

On the Cyclotomic Quantum Algebra of Time Perception

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Abstract.

I develop the idea that time perception is the quantum counterpart to time measurement. Phase-locking and prime number theory were proposed as the unifying concepts for understanding the optimal synchronization of clocks and their $1/f$ frequency noise. Time perception is shown to depend on the thermodynamics of a quantum algebra of number and phase operators already proposed for quantum computational tasks, and to evolve according to a Hamiltonian mimicking Fechner's law. The mathematics is Bost and Connes quantum model for prime numbers. The picture that emerges is a unique perception state above a critical temperature and plenty of them allowed below, which are parametrized by the symmetry group for the primitive roots of unity. Squeezing of phase fluctuations close to the phase transition temperature may play a role in memory encoding and conscious activity.

1. Introduction

In a recent issue of this journal B. Flanagan [1] discusses parallels between the perception of color and its concomitant quantum field. We follow that line of reasoning by proposing the concept of phase-locking to parallel classical measurements on oscillators with the quantum perception of time. Period measurements of a test oscillator against a reference one is usually performed close to baseband, thanks to a non-linear mixing element and a low pass filter. The resulting beat note in units of the frequency of the reference oscillator results from the continued fraction expansion of the input frequency ratio. The beat frequency exhibits variability, with $1/f$ power spectrum, that we explained from phase-locking of the input oscillators. We could model the effect by considering a discrete coupling coefficient versus time, related to the logarithm of prime numbers and also to the Riemann zeta function and its critical zeros [2].

Time evolution in human classical oscillators such as the circadian rhythm in plants, the heart rate of melatonin secretion should obey the same rules because they are slaved to the lightning environment or to internal pacemakers. But does time perception resort to the arithmetic above? Our postulate is that our mind still uses phase-locking, but in a discrete algebraic way, from quantum finite fields lying in the brain. The object under control by mental states would be $\mathcal{Z}_q = \mathcal{Z}/q\mathcal{Z}$, the ring of integers of modulo q , with q the finite dimension of the quantum field. In particular we claim the ability of our mind to lock to the largest cyclic subgroup in \mathcal{Z}_q . The human ability to perceive the greatest common divisor in the frequencies of two sounds, instead of their beat frequency, is well known, as is the ability to implicitly manage with continued fraction expansions in the musical design of well tempered scales [3]. On classical computers these tasks require a polynomial time. In contrast, finding the primitive roots of an algebraic equation $a^\alpha = 1 \pmod{q}$, on which periodicity query depends, requires exponential time. Thus our intuitive sense of time, of prime numbers, of primitive roots should result from the ability of our mind to perform some sort of quantum computation. So the $1/f$ noise effects observed in human cognition [4] would be related to the $1/f^\gamma$ noise observed in \mathcal{Z}_q about the period of its largest cyclic subgroup.

The best possible introduction to our research is still in the visionary Poincaré words [5]:

The Physical Continuum. We are next led to ask if the idea of the mathematical continuum is not simply drawn from experiment. If that be so, the rough data of experiment, which are our sensations, could be measured. We might, indeed, be tempted to believe that this is so, for in recent times there has been an attempt to measure them, and a law has even been formulated, known as Fechner's law, according to which sensation is proportional to the logarithm of the stimulus. But if we examine the experiments by which the endeavor has been made to establish this law, we shall be led to a diametrically opposite conclusion. It has, for instance, been observed that a weight A of 10 grammes and a weight B of 11 grammes produced identical sensations, that the weight B could no longer be distinguished from a weight C of 12 grammes, but

that the weight A was readily distinguished from the weight C . Thus the rough results of the experiments may be expressed by the following relations: $A = B, B = C, A < C$, which may be regarded as the formula of the physical continuum. But here is an intolerable disagreement with the law of contradiction, and the necessity of banishing this disagreement has compelled us to invent the mathematical continuum. We are therefore forced to conclude that this notion has been created entirely by the mind, but it is experiment that has provided the opportunity. We cannot believe that two quantities which are equal to a third are not equal to one another, and we are thus led to suppose that A is different from B , and B from C , and that if we have not been aware of this, it is due to the imperfections of our senses.

The Creation of the Mathematical Continuum: First Stage. So far it would suffice, in order to account for facts, to intercalate between A and B a small number of terms which would remain discrete. What happens now if we have recourse to some instrument to make up for the weakness of our senses? If, for example, we use a microscope? Such terms as A and B , which before were indistinguishable from one another, appear now to be distinct: but between A and B , which are distinct; is intercalated another new term D , which we can distinguish neither from A nor from B . Although we may use the most delicate methods, the rough results of our experiments will always present the characters of the physical continuum with the contradiction which is inherent in it. We only escape from it by incessantly intercalating new terms between the terms already distinguished, and this operation must be pursued indefinitely. We might conceive that it would be possible to stop if we could imagine an instrument powerful enough to decompose the physical continuum into discrete elements, just as the telescope resolves the Milky Way into stars. But this we cannot imagine; it is always with our senses that we use our instruments; it is with the eye that we observe the image magnified by the microscope, and this image must therefore always retain the characters of visual sensation, and therefore those of the physical continuum.

