(\Gamma) \end{cases}$$

is surjective, with kernel $I_{\mathfrak{g}_w}$. In other words, the functions $\phi_{\gamma,0}, (\phi_{\gamma,e})_{e \in \mathbf{P}(\Gamma)}$ define a closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1} = \text{Spec}(A[\Gamma])$ inducing a Z -isomorphism $\psi_\gamma : W_\gamma \xrightarrow{\sim} V_{\mathfrak{g}_w}$.

Proof. By construction, $W_\gamma \times_Z Z_* \simeq \text{Spec}(A_h[\phi_{\gamma,0}])$. Given two different leaves f and f' of Γ with first common ancestor $e \in \mathbf{P}(\Gamma)$, it follows from proposition 3.10 that $\phi_{\gamma, \text{Par}(e)}$ takes distinct values on the curves C_f and $C_{f'}$. Thus i_γ separates the irreducible components of $q_\gamma^{-1}(z_0)$. If $f \in \text{Leaves}(\Gamma)$ then $\phi_{\gamma, \text{Par}(f)}$ restricts to a coordinate function on C_f . This proves that i_γ is a closed embedding of Z -schemes. By (1) in proposition 3.10, the relation

$$\Delta_{0,e}(\mathfrak{g}_w) \left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e \in \mathfrak{P}(\Gamma)} \right) = h\phi_{\gamma,e} - Q_e(\mathfrak{g}_w) \left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e \in \mathfrak{P}(\Gamma)} \right) = 0$$

holds in B_γ for every $e \in \mathbf{P}(\Gamma)$. In turn, (1.5) implies that

$$h\Delta_{e',e}(\mathfrak{g}_w) \left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e \in \mathfrak{P}(\Gamma)} \right) = 0 \quad (e, e') \in \mathbf{P}(\Gamma) \times \text{Anc}_\Gamma(e)$$

and so, $\Delta_{e',e}(\mathfrak{g}_w) \left(\phi_0, (\phi_e)_{e \in \mathfrak{P}(\Gamma)} \right) = 0$ as B_γ is an integral A -algebra. Since i_γ maps the irreducible components of $q_\gamma^{-1}(z_0)$ bijectively on the irreducible components of $q_{\mathfrak{g}_w}^{-1}(z_0)$ (see proposition 2.11), we deduce that i_γ induces a bijective morphism $\psi_\gamma : W_\gamma \rightarrow V_{\mathfrak{g}_w}$. By virtue of Zariski's Main Theorem, ψ_γ is an isomorphism as $V_{\mathfrak{g}_w}$ is nonsingular (see corollary 2.12), whence normal. \square

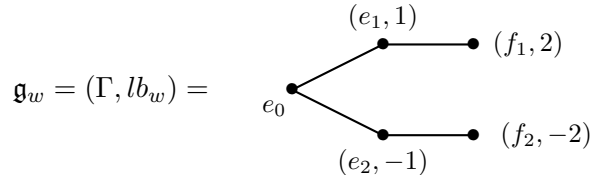
Example 3.18. We again consider the surface $V \subset \text{Spec}(\mathbb{C}[x, y, z])$ with equation

$$x^2z = y^2 - 2xy - 1$$

introduced in example 3.8. Starting with the weighted rooted tree $\gamma = (\Gamma, w)$ corresponding to the pair

$$(pr_x : V \rightarrow Z, pr_{x,y} : V \rightarrow \mathbb{A}_Z^1),$$

the above procedure gives the following fine-labelled rooted tree



Thus $V_{\mathfrak{g}_w} \subset \text{Spec}(A[X_0][X_{e_0}, X_{e_1}, X_{e_2}])$ is the surface with equations

$$\begin{cases} hX_{e_0} = (X_0^2 - 1), & hX_{e_1} = (X_0 + 1)(X_{e_0} - 2), & hX_{e_2} = (X_0 - 1)(X_{e_0} + 2) \\ (X_0 - 1)X_{e_1} = X_{e_0}(X_{e_0} - 2), & (X_0 + 1)X_{e_2} = X_{e_0}(X_{e_0} + 2) \end{cases}.$$

This shows that in general, the embedding of a GDS V obtained by the above procedure is not "the best possible". However, note that the \mathbb{C} -algebras homomorphism

$$A[X_0, X_{e_0}, X_{e_1}, X_{e_2}] \rightarrow \mathbb{C}[x, x^{-1}y, z], \begin{cases} h \mapsto x, & X_0 \mapsto y \\ X_{e_0} \mapsto xz + 2y \\ X_{e_1} \mapsto x^{-1}(xz + 2y - 2)(y + 1) \\ X_{e_2} \mapsto x^{-1}(xz + 2y + 2)(y - 1) \end{cases}$$

induces an isomorphism $V \xrightarrow{\sim} V_{\mathfrak{g}}$.

The following result is complementary to proposition 3.10.

