

# LIE GROUPS AND MECHANICS, AN INTRODUCTION

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ABSTRACT. The aim of this paper is to present aspects of the use of Lie groups in mechanics. We start with the motion of the rigid body for which the main concepts are extracted. In a second part, we extend the theory for an arbitrary Lie group and in a third section we apply these methods for the diffeomorphism group of the circle with two particular examples: the Burger equation and the Camassa-Holm equation.

## INTRODUCTION

The aim of this article is to present aspects of the use of Lie groups in mechanics. In a famous article [1], Arnold showed that the motion of the rigid body and the motion of an incompressible, inviscid fluid have the same structure. Both correspond to the geodesic flow of a one-sided invariant metric on a Lie group. From a rather different point of view, Jean-Marie Souriau has pointed out in the seventies [25] the fundamental role played by Lie groups in mechanics and especially by the dual space of the Lie algebra of the group and the coadjoint action. We aim to discuss some aspects of these notions through examples in finite and infinite dimension. The article is divided in three parts. In Section 1 we study in detail the motion of an  $n$ -dimensional rigid body. In the second section, we treat the geodesic flow of left-invariant metrics on an arbitrary Lie group (of finite dimension). This permits us to extract the abstract structure from the case of the motion of the rigid body which we presented in Section 1. Finally, in the last section, we study the geodesic flow of  $H^k$  right-invariant metrics on  $Diff(\mathbb{S}^1)$ , the diffeomorphism group of the circle, using the approach developed in Section 2. Two values of  $k$  have significant physical meaning in this example:  $k = 0$  corresponds to the inviscid Burgers equation [16] and  $k = 1$  corresponds to the Camassa-Holm equation [3, 4].

## 1. THE MOTION OF THE RIGID BODY

1.1. **Rigid body.** In classical mechanics, a *material system*  $(\Sigma)$  in the ambient space  $\mathbb{R}^3$  is described by a *positive measure*  $\mu$  on  $\mathbb{R}^3$  with compact support. This measure is called the *mass distribution* of  $(\Sigma)$ .

- If  $\mu$  is proportional to the Dirac measure  $\delta_P$ ,  $(\Sigma)$  is the *massive point*  $P$ , the multiplicative factor being the mass  $m$  of the point.
- If  $\mu$  is absolutely continuous with respect to the Lebesgue measure  $\lambda$  on  $\mathbb{R}^3$ , then the Radon-Nikodym derivative of  $\mu$  with respect to  $\lambda$  is the mass density of the system  $(\Sigma)$ .

In the *Lagrangian formalism of Mechanics*, a *motion* of a material system is described by a smooth path  $\varphi^t$  of *embeddings* of the *reference state*  $\Sigma = \text{Supp}(\mu)$  in the ambient space. A material system  $(\Sigma)$  is *rigid* if each map  $\varphi$  is the restriction to  $\Sigma$  of an isometry  $g$  of the Euclidean space  $\mathbb{R}^3$ . Such a condition defines what one calls a *constitutive law of motion* which restricts the space of *probable* motions to that of *admissible* ones.

In the following section, we are going to study the motions of a rigid body  $(\Sigma)$  such that  $\Sigma = \text{Supp}(\mu)$  spans the 3 space. In that case, the manifold of all possible configurations of  $(\Sigma)$  is completely described by the 6-dimensional *bundle* of frames of  $\mathbb{R}^3$ , which we denote  $\mathcal{R}(\mathbb{R}^3)$ . The group  $D_3$  of orientation-preserving isometries of  $\mathbb{R}^3$  acts simply and transitively on that space and we can identify  $\mathcal{R}(\mathbb{R}^3)$  with  $D_3$ . Notice, however, that this identification is not canonical – it depends of the choice of a "reference" frame  $\mathfrak{R}_0$ .

Although the physically meaningful rigid body mechanics is in dimension 3, we will not use this peculiarity in order to distinguish easier the main underlying concepts. Hence, in what follows, we will study the motion of an  $n$ -dimensional rigid body.

Moreover, since we want to insist on concepts rather than struggle with heavy computations, we will restrain our study to motions of a rigid body having a fixed point. This reduction can be justified physically by the possibility to describe the motion of an isolated body in an *inertial frame* around its *center of mass*. In these circumstances, the configuration space reduces to the group  $SO(n)$  of isometries which fix a point.

**1.2. Lie algebra of the rotation group.** The Lie algebra  $\mathfrak{so}(n)$  of  $SO(n)$  is the space of all skew-symmetric  $n \times n$  matrices<sup>1</sup>. There is a canonical inner product, the so-called *Killing form* [25]

$$\langle \Omega_1, \Omega_2 \rangle = -\frac{1}{2} \text{tr}(\Omega_1 \Omega_2)$$

which permit us to identify  $\mathfrak{so}(n)$  with its dual space  $\mathfrak{so}(n)^*$ .