The goal of the paper is to connect Poincaré physical continuum to the remarkable quantum phase model due to J.B. Bost and A. Connes [6]: *In this paper we construct a C^* -dynamical system whose partition function is the Riemann ζ function. It admits the ζ function as partition function and the Galois group $\text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ of the cyclotomic extension \mathbb{Q}^{cycl} of \mathbb{Q} as symmetry group. Moreover, it exhibits a phase transition with spontaneous symmetry breaking at inverse temperature $\beta = 1$. The original motivation for these results comes from the work of B. Julia.*

2. Classical phase-locking

Poincaré physical continuum can be figured out by the concept of phase-locking of clocks. The oscillators in our body, from the skin to the nerve cells, are engaged in our sensation of weight. Two groups of oscillators compare the physical strengths of the stimuli and react as A and B by locking at values $\frac{A}{B} = \frac{1}{1}$, or $\frac{A}{B} = \frac{1}{2} \dots$, or more generally at some simple rational ratio $\frac{A}{B} = \frac{p}{q}$, where p and q are coprime to each

other, i.e. with greatest common divisor $(p, q) = 1$. One model of the phase-locking phenomenon between coupled oscillators of angular frequency ω_0 and ω is the Adler's equation [2]

$$\frac{d\Phi}{dt} + K \sin \Phi = \Delta\omega. \quad (1)$$

It is achieved for example from a non-linear phase detector using electronic diode mixers and a sustaining loop which shifts the frequency of a voltage controlled oscillator. The same physical mechanism also occurs in phase-locked lasers and in feedbacks with biologic oscillators. The symbol Φ in (1) is the phase difference between the oscillators, $\Delta\omega = \omega - \omega_0$ is the detuning frequency and K is the strength of the interaction. This model leads to a mean frequency $\langle \dot{\Phi} \rangle = 0$ which is zero inside the phase-locked zone $|\Delta\omega| < 2K$ and which is $\langle \dot{\Phi} \rangle = \Delta\omega(1 - K^2/(\Delta\omega)^2)^{1/2}$ outside that zone. Note that if the oscillators have frequencies far apart, then $\langle \dot{\Phi} \rangle \rightarrow \Delta\omega$, so that the strength of "sensation" is proportional to the "physical" strength. At that stage the model can only discriminate if two physical strengths are equal or not. For a better operation of phase-locking the harmonics generated in the interaction have to be accounted for. Experimentally one finds $1/f$ frequency fluctuations about the unperturbed signal. Their range can be found easily by differentiating the shifted frequency [2]. Technical details of the frequency lockings between interacting harmonics need to have recourse to the theory of continued fraction expansions. A simple model of the phase-lockings is the so-called Arnold map

$$\Phi_{n+1} = \Phi_n + 2\pi\Omega - c \sin \Phi_n, \quad (2)$$

where the coupling coefficient is $c = \frac{K}{\omega_0}$. Such a non-linear map is studied by introducing the winding number $\nu = \lim_{n \rightarrow \infty} (\Phi_n - \Phi_0)/(2\pi n)$. The limit exists everywhere as long as $c < 1$. The curve ν versus Ω is a devil's staircase with steps attached to the rational values $\Omega = \frac{p}{q}$ and the width increases with the coupling coefficient c . The structure in steps corresponds to the Poincaré physical continuum.

It is well known that the phase-locking zones may overlap if $c > 1$ leading to chaos from quasi-periodicity [2]. However, this type of chaos doesn't resort to $1/f$ noise since the phase-locked loops generally operate at quite small values of the coupling c . To account for the $1/f$ noise we refined Arnold's equation by introducing a discrete time dependance on c able to reflect the tiny energy exchanges occurring from time to time and needed for phase synchronization. We postulated a modulation of c by the Mangoldt function $\Lambda(n)$ of prime number theory. It is defined as

$$\Lambda(n) = \begin{cases} \ln b & \text{if } n = b^k, b \text{ a prime,} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Mangoldt function arises in prime number theory as the logarithmic derivative of Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$, $\Re(s) \geq 1$ since $d \ln \zeta(s)/ds = \sum_{n \geq 1} \Lambda(n)n^{-s}$. Riemann zeta function can be extended analytically to the whole plane of the complex variable s , except for a pole at $s = 1$. The zeta function also shows trivial zeros on the negative axis, at $s = -2l$, l integer, and an infinite number of very irregularly