Proposition 3.19. *For every GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ defined by a fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, the weighted tree γ associated to the pair $(q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z, pr_{X_0} : V_{\mathfrak{g}} \rightarrow \mathbb{A}_Z^1)$ has the same underlying tree Γ as \mathfrak{g} , and the GDS $V_{\mathfrak{g}}$ is Z -isomorphic to W_{γ} via the closed embedding $i_{\gamma} : W_{\gamma} \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1}$.*

Proof. By construction of the embedding $i_{\gamma} : W_{\gamma} \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1}$, the canonical Z -morphism $\phi_{0,\gamma} : W_{\gamma} \rightarrow \mathbb{A}_Z^1$ factors as $\phi_{\gamma,0} = pr_{X_0} \circ i_{\gamma}$ (see 3.17). So the statement follows from remark 3.16. \square

4. $\mathbb{G}_{a,Z}$ -ACTIONS ON GDS 'S OVER Z

In this subsection we study $\mathbb{G}_{a,Z}$ -actions on a GDS $q : V \rightarrow Z$. In view of the well known correspondence between algebraic $\mathbb{G}_{a,Z}$ -actions on V and locally nilpotent A -derivations of the algebra $B = H^0(Z, q_*\mathcal{O}_V)$, this is the same as to study the set $\text{LND}_A(B)$ of these locally nilpotent derivations.

$\mathbb{G}_{a,Z}$ -actions on the GDS 's $q_{\gamma} : W_{\gamma} \rightarrow Z$.

Here we recall following [5] the description of $\mathbb{G}_{a,Z}$ -actions on a GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$ defined from a weighted rooted tree $\gamma = (\Gamma, w)$. By construction (see 3.4), the morphism $q_{\gamma} : W_{\gamma} \rightarrow Z$ factors through $\rho_{\gamma} : W_{\gamma} \rightarrow X_{\gamma}$, in such a way that $W_{\gamma}|_{X_f}$ is isomorphic to the restriction of the line bundle $p : L_{\gamma} = \mathbf{Spec}(\mathbf{S}(\mathcal{L}_{\gamma})) \rightarrow X_{\gamma}$ over X_f , $f \in \text{Leaves}$. This line bundle L_{γ} has a natural structure of group scheme as it represents the group functor

$$(\text{Sch}/_{X_{\gamma}}) \rightarrow (\text{Grp}), \left(Y \xrightarrow{f} X_{\gamma} \right) \mapsto H^0(Y, f^*\mathcal{L}_{\gamma}^{\vee}),$$

and W_{γ} comes naturally equipped with a left action $\xi_{\gamma} : L_{\gamma} \times_{X_{\gamma}} W_{\gamma} \rightarrow W_{\gamma}$ making $\rho_{\gamma} : W_{\gamma} \rightarrow X_{\gamma}$ a principal homogeneous L_{γ} -bundle.

4.1. Every nonzero section $s \in H^0(X_{\gamma}, \mathcal{L}_{\gamma}^{\vee})$ gives rise to a group homomorphism

$$\phi_s : \mathbb{G}_{a,X_{\gamma}} = \mathbf{Spec}(\mathcal{O}_{X_{\gamma}}[T]) \rightarrow L_{\gamma},$$

whence to a nontrivial $\mathbb{G}_{a,X_{\gamma}}$ -action $\mu_s = \xi_{\gamma} \circ (\phi_s \times \text{Id})$ on W_{γ} . By proposition 3.10 in [5], every nontrivial $\mathbb{G}_{a,Z}$ -action on $q_{\gamma} : W_{\gamma} \rightarrow Z$ is induced by such a nonzero section s . Since

$$K(X_\gamma) \simeq K(Z) \simeq \text{Frac}(A),$$

$$H^0(X_\gamma, \mathcal{L}_\gamma^\vee) \simeq \left\{ g \in K(X_\gamma), \text{div}(g) - \sum_{f \in \text{Leaves}(\Gamma)} n_f x_f \geq 0 \right\} \cup \{0\}$$

and so, nonzero sections $s \in H^0(X_\gamma, \mathcal{L}_\gamma^\vee)$ are in one-to-one correspondence with regular functions $h^m g \in A$, where $m \geq h(\Gamma) = \max(n_f)_{f \in \text{Leaves}(\Gamma)}$ and $g \in A \setminus hA$.

Remark 4.2. Let $s \in H^0(X_\gamma, \mathcal{L}_\gamma^\vee)$ be a nonzero section corresponding to a regular function $h^m g \in A$. Over X_f , the multiplication by h^{n_f} induces an isomorphism of \mathcal{O}_{X_f} -algebras $\tau_f : \mathcal{A}_\gamma|_{X_f} \xrightarrow{\sim} \mathcal{O}_{X_f}[U]$ such that the group co-action corresponding to the restriction of the $\mathbb{G}_{a,Z}$ -action μ_s on $W_\gamma|_{X_f}$ is given by the \mathcal{O}_{X_f} -algebras homomorphism

$$\mathcal{O}_{X_f}[U] \rightarrow \mathcal{O}_{X_f}[U] \otimes_{\mathcal{O}_{X_f}} \mathcal{O}_{X_f}[T], U \mapsto U \otimes 1 + h^{m-n_f} g \otimes T$$

In other words, the restriction of μ_s to $W_\gamma|_{X_f}$ is simply a translation, twisted by $h^{m-n_f} g$.

We also recall the following useful result.

Proposition 4.3. ([5, Proposition 3.12]) *For a GDS $q_\gamma : W_\gamma \rightarrow Z$ defined by a weighted rooted tree $\gamma = (\Gamma, w)$, the following are equivalent.*

- 1) W_γ admits a free $\mathbb{G}_{a,Z}$ -action.
- 2) All the leaves of Γ are at the same level.
- 3) The canonical sheaf ω_{W_γ} of W_γ is trivial.