For  $x$  and  $y$  in  $\mathbb{R}^n$ , we define

$$L^*(x, y)(\Omega) = (\Omega x) \cdot y, \quad \Omega \in \mathfrak{so}(n)$$

which is skew-symmetric in  $x, y$  and defines thus a linear map

$$L^* : \bigwedge^2 \mathbb{R}^n \rightarrow \mathfrak{so}(n)^* .$$

This map is injective and is therefore an isomorphism between  $\mathfrak{so}(n)^*$  and  $\bigwedge^2 \mathbb{R}^n$ , which have the same dimension. Using the identification of  $\mathfrak{so}(n)^*$  with  $\mathfrak{so}(n)$ , we check that the element  $L(x, y)$  of  $\mathfrak{so}(n)$  corresponding to  $L^*(x, y)$  is the matrix

$$(1) \quad L(x, y) = yx^t - xy^t .$$

where  $x^t$  stands for the transpose of the column vector  $x$ .

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<sup>1</sup>In dimension 3, we generally identify the Lie algebra  $\mathfrak{so}(3)$  with  $\mathbb{R}^3$  endowed with the Lie bracket given by the cross product  $\omega_1 \times \omega_2$ .

**1.3. Kinematics.** The location of a point  $a$  of the body  $\Sigma$  is described by the column vector  $r$  of its coordinates in the frame  $\mathfrak{R}_0$ . At time  $t$ , this point occupies a new position  $r(t)$  in space and we have  $r(t) = g(t)r$ , where  $g(t)$  is an element of the group  $SO(3)$ . In the *Lagrangian formalism*, the velocity  $\mathbf{v}(a, t)$  of point  $a$  of  $\Sigma$  at time  $t$  is given by

$$\mathbf{v}(a, t) = \frac{\partial}{\partial t} \varphi(a, t) = \dot{g}(t) r.$$

The kinetic energy  $K$  of the body  $\Sigma$  at time  $t$  is defined by

$$(2) \quad K(t) = \frac{1}{2} \int_{\Sigma} \|\mathbf{v}(a, t)\|^2 d\mu = \frac{1}{2} \int_{\Sigma} \|\dot{g} r\|^2 d\mu = \frac{1}{2} \int_{\Sigma} \|\Omega r\|^2 d\mu$$

where  $\Omega = g^{-1} \dot{g}$  lies in the Lie algebra  $\mathfrak{so}(n)$ .

**Lemma 1.1.** *We have  $K = -\frac{1}{2} \operatorname{tr}(\Omega J \Omega)$ , where  $J$  is the symmetric matrix with entries*

$$J_{ij} = \int_{\Sigma} x_i x_j d\mu.$$

*Proof.* Let  $L : \bigwedge^2 \mathbb{R}^n \rightarrow \mathfrak{so}(n)$  be the operator defined by (1). We have

$$(3) \quad L(r, \Omega r) = (r r^t) \Omega + \Omega (r r^t) \quad \Omega \in \mathfrak{so}(n), \quad r \in \mathbb{R}^n,$$

where  $r r^t$  is the symmetric matrix with entries  $x_i x_j$ . Therefore

$$(4) \quad (\Omega r) \cdot (\Omega r) = L^*(r, \Omega r) \Omega = -\frac{1}{2} \operatorname{tr} (L(r, \Omega r) \Omega) = -\operatorname{tr} (\Omega (r r^t) \Omega),$$

which leads to the claimed result after integration.  $\square$

The kinetic energy  $K$  is therefore a positive quadratic form on the Lie algebra  $\mathfrak{so}(n)$ . A linear operator  $A : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ , called the *inertia tensor* or the *inertia operator*, is associated to  $K$  by means of the relation

$$K = \frac{1}{2} \langle A(\Omega), \Omega \rangle, \quad \Omega \in \mathfrak{so}(n).$$

More precisely, this operator is given by

$$(5) \quad A(\Omega) = J \Omega + \Omega J = \int_{\Sigma} (\Omega r r^t + r r^t \Omega) d\mu.$$