spaced zeros on the vertical axis $\Re s = \frac{1}{2}$. Riemann hypothesis states that indeed all zeros are located on that critical axis. In addition the averaged Mangoldt function can be expressed explicitly in terms of critical zeros. Numerically the averaged Mangoldt function deviates from 1 with a random looking term, having a $1/f$ type power spectral density. This fact motivated us to put it as a modulating term of c . In Sect.3.3 below, it will be shown how the pole of the Riemann zeta function is associated with the critical temperature T_c in the phase transition model of time perception, and how phase fluctuations of the Mangoldt function type occur close to T_c .

3. Quantum phase-locking

3.1. Quantum phase states

Our hypothesis is that mental states of human perception still rely on phase-locking but with an important feature added: the discreteness of phase states, which adds to the discreteness of frequency ratios between the interacting oscillators. What characterizes best the quantum time perception in contrast to classical time measurement is the existence of quantum operators sustaining this algebra of phase states, and a properly chosen evolution Hamiltonian operator. Otherwise our approach is quantum field theory and quantum statistical mechanics.

Let us start with the familiar Fock states of the quantized electromagnetic field (the photon occupation states) $|n\rangle$ which live in an infinite dimensional Hilbert space. They are orthogonal to each other: $\langle n|m\rangle = \delta_{mn}$, where δ_{mn} is the Dirac symbol. The states form a complete set: $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$.

The annihilation operator \hat{a} removes one photon from the electromagnetic field

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad n = 1, 2, \dots \quad (4)$$

Similarly the creation operator \hat{a}^\dagger adds one photon: $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $n = 0, 1, \dots$. These operators follow the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The operator $N = \hat{a}^\dagger\hat{a}$ represents the particle number operator and satisfies the eigenvalue equation $N|n\rangle = n|n\rangle$.

The algebra can be simplified by removing the normalization factor \sqrt{n} in (4) defining thus the shift operator E as

$$E|n\rangle = |n-1\rangle, \quad n = 1, 2, \dots \quad (5)$$

Similarly $E^\dagger|n\rangle = |n+1\rangle$, $n = 0, 1, \dots$. The operator E has been known as the (exponential) phase operator

$$E = e^{i\Phi} = (N+1)^{-1/2}\hat{a} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \quad (6)$$

The problem with this definition is that the operator E is only one-sided unitary since $EE^\dagger = 1$, $E^\dagger E = 1 - |0\rangle\langle 0|$, due to the vacuum-state projector $|0\rangle\langle 0|$. The problem of defining a Hermitian quantum phase operator was already encountered by the founder

of quantum field theory P.A.M. Dirac [7]. The eigenvectors of E have been known as the Susskind-Glogower phase states; they satisfy the eigenvalue equation

$$E|\psi\rangle = e^{i\psi}|\psi\rangle, \quad \text{with } |\Psi\rangle = \sum_{n=0}^{\infty} e^{in\psi}|n\rangle. \quad (7)$$

They are non-orthogonal to each other and form an overcomplete basis which solves the identity operator since $\frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi |\psi\rangle\langle\psi| = 1$. When properly normalized the phase states $|\psi\rangle$ in (7) are a special case of the $SU(1, 1)$ Perelomov coherent states [8],[9]. The operator $\cos\Phi = \frac{1}{2}(E + E^\dagger)$ is used in the theory of Cooper pair box with a very thin junction when the junction energy $E_J \cos\Phi$ is higher than the electrostatic energy [10].

Further progress in the definition of phase operator was obtained by Pegg and Barnett [7] in the context of a finite Hilbert space. From now we consider a finite basis $|n\rangle$ of number states, with the index $n \in \mathcal{Z}_q = \mathcal{Z}/q\mathcal{Z}$, the ring of integers modulo q . The phase states are now defined from the discrete Fourier transform (or more precisely from the quantum Fourier transform since the superposition is on Fock states not on real numbers)

$$|\theta_p\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp(2i\pi \frac{p}{q} n) |n\rangle. \quad (8)$$

The states are eigenstates of the Hermitian phase operator Θ_q such that $\Theta_q|\theta_p\rangle = \theta_p|\theta_p\rangle$ where $\Theta_q = \sum_{p=0}^{q-1} \theta_p |\theta_p\rangle\langle\theta_p|$ and $\theta_p = \theta_0 + 2\pi p/q$ with θ_0 a reference angle. It is implicit in the definition (8) that the Hilbert space is of finite dimension q . The states $|\theta_p\rangle$ form an orthonormal set and in addition the projector over the subspace of phase states is $\sum_{p=0}^{q-1} |\theta_p\rangle\langle\theta_p| = 1_q$, where 1_q is the unitary operator. Given a state $|F\rangle$ one can write a probability distribution $|\langle\theta_p|F\rangle|^2$ which may be used to compute various moments, e.g. expectation values, variances. The key element of the formalism is that first the calculations are done in the subspace of dimension q , then the limit $q \rightarrow \infty$ is taken.