Example 4.4. In [1] Bandman and Makar-Limanov proved that the canonical sheaf of the surface V of example 1.16 is not trivial by constructing certain explicit global holomorphic 2-forms on V . On the other hand, the fine-labelled tree \mathfrak{g} corresponding to this surface has leaves at levels 1 and 2. Therefore, propositions 3.19 and 4.3 implies the same result. This shows in particular that there is no free $\mathbb{G}_{a,Z}$ -action on V (see also example 4.5 below).

$\mathbb{G}_{a,Z}$ -actions on the GDS's $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$.

In this subsection, we prove that every $\mathbb{G}_{a,Z}$ -action on a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ defined by a fine-labelled rooted tree extends to a $\mathbb{G}_{a,Z}$ -action on the ambient space $\mathbb{A}_Z^{d(\Gamma)+1} = \text{Spec}(A[\Gamma])$.

Example 4.5. The fine-labelled tree \mathfrak{g} of example 1.16 corresponds to the Bandman and Makar-Limanov surface $V \subset \text{Spec}(A[X_0, X_{e_0}, X_{e_1}])$ with equations

$$hX_{e_0} = X_0(X_0^2 - 1), \quad X_0X_{e_1} - X_{e_0}(X_{e_0}^2 - 1), \quad hX_{e_1} = (X_0^2 - 1)(X_{e_0}^2 - 1).$$

Consider the following locally nilpotent A -derivation $\partial = \tilde{\partial}_{\mathfrak{g},2}$ of $A[\Gamma]$ given via

$$\begin{cases} \partial(X_0) = h^2, & \partial(X_{e_0}) = h(3X_0^2 - 1) \\ \partial(X_{e_1}) = 2hX_0(X_{e_0}^2 - 1) + 2(X_0^2 - 1)(3X_0^2 - 1)X_{e_0} \end{cases}.$$

In [1, p. 579], Bandman and Makar-Limanov proved that ∂ induces a locally nilpotent A -derivation $\partial_{\mathfrak{g},2}$ of $B_{\mathfrak{g}}$. Note that corresponding $\mathbb{G}_{a,Z}$ -action restricts to a free action on the complement of the irreducible components

$$C_{f_1} = \{h = 0, X_0 = 1\} \cap V \quad \text{and} \quad C_{f_2} = \{h = 0, X_0 = -1\} \cap V$$

of $q_{\mathfrak{g}}^{-1}(z_0)$.

More generally, the following proposition asserts that a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ admits canonical $\mathbb{G}_{a,Z}$ -actions which are the restrictions to $V_{\mathfrak{g}}$ of certain $\mathbb{G}_{a,Z}$ -actions on $\mathbb{A}_Z^{d(\Gamma)+1}$.

Proposition 4.6. *For every $m \geq h(\Gamma)$, the A -derivation $\tilde{\partial}_{\mathfrak{g},m}$ of $A_h[\Gamma]$ defined recursively by*

$$\tilde{\partial}_{\mathfrak{g},m} = h^m \frac{\partial}{\partial X_0} + h^{-1} \sum_{e \in \mathfrak{P}(\Gamma)} \tilde{\partial}_{\mathfrak{g},m}(Q_e(\mathfrak{g})) \frac{\partial}{\partial X_e}$$

restricts to a triangular derivation of $A[\Gamma]$. It induces a locally nilpotent A -derivation $\partial_{\mathfrak{g},m}$ of the A -algebra $B_{\mathfrak{g}} = A[\Gamma]/I_{\mathfrak{g}}$.

Proof. Given a node $e \in \mathbf{P}(\Gamma)$ at level $k < h(\Gamma)$, $Q_e(\mathfrak{g})$ only involves the variables X_0 and $X_{e'}$, $e' \in \text{Anc}(e)$ (see definition 1.9). We conclude recursively that

$$\tilde{\partial}_{\mathfrak{g},m}(X_e) = h^{-1} \sum_{e' \in \text{Anc}(e) \cup \{0\}} \frac{\partial Q_e(\mathfrak{g})}{\partial X_{e'}} \tilde{\partial}_{\mathfrak{g},m}(X_{e'}) \in h^{m-k-1} A[X_0, (X_{e'})_{e' \in \text{Anc}(e)}].$$

Thus $\tilde{\partial}_{\mathfrak{g},m}$ restricts to a triangular A -derivation of $A[\Gamma]$ as $m \geq h(\Gamma)$. By construction, $\tilde{\partial}_{\mathfrak{g},m}$ annihilates $\Delta_{0,e}(\mathfrak{g})$ for every $e \in \mathfrak{P}(\Gamma)$. Given $(e, e') \in \mathbf{P}(\Gamma) \times \text{Anc}(e)$, we deduce from (1.5) that

$$h \tilde{\partial}_{\mathfrak{g},m}(\Delta_{e',e}(\mathfrak{g})) = \tilde{\partial}_{\mathfrak{g},m}(h \Delta_{e',e}(\mathfrak{g})) \in I_{\mathfrak{g}}.$$

Since $B_{\mathfrak{g}}$ is an integral A -algebra, we conclude that $\tilde{\partial}_{\mathfrak{g},m}(\Delta_{e',e}(\mathfrak{g})) \in I_{\mathfrak{g}}$ for otherwise h is a zero divisor in $B_{\mathfrak{g}}$. Thus $\tilde{\partial}_{\mathfrak{g},m}(I_{\mathfrak{g}}) \subset I_{\mathfrak{g}}$ and so, $\tilde{\partial}_{\mathfrak{g},m}$ induces a locally nilpotent A -derivation $\partial_{\mathfrak{g},m}$ on $B_{\mathfrak{g}}$. \square

In [6], Fieseler exploited the vector field associated to a \mathbb{C}_+ -action to introduce the concept of fixed point order of a \mathbb{C}_+ -action along an invariant subscheme. Here we give an equivalent definition in terms of locally nilpotent derivations.