*Remark.* In dimension 3 the identification between a skew-symmetric matrix  $\Omega$  and a vector  $\omega$  is given by  $\omega_1 = -\Omega_{23}$ ,  $\omega_2 = \Omega_{13}$  and  $\omega_3 = -\Omega_{12}$ . If we look for a symmetric matrix  $I$  such that  $A(\Omega)$  correspond to the vector  $I\omega$ , we find that

$$I = \int_{\Sigma} \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} d\mu,$$

which gives the formula used in Classical Mechanics.  $\diamond$

**1.4. Angular momentum.** In classical mechanics, we define the *angular momentum* of the body as the following 2-vector<sup>2</sup>

$$\mathcal{M}(t) = \int_{\Sigma} (gr) \wedge (\dot{gr}) d\mu .$$

**Lemma 1.2.** *We have  $L(\mathcal{M}) = gA(\Omega)g^{-1}$ .*

*Proof.* A straightforward computation shows that

$$L(gr, \dot{gr}) = g\Omega r r^t g^{-1} + g r r^t \Omega g^{-1} .$$

Hence

$$L(\mathcal{M}) = \int_{\Sigma} L(gr, \dot{gr}) d\mu = gA(\Omega)g^{-1} .$$

□

**1.5. Equation of motion.** If there are no external actions on the body, the spatial angular momentum is a constant of the motion,

$$(6) \quad \frac{d\mathcal{M}}{dt} = 0 .$$

Coupled with the relation  $L(\mathcal{M}) = gA(\Omega)g^{-1}$ , we deduce that

$$(7) \quad A(\dot{\Omega}) = A(\Omega)\Omega - \Omega A(\Omega)$$

which is the generalization in  $n$  dimensions of the traditional *Euler equation*. Notice that if we let  $M = A(\Omega)$ , this equation can be rewritten as

$$(8) \quad \dot{M} = [M, \Omega] .$$

**1.6. Integrability.** Equation (8) has the peculiarity that the eigenvalues of the matrix  $M$  are preserved in time. Usually, integrals of motion help to integrate a differential equation. The *Lax pairs* technique [19] is a method to generate such integrals. Let us summarize briefly this technique for finite dimensional vector spaces. Let  $\dot{u} = F(u)$  be an ordinary differential equation in a vector space  $E$ . Suppose that we were able to find a smooth map  $L : E \rightarrow \text{End}(F)$ , where  $F$  is another vector space of finite dimension, with the following property: if  $u(t)$  is a solution of  $\dot{u} = F(u)$ , then the operators  $L(t) = L(u(t))$  remain conjugate with each other, that is, there is a one-parameter family of invertible operators  $P(t)$  such that

$$(9) \quad L(t) = P(t)^{-1}L(0)P(t) .$$

In that case, differentiating (9), we get

$$(10) \quad \dot{L} = [L, B]$$

where  $B = P^{-1}\dot{P}$ . Conversely, if we can find a smooth one-parameter family of matrices  $B(t) \in \text{End}(F)$ , solutions of equation (10), then (9) is satisfied with  $P(t)$  a solution of  $\dot{P} = PB$ . If this is the case, then the eigenvalues, the trace and more generally all conjugacy invariants of  $L(u)$  constitute a set of integrals for  $\dot{u} = F(u)$ .

A *Hamiltonian system* on  $\mathbb{R}^{2N}$  is called *completely integrable* if it has  $N$  integrals *in involution* that are functionally independent almost everywhere.

<sup>2</sup>In the Euclidean 3-space, 2-vectors and 1-vectors coincide. This is why, usually, one consider the angular momentum as a 1-vector.

A theorem of Liouville describes in that case, at least qualitatively, the dynamics of the equation. This is the reason why it is so important to find integrals of motions of a given differential equation.

Using the Lax pairs technique, Manakov [21] proved the following theorem

**Theorem 1.3.** *Given any  $n$ , equation (8) has*

$$N(n) = \frac{1}{2} \left[ \frac{n}{2} \right] + \frac{n(n-1)}{4}$$

*integrals of motion in involution. The equation of motion of an  $n$ -dimensional rigid body is completely integrable.*

*Sketch of proof.* The proof is based on the following basic lemma.

**Lemma 1.4.** *Euler's equations (8) of the dynamics of an  $n$ -dimensional rigid body have, for any  $n$ , a representation in Lax's form in matrices, linearly dependent on a parameter  $\lambda \in \mathbb{C}$ , given by  $L_\lambda = M + J^2\lambda$  and  $B_\lambda = \Omega + J\lambda$ .*

Hence, the polynomials  $P_k(\lambda) = \text{tr}(M + J^2\lambda)^k$ , ( $k = 2, \dots, n$ ) are time-independent and the coefficients  $P_k(\lambda)$  are integrals of motion. Since  $M$  is skew-symmetric and  $J$  is symmetric, the coefficient of  $\lambda^s$  in  $P_k(\lambda)$  is nonzero, provided  $s$  has the same parity as  $k$ . The calculation of  $N(n)$  here presents no difficulties.  $\square$

## 2. GEODESIC FLOW ON A LIE GROUP

In this section, we are going to study the geodesic flow of a left invariant metric on a Lie group of finite dimension. Our aim is to show that all the computations performed in Section 1 are a very special case of the theory of one-sided invariant metrics on a Lie group. Later on, we will use these techniques to handle partial differential equations. We refer to [2] from where materials of this section come from and to Souriau's book [25] for a thorough discussion of the role played by the dual of the Lie algebra in mechanics and physics.