3.2. Primitive roots and $1/f$ noise

Let us consider the ring \mathcal{Z}_q . An important point about this set is that it is endowed with cyclic properties. If $q = p$, a prime number, then the period is $p - 1$; in any case they are cyclic subgroups in \mathcal{Z}_q . This can be shown by the examination of the algebraic equation

$$a^\alpha = 1 \pmod{q}. \quad (9)$$

Let us ask the question: what is the cycle of largest period in \mathcal{Z}_q ? To find it one should look at the primitive roots which are defined as the solution of (9) such that the equation is wrong for any $1 \leq \alpha < q - 1$ and true only for $\alpha = q - 1$. If $q = b$, a prime number, and $b = 7$, the largest period is thus $\phi(b) = b - 1 = 6$, and the cycle is as given in Table 1. If $q = 2$, or 4, or $q = b^r$, a power of a prime number > 2 , or $q = 2b^r$, twice the power of a prime number > 2 , then a primitive root exists, and the largest cycle in the group is $\phi(q)$. For example $a = 2$ and $q = 3^2$ leads to the period $\phi(9) = 6 < q - 1 = 8$, as it is shown in Table 2.

Table 1. $(Z/7Z)^*$ is a cyclic group of order $\phi(7) = 6$.

α	1	2	3	4	5	6	7	8
3^α	3	2	6	4	5	1	3	2

Table 2. $(Z/3^2Z)^*$, is a cyclic group of order $\phi(9) = 6$.

α	1	2	3	4	5	6	7	8
2^α	2	4	8	7	5	1	2	4

Otherwise there is no primitive root. The period of the largest cycle in $Z_q = Z/qZ$ can still be calculated and is called the Carmichael Lambda function $\lambda(q)$. It is shown in Table 3 for the case $a = 3$ and $q = 8$. It is $\lambda(8) = 2 < \phi(8) = 4 < 8 - 1 = 7$.

Table 3. $(Z/8Z)^*$ has a largest cyclic group of order $\lambda(8) = 2$.

α	1	2	3	4	5	6	7	8
3^α	3	1	3	1	3	1	3	1

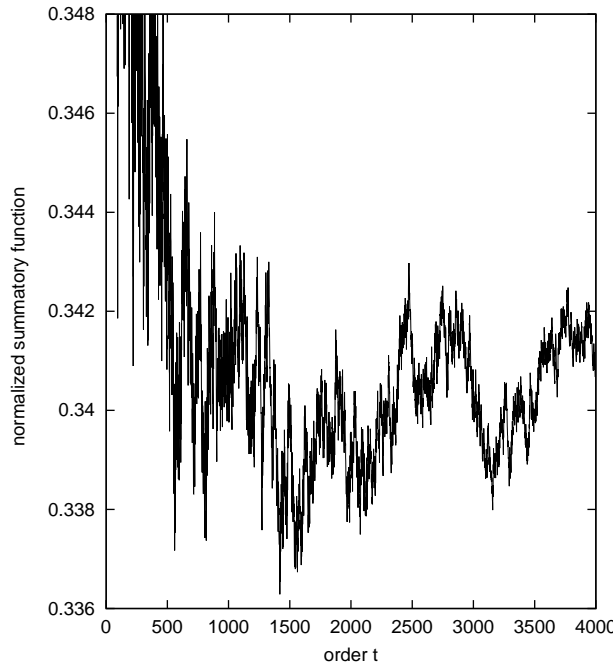


Figure 1. Normalized Carmichael lambda function: $(\sum_1^t \lambda(n))/t^{1.90}$.