Definition 4.7. Let $V = \text{Spec}(B)$ be an integral affine scheme over a field k of characteristic 0. Given a nontrivial $\mathbb{G}_{a,k}$ -action $\alpha : \mathbb{G}_{a,k} \times V \rightarrow V$ corresponding to a locally nilpotent derivation $\partial \in \text{LND}_k(B)$ and an invariant subscheme $Y \subset V$ with defining ideal $I_Y \subset B$, the *fixed point order* $\mu(\alpha, Y)$ of α along Y is the maximal number $n \geq 0$ such that $\partial(B) \subset I_Y^n B$.

We have the following result.

Lemma 4.8. *For every $m \geq h(\Gamma)$ the $\mathbb{G}_{a,Z}$ -action $\alpha_{\mathfrak{g},m}$ on $V_{\mathfrak{g}}$ corresponding to the derivation $\partial_{\mathfrak{g},m}$ restricts to a free action on $V_{\mathfrak{g}} \times_Z Z_*$. It has fixed point order $\mu(\alpha_{\mathfrak{g},m}, C_e) = m - n_e$ along an irreducible component $C_e \subset q_{\mathfrak{g}}^{-1}(z_0)$ corresponding to a leaf e of Γ at level n_e .*

Proof. Clearly, $\alpha_{\mathfrak{g},m}$ restricts to a free action on $V_{\mathfrak{g}} \times_Z Z_* \simeq \text{Spec}(A_h[X_0])$ (see 2.4) as $\tilde{\partial}_{\mathfrak{g},m}(X_0) = h^m$. With the notation of (1.1), the ideal $\tilde{I}_{\mathfrak{g}}(e) \subset B_{\mathfrak{g}} = A[\Gamma]/I_{\mathfrak{g}}$ of the curve C_e is generated by the image of the ideal

$$I_{\mathfrak{g}}(e) = (I_{\mathfrak{g}}, h, X_0 - lb(e_1), \dots, X_{e_{n_e-2}} - lb(e_{n_e})) \subset A[\Gamma]$$

in $B_{\mathfrak{g}}$ (see proposition 2.11). By definition of $\tilde{\partial}_{\mathfrak{g},m}$,

$$\tilde{\partial}_{\mathfrak{g},m}(h, X_{e_0}, \dots, X_{e_{n_e-1}}) \subset I_{\mathfrak{g}}^{m-n_e}(e) \setminus I_{\mathfrak{g}}^{m-n_e+1}(e).$$

For $e' \in \mathbf{P}(\Gamma) \setminus (\downarrow e)$ there exists an indice k , $0 \leq k \leq n_e - 1$ such that e_k is the first common ancestor of e' and e . Letting $e'' = \text{Child}(e_k) \cap (\downarrow e')$, we have $Q_{e'} = R_{e''} Q_{e_k, e'}$, where $Q_{e_k, e'}$

only involves the variables corresponding to nodes in $[e_k, \text{Par}(e')]_{\Gamma}$. Since $(X_{e_{k-1}} - lb(e_{k+1}))$ divides $R_{e'}$ we conclude by induction that

$$\tilde{\partial}_{\mathfrak{g},m}(X_{e'}) = h^{-1}\tilde{\partial}_{\mathfrak{g},m}(Q_{e'}) \in I_{\mathfrak{g}}^{m-k-1}(e) \subset I_{\mathfrak{g}}^{m-n_e}(e).$$

Thus $\partial_{\mathfrak{g},m}(B_{\mathfrak{g}}) \subset \tilde{I}_{\mathfrak{g}}^{m-n_e}(e) \setminus \tilde{I}_{\mathfrak{g}}^{m-n_e-1}(e)$ and so, $\mu(\alpha_{\mathfrak{g},m}, C_e) = m - n_e$. \square

This leads to the following description.

Theorem 4.9. *Every nontrivial $\mathbb{G}_{a,Z}$ -action on a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ is induced by a locally nilpotent derivation $g\partial_{\mathfrak{g},m}$ of $B_{\mathfrak{g}}$, where $m \geq h(\Gamma)$ and $g \in A \setminus hA$.*

Proof. Since A is contained in the kernel of $\partial_{\mathfrak{g},m}$, $g\partial_{\mathfrak{g},m}$ is again locally nilpotent, whence induces a nontrivial $\mathbb{G}_{a,Z}$ -action α on $V_{\mathfrak{g}}$. For every $e \in \text{Leaves}(\Gamma)$, the open subset

$$V_e = (V_{\mathfrak{g}} \setminus q_{\mathfrak{g}}^{-1}(z_0)) \cup C_e$$

is Z -isomorphic to $\mathbb{A}_Z^1 = \text{Spec}(A[U])$ (see 3.3). By lemma 4.8, $\mu(\alpha, C_e) = m - n_e$, where $n_e = l(\downarrow e)$. We conclude similarly that $\mu(\alpha, q_{\mathfrak{g}}^{-1}(z)) = \text{ord}_z(g)$ for every $z \in Z_*$. Letting $\mathbb{G}_{a,Z} = \text{Spec}(A[T])$, this means that the group co-action corresponding to the restriction of α to V_f is given by the A -algebras homomorphism

$$A[U] \rightarrow A[U] \otimes_A A[T], U \mapsto U \otimes 1 + h^{m-n_e}g \otimes T.$$

By proposition 3.19, the GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$ defined by the weighted rooted tree $\gamma = (\Gamma, w)$ corresponding to the pair $(q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z, pr_{X_0} : V_{\mathfrak{g}} \rightarrow \mathbb{A}_Z^1)$ is isomorphic to $V_{\mathfrak{g}}$ via the open embedding $i_{\gamma} : W_{\gamma} \hookrightarrow \text{Spec}(A[\Gamma])$. We deduce from remark 4.2 that this isomorphism is equivariant when we equip $V_{\mathfrak{g}}$ with α and W_{γ} with the action μ_s corresponding to $h^m g \in A$ (see 4.1). Since every nontrivial $\mathbb{G}_{a,Z}$ -action on W_{γ} is of the form μ_s for some section $s \in H^0(X_{\gamma}, \mathcal{L}_{\gamma}^{\vee})$ corresponding to a regular function $h^m g \in A$, where $m \geq h(\Gamma)$ and $g \in A \setminus hA$, we conclude that every $\mathbb{G}_{a,Z}$ -action on $V_{\mathfrak{g}}$ is induced by a locally nilpotent derivation $g\partial_{\mathfrak{g},m}$ for some $g \in A \setminus hA$. \square

As a consequence of theorems 3.1 and 4.9, we obtain the following result.

Corollary 4.10. *Any GDS $q : V \rightarrow Z$ can be embedded into an affine space \mathbb{A}_Z^N in such a way that every $\mathbb{G}_{a,Z}$ -action on V extends to a $\mathbb{G}_{a,Z}$ -action on \mathbb{A}_Z^N .*

5. GDS'S WITH A TRIVIAL MAKAR-LIMANOV INVARIANT

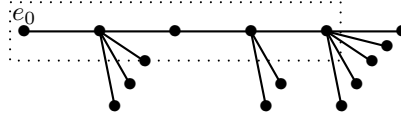
In this section we characterize GDS's $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ with a trivial Makar-Limanov invariant. By theorem 7.7 in [5], every normal affine surface S with a trivial Makar-Limanov invariant is a cyclic quotient of a GDS V . In case that S is a log \mathbb{Q} -homology planes with a trivial Makar-Limanov invariant, we construct this GDS V and the cyclic group action explicitly.

Embeddings of GDS's with a trivial Makar-Limanov invariant.

We recall that an (*oriented*) *comb* of height n is a rooted tree Γ such that for every node $e \in N(\Gamma) \setminus \text{Leaves}(\Gamma)$ of degree $\deg_{\Gamma}(e) \geq 1$, all but possibly one of the children of e are leaves of Γ . This means equivalently that

$$(5.1) \quad C_{\Gamma} = (N(\Gamma) \setminus \text{Leaves}(\Gamma)) = \{e_0 < \dots < e_{n-1}\}$$

is either empty or a chain of length $n - 1$.


 A comb rooted in e_0 .

We have the following result.

Theorem 5.1. *A GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z$ has a trivial Makar-Limanov invariant if and only if $\mathfrak{g} = (\Gamma, lb)$ is a fine-labelled comb.*

Proof. By proposition 3.19, $V_{\mathfrak{g}}$ is Z -isomorphic to the GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$ defined by the weighted rooted tree γ corresponding to the pair $(q_{\mathfrak{g}} : V_{\mathfrak{g}} \rightarrow Z, pr_{X_0} : V_{\mathfrak{g}} \rightarrow \mathbb{A}_{\mathbb{Z}}^1)$. Since γ has the same underlying tree Γ as \mathfrak{g} , the statement follows from theorem 7.2 in [5] which asserts that a GDS $q_{\gamma} : W_{\gamma} \rightarrow Z$ has a trivial Makar-Limanov invariant if and only if Γ is a comb. \square

5.2. Given a nontrivial fine-labelled comb $\mathfrak{g} = (\Gamma, lb)$ of height n , the ideal $I_{\mathfrak{g}} \subset A[\Gamma] = A[X_0][X_{e_0}, \dots, X_{e_{n-1}}]$ of $V_{\mathfrak{g}}$ is generated by the polynomials

$$(5.2) \quad \begin{cases} \Delta_{0, e_i}(\mathfrak{g}) &= hX_{e_i} - Q_{e_i}(\mathfrak{g}) & 0 \leq i \leq n-1 \\ \Delta_{e_j, e_i}(\mathfrak{g}) &= (X_{e_{j-1}} - lb(e_{j+1}))X_{e_i} - X_{e_j}Q_{e_j, e_i}(\mathfrak{g}) & 0 \leq j < i \leq n-1, \end{cases}$$