Fig. 1 shows the properly normalized period for the cycles in $Z_q = Z/qZ$. Its fractal character can be appreciated by looking at the corresponding power spectral density (FFT) shown in Fig. 2. It has the form of a $1/f^\gamma$ noise, with $\gamma = 0.70$. For

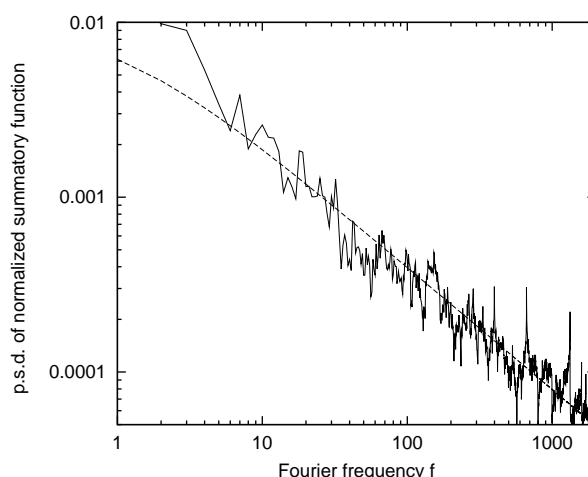


Figure 2. FFT of the normalized Carmichael lambda function. The straight line has slope -0.70 .

a more refined link between primitive roots, cyclotomy and Ramanujan sums see also [11].

Let us remind that the algebraic equation (9) is used as a numerical trap door for public key encryption (the RSA cryptosystem) [3], p.125. While it is easy to multiply 1000-digit numbers within a time of few milliseconds on a modern classical computer, the opposite task of factorizing is impossible in a reasonable time. This impossibility would be lifted on a quantum computer thanks to Schor's algorithm of finding periods efficiently in \mathcal{Z}_q (see Sect.3.3).

3.3. Quantum computation and the Bost and Connes algebra

The quantum operator corresponding to (9) is another shift μ_a in the space of number states, where $a \in \mathcal{Z}_q$ and $(a, n) = 1$. It is defined as

$$\mu_a |n\rangle = |an(\bmod q)\rangle, \quad (10)$$

It is multiplicative: $\mu_k \mu_l = \mu_{kl}$ and $\mu_k^* \mu_k = 1$, $k, l \in \mathcal{N}$. The eigenvalues and eigenvectors of the operator (10) allow to define the order $r = \text{ord}_q(a)$ of $a \pmod{q}$ which is the smallest exponent such that (9) is satisfied. When the order is $r = \phi(q)$ then a is a primitive root. This results from Euler theorem which states that $a^{\phi(q)} = 1 \pmod{q}$. If the order is the same for any a such that $(a, q) = 1$ it is also called a universal exponent. Thus $\phi(q)$ is a universal exponent. It is the smallest one if a is a primitive root, otherwise there is a smaller one which is Carmichael lambda function introduced above. Let us summarize

$$\text{ord}_q(a) \leq \lambda(q) \leq \phi(q) \leq q - 1. \quad (11)$$

The eigenvalues of operator (10) are of the form $\exp(2i\pi\frac{k}{r})$ and the corresponding eigenvectors are given as a quantum Fourier transform [12], p. 22.

$$|u_k\rangle = \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} \exp(-2i\pi\frac{kj}{r}) |a^j \bmod q\rangle. \quad (12)$$

We observe that (12) acts on the subspace of dimension r (the least period) from the powers of a in (9). One sees that the u_k 's acquire the status of phase states in the cyclic subspace of Z_q of dimension r .

The operator μ_a is used in the context of quantum computation in relation to Schor's algorithm [12]. The goal is to factor integers efficiently (in polynomial time instead of the exponential time needed on the classical computer). It is in fact related to the efficient estimation of the period r taken from the eigenvalue $\exp(2i\pi\frac{k}{r})$, and to the efficient implementation of quantum Fourier transforms on the quantum computer.

The shift operator (10) which quantizes (9) plays the role of multiplication in the language of operators. The addition operator is easier to define

$$e_p|n\rangle = \exp(2i\pi\frac{p}{q}n)|n\rangle. \quad (13)$$

It encodes the individuals in the quantum Fourier transform (8). These individuals are eigenvalues of the operator e_p . One gets $e_0 = 1$, $e_p^* = e_{-p}$, $e_l e_m = e_{l+m}$. Both operators μ_a and e_p form an algebra $A = (\mu_a, e_p)$. It was used by Bost and Connes [6],[13] to build a remarkable quantum statistical model undergoing a phase transition.

3.4. The "quantum Fechner law" and its thermodynamics

Let us now make use of the concepts of quantum statistical mechanics. Given an observable Hermitian operator M and a Hamiltonian H_0 one has the evolution $\sigma_t(M)$ versus time t

$$\sigma_t(M) = e^{itH_0} M e^{-itH_0}, \quad (14)$$

and the expectation value of M at inverse temperature $\beta = \frac{1}{kT}$ is the unique Gibbs state

$$\text{Gibbs}(M) = \text{Trace}(M e^{-\beta H_0}) / \text{Trace}(e^{-\beta H_0}). \quad (15)$$