These equations can be put in the following more convenient form. Letting

$$P_i(T) = (T - \lambda_{i,1})\tilde{P}_i(T) = (T - lb(e_i))S_{e_{i-1}}^{\{e_i\}}(\mathfrak{g})(T) \in \mathbb{C}[T], \quad 1 \leq i \leq n,$$

it is easily seen that $V_{\mathfrak{g}}$ is isomorphic to the surface

$$V_{P_1, \dots, P_n} \subset \mathbb{A}_{\mathbb{C}}^{n+2} = \text{Spec}(\mathbb{C}[x][y_1, \dots, y_{n+1}])$$

with equations

$$\begin{cases} xy_{i+1} &= \left(\prod_{k=1}^{i-1} \tilde{P}_k(y_k) \right) P_i(y_i) & 1 \leq i \leq n \\ (y_{j-1} - \lambda_{j-1,1})y_{i+1} &= y_j \left(\prod_{k=j}^{i-1} \tilde{P}_k(y_k) \right) P_i(y_i) & 2 \leq j \leq i \leq n \end{cases},$$

where, by convention, $\prod_{k=j}^{i-1} \tilde{P}_k(y_k) = 1$ if $i = j$. In particular, if \mathfrak{g} is a fine-labelled comb of

height 1 then $V_{\mathfrak{g}}$ is the ordinary Danielewski surface with equation $xy_2 - P_1(y_1) = 0$.

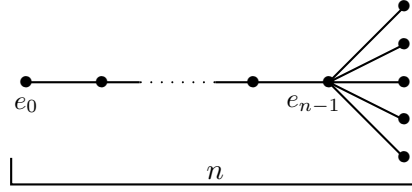
Remark 5.3. Note that for every collection of nonconstant polynomials P_1, \dots, P_n with simple roots, the surface $V = V_{P_1, \dots, P_n}$ admits two natural \mathbb{A}^1 -fibrations $pr_x : V \rightarrow \text{Spec}(\mathbb{C}[x])$ and $pr_{y_{n+1}} : V \rightarrow \text{Spec}(\mathbb{C}[y_{n+1}])$. By construction $pr_x : V \rightarrow \text{Spec}(\mathbb{C}[x])$ is a GDS over Z . However, it may happen that $pr_{y_{n+1}} : V \rightarrow \text{Spec}(\mathbb{C}[y_{n+1}])$ is not a GDS, due to the fact that the fiber $pr_{y_{n+1}}^{-1}(0)$ is not reduced (see example 5.5 below)

Proposition 4.3 implies the following result due to Bandman and Makar-Limanov [1, Theorem 3].

Theorem 5.4. *For a GDS $q : V \rightarrow Z$ with a trivial Makar-Limanov invariant, the following are equivalent.*

- 1) V admits a free $\mathbb{G}_{a,Z}$ -action.
- 2) The canonical sheaf ω_V of V is trivial.
- 3) V is isomorphic to an ordinary Danielewski surface $V_{P,1} \subset \mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ with the equation $xz - P(y) = 0$ for a certain nonconstant polynomial P with simple roots.

Proof. By virtue of proposition 4.3, (1) and (2) are equivalent. Clearly, (3) \Rightarrow (2). So it remains to show that (2) \Rightarrow (3). By theorem 3.1, there exists a fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$ and a Z -isomorphism $\psi : V \xrightarrow{\sim} V_{\mathfrak{g}}$. By theorem 5.4, \mathfrak{g} is a comb as V has a trivial Makar-Limanov invariant. Since ω_V is trivial, proposition 4.3 implies that all the leaves of Γ are at the same level $n \geq 0$. Therefore Γ has the following structure:



If $n = 0$ then $V_{\mathfrak{g}} \simeq \mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(A[X_0])$. Otherwise, letting $X_{e_{-1}} = X_0$, equations (5.2) simplify as

$$\begin{cases} hX_{e_i} &= X_{e_{i-1}} - lb(e_{i+1}) & 0 \leq i \leq n-2 \\ hX_{e_{n-1}} &= \prod_{f \in \text{Child}(e_{n-1})} (X_{e_{n-2}} - lb(f)) \end{cases},$$

and so, the statement follows. \square

By [8], any two free \mathbb{C}_+ -actions on an ordinary Danielewski surface $V \subset \text{Spec}(\mathbb{C}(x, y, z))$ with equation $xz - P(y) = 0$ are conjugated under the action of the automorphism group $\text{Aut}(V)$ of V . In particular, every two \mathbb{A}^1 -fibrations $q : V \rightarrow Z$ and $\tilde{q} : V \rightarrow \tilde{Z}$ are conjugated, in the sense that there exists an isomorphism $\bar{\theta} : Z \xrightarrow{\sim} \tilde{Z}$ and an automorphism $\theta : V \xrightarrow{\sim} V$ of V such that $\bar{\theta} \circ q = \tilde{q} \circ \theta$. In [2], Daigle and Russel give examples of nonconjugated \mathbb{A}^1 -fibrations on a log \mathbb{Q} -homology plane with a trivial Makar-Limanov invariant. In case that $q : V \rightarrow Z$ is a GDS, the following example shows that there may also exist nonconjugated \mathbb{A}^1 -fibrations on V .

Example 5.5. By example 1.16, the Bandman and Makar-Limanov surface $V \subset \text{Spec}(\mathbb{C}[x, y, z, u])$ with equations

$$xz = y(y^2 - 1), \quad yu = z(z^2 - 1), \quad xu = (y^2 - 1)(z^2 - 1)$$

is a GDS via $pr_x : V \rightarrow Z = \text{Spec}(\mathbb{C}[x])$. It is also a GDS via $pr_u : V \rightarrow \tilde{Z} = \text{Spec}(\mathbb{C}[u])$. The \mathbb{C} -algebra involution

$$\mathbb{C}[x, u][y, z] \rightarrow \mathbb{C}[x, u][y, z], \quad x \leftrightarrow u, \quad y \leftrightarrow z,$$

induces an automorphism $\theta : V \xrightarrow{\sim} V$. Letting $\bar{\theta} : Z \xrightarrow{\sim} \tilde{Z}$ be the isomorphism corresponding to the induced isomorphism $\mathbb{C}[u] \xrightarrow{\sim} \mathbb{C}[x]$, we conclude that $\bar{\theta} \circ pr_x = pr_u \circ \theta$.

Similarly, the surface $V' \subset \mathbb{C}[x, y, z, u]$ with equations

$$xz = y(y - 1), \quad yu = z^2, \quad xu = (y - 1)z$$

is a *GDS* $pr_x : V' \rightarrow Z = \text{Spec}(\mathbb{C}[x])$. The natural second \mathbb{A}^1 -fibration $pr_u : V' \rightarrow \tilde{Z}$ restricts to the trivial line bundle over $\tilde{Z} \setminus \{0\}$ but the fiber

$$pr_u^{-1}(0) \simeq \text{Spec}(\mathbb{C}[x, y, z] / (z^2, (y-1)z, xz - y(y-1)))$$

is not reduced. Therefore, there is no automorphism $\theta : V' \xrightarrow{\sim} V'$ satisfying $\bar{\theta} \circ pr_x = pr_u \circ \theta$.

Log \mathbb{Q} -homology planes with a trivial Makar-Limanov invariant.

A log \mathbb{Q} -homology plane is a normal affine surface S with log-terminal singularities and with vanishing singular homology groups $H_i(S, \mathbb{Q})$ for all $i > 0$. In this subsection we prove the following result (compare with [11, Theorem 3.4]).

Theorem 5.6. *Every log \mathbb{Q} -homology plane $S \not\cong \mathbb{A}_{\mathbb{C}}^2$ with a trivial Makar-Limanov invariant is isomorphic to the quotient of an ordinary Danielewski surface $V \subset \text{Spec}(\mathbb{C}[x, t, z])$ with equation $xz = t^n - 1$ by a \mathbb{Z}_m -action $(x, t, z) \mapsto (\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z)$, where ε is a primitive m -th root of unity, n divides m and $\gcd(q, m/n) = 1$.*

The proof is given in 5.7-5.8 below. Since S has a trivial Makar-Limanov invariant, it admits an \mathbb{A}^1 -fibration $\rho : S \rightarrow Y \simeq \text{Spec}(\mathbb{C}[y])$ such that all but possibly one closed fiber, say $\rho^{-1}(y_0)$, where $y_0 = (y) \in Y$, are isomorphic to $\mathbb{A}_{\mathbb{C}}^1$ (see e.g. [4, Proposition 2.15]). In turn, lemma 1.2 in [6] and [5, 2.5] imply that ρ restricts to a trivial line bundle over $Y_* = Y \setminus \{y_0\}$. By Theorem 4.3.1 in [10] p.231, $\rho^{-1}(y_0) = mC$, where $m \geq 1$ and $C \simeq \mathbb{A}_{\mathbb{C}}^1$. In particular, there exists k , $0 \leq k \leq m-1$, such that the canonical sheaf ω_S of S is isomorphic to $\mathcal{O}_S(kC)$.

5.7. If $m = 1$ then $S \simeq \mathbb{A}_Y^1$ (see e.g. [10, Theorem 4.3.1 p.231]). Otherwise, if $m \geq 2$ then we let $\theta : Z \simeq \mathbb{A}_{\mathbb{C}}^1 \rightarrow Y$ be a Galois covering of order m , unramified over Y_* and totally ramified over y_0 . Up to an isomorphism, we can suppose that θ is induced by the \mathbb{C} -algebras homomorphism $\theta^* : \mathbb{C}[y] \rightarrow \mathbb{C}[x, y] / (x^m - y) \simeq \mathbb{C}[x]$, where the Galois group \mathbb{Z}_m of m -th roots of unity acts on $\mathbb{C}[x, y]$ via $\varepsilon \cdot f(x, y) = f(\varepsilon^{-1}x, y)$. The group \mathbb{Z}_m acts on the normalization V of the fiber product $S \times_Y Z$ and $S \simeq V/\mathbb{Z}_m$. This surface V inherits an \mathbb{A}^1 -fibration $q : V \rightarrow Z$ with reduced fibers, induced by the projection $pr_Z : S \times_Y Z \rightarrow Z$. Letting $z_0 = (x) \in Z$, the group \mathbb{Z}_m acts transitively on the set of irreducible components C_1, \dots, C_n of $q^{-1}(z_0)$, where n divides m (see e.g. [6, Theorem 1.7]). Since $\rho : S \rightarrow Y$ restricts to a trivial line bundle over Y_* , we conclude that $q : V \rightarrow Z$ restricts to a trivial line bundle over $Z_* = Z \setminus \{z_0\}$ and so, $q : V \rightarrow Z$ is a *GDS* over Z . By construction, the quotient morphism $\phi : V \rightarrow S$ is étale over S_{reg} , whence is étale in codimension 1 as S is normal. Thus the canonical sheaf ω_V of V is isomorphic to $\phi^*(\omega_S) \simeq q^*(\mathcal{O}_Z(kz_0))$, hence is trivial. Moreover, we deduce from [12] that the Makar-Limanov invariant of V is trivial.