For the more general case of an algebra of observables A , the Gibbs state is replaced by the so-called Kubo-Martin-Schwinger (or KMS_β) state. One introduces a 1-parameter group σ_t of automorphisms of A so that for any $t \in \mathbb{R}$ and $x \in A$

$$\sigma_t(x) = e^{itH_0} x e^{-itH_0}, \quad (16)$$

but the equilibrium state remains unique only if some conditions regarding the evolution of the operators $x \in A$ are satisfied: the so-called KMS conditions [6]. It happens quite often that there is a unique equilibrium state above a critical temperature T_c and a coexistence of many equilibrium states at low temperature $T < T_c$. There is thus a spontaneous symmetry breaking at the critical temperature T_c . A simple example is the phase diagram for a ferromagnet. At temperature larger than $T \simeq 10^3 K$ the disorder dominates and the thermal equilibrium state is unique, while for $T < T_c$ the individual

magnets tend to align each other, which in the three dimensional space \mathcal{R}^3 of the ferromagnet yields a set of extremal equilibrium phases parametrized by the symmetry group $SO(3)$ of rotations in \mathcal{R}^3 .

In Bost and Connes approach the Hamiltonian operator is given as the logarithm of the number operator $H_0 = \ln N$, i.e. it is a kind of “quantum Fechner law” since the “sensor” operator H_0 is the logarithm of the “physical” operator N , in contrast to the ordinary Hamiltonian $H_0 = N + \frac{1}{2}$ of the harmonic oscillator. The dynamical system (A, σ_t) is defined by its action on the Fock states

$$H_0|n\rangle = \ln n|n\rangle. \quad (17)$$

Using the relations $e^{-\beta H_0}|n\rangle = e^{-\beta \ln n}|n\rangle = n^{-\beta}|n\rangle$, it follows that the partition function of the model at the inverse temperature β is

$$\text{Trace}(e^{-\beta H_0}) = \sum_{n=1}^{\infty} n^{-\beta} = \zeta(\beta), \quad \Re\beta > 1. \quad (18)$$

where $\zeta(\beta)$ is the Riemann zeta function introduced at the end of Sect.2. Applying (16) to the Hamiltonian H_0 in (17) one gets

$$\sigma_t(\mu_a) = a^{it}\mu_a \quad \text{and} \quad \sigma_t(e_p) = e_p. \quad (19)$$

One observes that under the action of σ_t the additive phase operator (13) is invariant, while the multiplicative shift operator (10) oscillates in time at angular frequency $\ln a$. It is found in [6],[13] that the pole $\beta = 1$ of the Riemann zeta function separates two dynamical regimes.

In the high temperature regime $0 < \beta \leq 1$ there is a unique KMS_β state of the dynamical system (A, σ_t) . The expectation value $\psi_\beta(\frac{p}{q})$ for the irreducible fractions $\frac{p}{q}$, $(p, q) = 1$ is

$$\psi_\beta\left(\frac{p}{q}\right) = \prod_{\substack{b \text{ prime} \\ k_b \neq 0}} b^{-k_b \beta} \frac{1 - b^{\beta-1}}{1 - b^{-1}}, \quad (20)$$

and the k_b are the exponents in the unique prime decomposition of the denominator q

$$q = \prod_{p \text{ prime}} p^{k_p}. \quad (21)$$

In the low temperature regime $\beta > 1$ things become more involved. The KMS_β state is no longer unique, as it is in the case of the low temperature ferromagnet. The symmetry group G of the dynamical system (A, σ_t) is a subgroup of the automorphisms of A which commutes with σ_t

$$w \circ \sigma_t = \sigma_t \circ w, \quad \forall w \in G, \quad \forall t \in \mathcal{R}, \quad (22)$$

where \circ means the composition law in G . When the temperature is high the system is disordered enough that there is no interaction between its constituents and the equilibrium state of the system does not see the action of the symmetry group G , which explains why the thermal state is unique. At lower temperature $T < T_c$ the constituents of the system may interact. The symmetry group G then permutes a family of extremal KMS_β states generating the possible states of the system after phase transition. The

symmetry group for the algebra $A = (\mu_a, e_p)$ and the “quantum Fechner Hamiltonian” H_0 in (17) is the Galois group $G = \text{Gal}(\mathcal{Q}^{\text{cycl}}/\mathcal{Q})$ of the cyclotomic extension $\mathcal{Q}^{\text{cycl}}$ of \mathcal{Q} . The field $\mathcal{Q}^{\text{cycl}}$ is the one generated on \mathcal{Q} by all the roots of unity. The Galois group G is the automorphism group of $\mathcal{Q}^{\text{cycl}}$ which preserves the sub-field \mathcal{Q} . It is isomorphic to $\mathcal{Z}_q = (\mathcal{Z}/q\mathcal{Z})^*$ which is the group of integers modulo q with the zero element removed.

One immediately sees that the action of the Galois group G commutes with the 1-parameter group defined in (19). Hence G permutes the extremal KMS_β states of (A, σ_t) . To describe these thermal states for $\beta > 1$ one starts from the action of operators μ_a and e_p on Fock states as given in (10) and (13), and one allows the permutation by the symmetry group G . For each $w \in G$, one has a representation π_w of the algebra A so that

$$\pi_w(\mu_a)|n\rangle = |an(\text{mod } q)\rangle, \quad (23)$$

$$\pi_w(e_p)|n\rangle = w(\exp(2i\pi n \frac{p}{q}))|n\rangle, \quad (24)$$

and it is shown in [6] that the state

$$\phi_{\beta,w}(x) = \zeta(\beta)^{-1} \text{Trace}(\pi_w(x)e^{-\beta H_0}), \quad x \in A \quad (25)$$

is a KMS_β state for (A, σ_t) . The action of G on A induces an action on these thermal states which permutes them. For $\beta > 1$ the expectation value acting on the phase operator e_p may be written as $\text{KMS}(e_p) = q^{-\beta} \prod_{\substack{p \text{ divides } q \\ p \text{ prime}}} \frac{1-p^{\beta-1}}{1-p^{-1}}$. But it can be also found that formula (20) is valid in the whole range of temperatures $0 \leq \beta \leq \infty$.

Plotting the thermal state (Fig. 3) one observes the following asymptotes. In the high temperature limit $\beta=0$ the thermal state tends to 1; in the low temperature range $\beta \gg 1$ the thermal state is well approximated by the function $\mu(q)/\phi(q)$, where the Möbius function $\mu(q)$ is 0 if the prime decomposition of q contains a square, 1 if $q = 1$ and $(-1)^k$ if q is the product of k distinct primes. Close to the critical temperature at $\beta = 1$ the fluctuations are squeezed and the thermal state is quite close to $\epsilon\Lambda(q)/q$, when $\epsilon = 1 - \beta \rightarrow 0$, where $\Lambda(q)$ is the Mangoldt function defined in (3). It is an unexpected surprise that phase fluctuations at the critical region of the quantum perception model resembles the ones of the classical phenomenological model of time measurements.

It is shown in our earlier papers [2], in eq. (38) that the normalized Möbius function $\mu(q)/\phi(q)$ is the dual to the Mangoldt function $\Lambda(n)$ when one projects on to the basis of Ramanujan sums. Ramanujan sums $c_q(n)$, which are defined as the sums of n^{th} powers of primitive roots of unity, generalize the cosine function, being quasi-periodic in n and aperiodic in q . The thermal states in the transition region can thus be considered as an unfolding through the dimensions q of the unique high temperature thermal state. This also suggests to consider the inverse temperature β and the dimension q of the Hilbert space as complementary to each other in the Heisenberg sense.

3.5. Time perception and memory recording

We tried to make clear that time perception is the quantum counterpart to time measurements. The quantum algebra of shift and clock operators together with the

“quantum Fechner law” lead to well defined thermal perception states, that should play some role in memory encoding and conscious activity. There seems to be a growing consensus that the mental activity in brain doesn’t relate to classical synchronization rules in neurons, but to some other non local mechanism at a smaller scale, possibly at the mesoscopic scale of microtubules within neurons [14],[15],[16].

Quantum field theory of many body systems and the mechanism of spontaneous breakdown of symmetry was already proposed to accommodate some features of the conscious brain function. Memory recording was represented by the ordering induced in the ground state by the condensation of modes after symmetry breaking of the rotational symmetry of the electrical dipoles of the water molecules. The recall mechanism was described by the excitation of these collective modes from the ground state under the action of an external input similar to the one which was previously produced by memory encoding. In a dissipative extension of the model two modes squeezed coherent states of the radiation field were found to play a role in extending considerably the memory capacity [17].

The set of collective modes in the ground state of Vitiello’s model is replaced in our model by the set of thermal phase states (the extremal KMS_β states). Transitions among the vacua are replaced by transitions among the KMS_β states under the action of the cyclotomic group. Vitiello’s model can be seen as an attempt to include the environment in the quantum model of brain, thus the idea of two entangled modes, one in the brain, the other one in the environment. Our approach seems more general being able to describe the environment thanks to a continuous temperature parameter β , and putting on the scene more general squeezed thermal states at the transition temperature. The discrete parameter q is the dimension of the Hilbert space of Fock states. It plays the role of a bandwidth in conscious and unconscious activity. A very wide bandwidth $q \rightarrow \infty$ would mean meditation, a small bandwidth would be associated to precise memory encoding, jumps between dimensions, with tiny energy needed, thanks to the ridges of the KMS versus q and β diagram (see. Fig. 3). Another closely related view of time perception based on cyclic properties of Galois Fields $GF(b^k)$ has been proposed [18]. This pencil model sheds new light on profoundly distorted perceptions of time characterizing a number of mental psychoses, drug-induced states, as well as many other “altered” states of consciousness. The high temperature thermal state could be associated to the ordinary sense of time, and the low temperature thermal states to altered states of time perception.