5.8. By theorem 5.4, V is isomorphic to an ordinary Danielewski surface $V_{P,1} \subset \text{Spec}(\mathbb{C}[x, t, z])$ with the equation $xz - P(t) = 0$ for a certain nonconstant polynomial P with n simple roots $t_1, \dots, t_n \in \mathbb{C}$. Since \mathbb{Z}_m acts transitively on the set of irreducible components of $q^{-1}(z_0)$ and t is locally constant on $q^{-1}(z_0)$ with the value t_i on C_i , $1 \leq i \leq n$, we can suppose that $t_{i+1} = \varepsilon^q t_i$ where ε is a primitive m -th root of unity and $\gcd(q, m/n) = 1$. Up to an isomorphism, we can further suppose that $t_1 = 1$ (see e.g. [8]). The canonical morphism $pr_{x,t} : V \rightarrow \mathbb{A}_{\mathbb{C}}^2 = \text{Spec}(\mathbb{C}[x, t])$ expresses V as the affine modification [7] of $\mathbb{A}_{\mathbb{C}}^2$ with the locus $(I, f) = ((x, P(t)), x)$. Clearly, $pr_{x,t}$ is \mathbb{Z}_m -equivariant when we equip $\mathbb{A}_{\mathbb{C}}^2$ with the \mathbb{Z}_m -action $(x, t) \mapsto (\varepsilon x, \varepsilon^q t)$. As $\Gamma(V, \mathcal{O}_V) \simeq \mathbb{C}[x, t, P(t)]/x$, we conclude that V is isomorphic to the ordinary Danielewski surface with equation $xz = t^n - 1$, equipped with the \mathbb{Z}_m -action $(x, t, z) \mapsto (\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z)$.

Remark 5.9. Note that $S \simeq V/\mathbb{Z}_m$ is nonsingular if and only if $n = m$. On the other hand, if $n = 1$ then S is isomorphic to a cyclic quotient of $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[x, t])$ by the \mathbb{Z}_m -action $(x, t) \mapsto (\varepsilon x, \varepsilon^q t)$, where $\gcd(q, m) = 1$. In particular, every affine toric surface nonisomorphic to $\mathbb{C} \times \mathbb{C}^*$ has a trivial Makar-Limanov invariant, see also [4, Proposition 2.11] for a geometrical proof.

Corollary 5.10. *Every log \mathbb{Q} -homology plane with a trivial Makar-Limanov invariant admits a nontrivial \mathbb{C}^* -action.*

Proof. Indeed, every ordinary Danielewski surface $V_{P,1} \subset \mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, t, z])$ with equation $xz - P(t) = 0$ admits a nontrivial \mathbb{C}^* -action induced by the \mathbb{C}^* -action on $\mathbb{A}_{\mathbb{C}}^3$ $\lambda \cdot (x, t, z) = (\lambda x, y, \lambda^{-1} z)$, $\lambda \in \mathbb{C}^*$. In case that $P(t) = t^n - 1$, this action commutes with the \mathbb{Z}_m -action $(x, t, z) \mapsto (\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z)$ and so, the quotient $S = V/\mathbb{Z}_m$ inherits a nontrivial \mathbb{C}^* -action. \square

REFERENCES

1. T. Bandman and L. Makar-Limanov, *Affine surfaces with $AK(S) = \mathbb{C}$* , Michigan J. Math. **49** (2001), 567–582.
2. D. Daigle and P. Russel, *On log \mathbb{Q} -homology planes and weighted projective planes*, To appear in Canadian J. Math.
3. W. Danielewski, *On a cancellation problem and automorphism groups of affine algebraic varieties*, Preprint Warsaw, 1989.
4. A. Dubouloz, *Completions of normal affine surfaces with a trivial Makar-Limanov invariant*, To appear in Michigan J. Math. (2004), Prepub. Inst. Fourier n°579, 2002.
5. A. Dubouloz, *Generalized Danielewski Surfaces*, E-print. math.AG/0401225 (2004), 26p.
6. K.H. Fieseler, *On complex affine surfaces with \mathbb{C}_+ -actions*, Comment. Math. Helvetici **69** (1994), 5–27.
7. S. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transformation Groups **4** (1999), 53–95.
8. L. Makar-Limanov, *On groups of automorphisms of a class of surfaces*, Israel J. Math. **69** (1990), 250–256.
9. L. Makar-Limanov, *On the group of automorphisms of a surface $x^n y = p(z)$* , Israel J. Math. **121** (2001), 113–123.
10. M. Miyanishi, *Open Algebraic Surfaces*, vol. 12, CRM Monograph Series, 2001.
11. M. Miyanishi and K. Masuda, *The additive group actions on \mathbb{Q} -Homology planes*, Ann. Inst Fourier **53** (2003), no. 2, 429–464.
12. W. Vasconcelos, *Derivations of commutative noetherian rings*, Math. Z. **112** (1969), 229–233.