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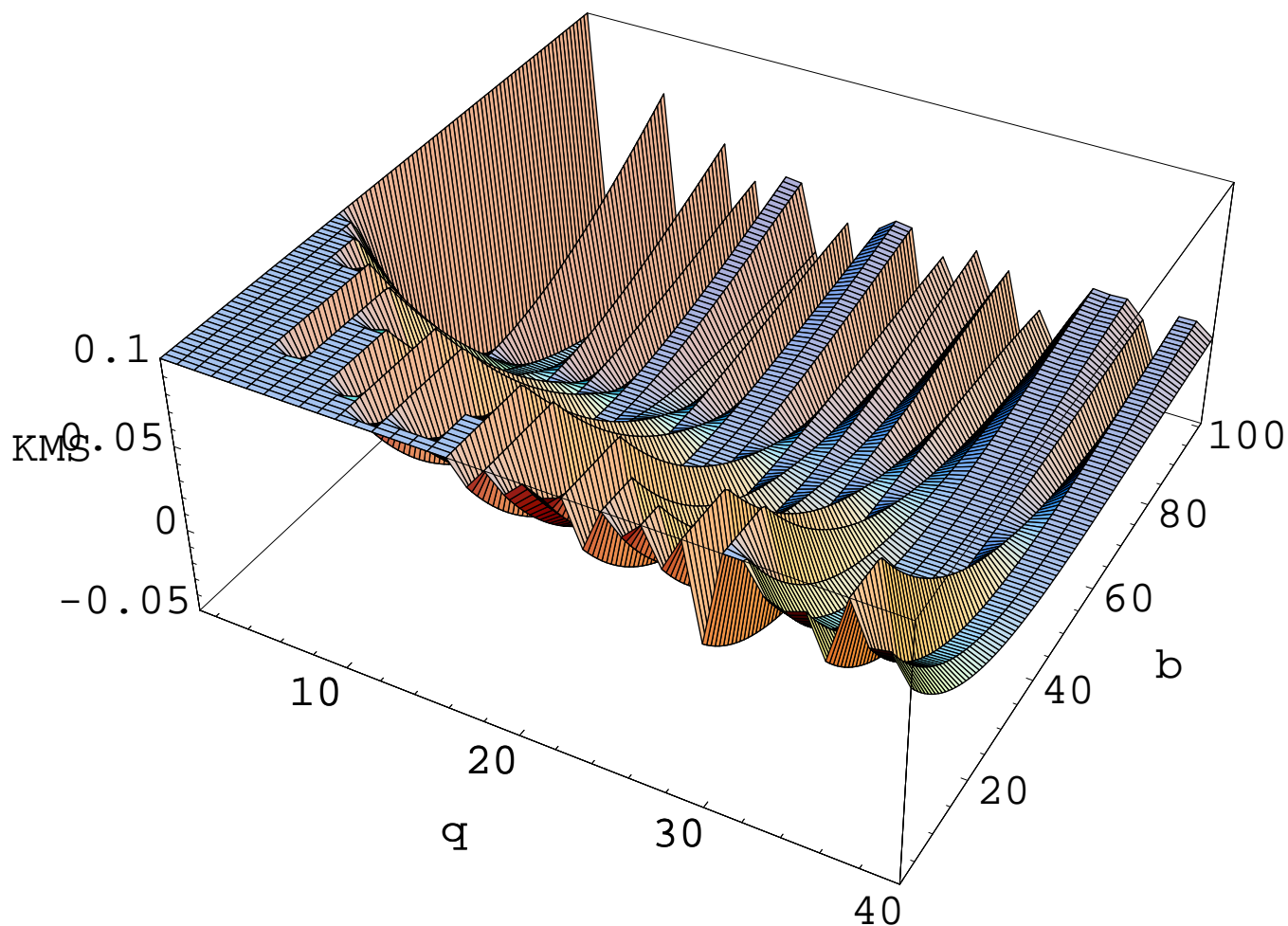


Figure 3. The KMS thermal state as given from (20). The inverse temperature scale is from $\beta = 0.5$ to 1.5. The dimension is from $q = 1$ to 40 as shown. Some parts of the graphics are truncated in the high temperature region $\beta < 1$. Thermal fluctuations are clearly reduced at the inverse critical temperature $\beta = 1$.

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