**2.1. Lie Groups.** A Lie group  $G$  is a group together with a smooth structure such that  $g \mapsto g^{-1}$  and  $(g, h) \mapsto gh$  are smooth. On  $G$ , we define the *right translations*  $R_h : G \rightarrow G$  by  $R_g(h) = hg$  and the *left translations*  $L_g : G \rightarrow G$  by  $L_g(h) = gh$ .

A Lie group is equipped with a canonical *vector-valued* one form, the so called *Maurer-Cartan form*  $\omega(X_g) = L_{g^{-1}}X_g$  which shows that the tangent bundle to  $G$  is trivial  $TG \simeq G \times \mathfrak{g}$ . Here  $\mathfrak{g}$  is the tangent space at the group unity  $e$ .

A *left-invariant* tensor is completely defined by its value at the group unity  $e$ . In particular, there is an isomorphism between the tangent space at the origin and left-invariant vector fields. Since the Lie bracket of such fields is again a left-invariant vector field, the Lie algebra structure on vector fields is inherited by the tangent space at the origin  $\mathfrak{g}$ . This space  $\mathfrak{g}$  is called the *Lie algebra* of the group  $G$ .

*Remark.* One could have defined the Lie bracket on  $\mathfrak{g}$  by pulling back the Lie bracket of vector fields by right translation. The two definitions differ just by a minus sign

$$[\xi, \omega]_R = -[\xi, \omega]_L. \quad \diamond$$

*Example.* The Lie algebra  $\mathfrak{so}(n)$  of the rotation group  $SO(n)$  consists of skew-symmetric  $n \times n$  matrices.  $\diamond$

**2.2. Adjoint representation of  $G$ .** The composition  $I_g = R_{g^{-1}}L_g : G \rightarrow G$  which sends any group element  $h \in G$  to  $ghg^{-1}$  is an automorphism, that is,

$$I_g(hk) = I_g(h)I_g(k).$$

It is called an *inner automorphism* of  $G$ . Notice that  $I_g$  preserves the group unity.

The differential of the inner automorphism  $I_g$  at the group unity  $e$  is called the *group adjoint operator*  $Ad_g$  defined by

$$Ad_g : \mathfrak{g} \rightarrow \mathfrak{g}, \quad Ad_g \omega = \left. \frac{d}{dt} \right|_{t=0} I_g(h(t)),$$

where  $h(t)$  is a curve on the group  $G$  such that  $h(0) = e$  and  $\dot{h}(0) = \omega \in \mathfrak{g} = T_e G$ . The *orbit* of a point  $\omega$  of  $\mathfrak{g}$  under the action of the adjoint representation is called an *adjoint orbit*. The adjoint operators form a representation of the group  $G$  (i.e.  $Ad_{gh} = Ad_g Ad_h$ ) which preserves the Lie bracket of  $\mathfrak{g}$ , that is,

$$[Ad_g \xi, Ad_g \omega] = Ad_g [\xi, \omega].$$

This is the *Adjoint representation* of  $G$  into its Lie algebra  $\mathfrak{g}$ .

*Example.* For  $g \in SO(n)$  and  $\Omega \in \mathfrak{so}(n)$ , we have  $Ad_g \Omega = g\Omega g^{-1}$ .  $\diamond$

**2.3. Adjoint representation of  $\mathfrak{g}$ .** The map  $Ad$ , which associates the operator  $Ad_g$  to a group element  $g \in G$ , may be regarded as a map from the group  $G$  to the space  $End(\mathfrak{g})$  of *endomorphisms* of  $\mathfrak{g}$ . The differential of the map  $Ad$  at the group unity is called the adjoint representation of the Lie algebra  $\mathfrak{g}$  into itself,

$$ad : \mathfrak{g} \rightarrow End(\mathfrak{g}), \quad ad_\xi \omega = \left. \frac{d}{dt} \right|_{t=0} Ad_{g(t)} \omega.$$

Here  $g(t)$  is a curve on the group  $G$  such that  $g(0) = e$  and  $\dot{g}(0) = \xi$ . Notice that the space  $\{ad_\xi \omega, \xi \in \mathfrak{g}\}$  is the tangent space to the adjoint orbit of the point  $\omega \in \mathfrak{g}$ .

*Example.* On the rotation group  $SO(n)$ , we have  $ad_\Xi \Omega = [\Xi, \Omega]$ , where  $[\Xi, \Omega] = \Xi\Omega - \Omega\Xi$  is the commutator of the skew-symmetric matrices  $\Xi$  and  $\Omega$ . As we already noticed, for  $n = 3$ , the vector  $[\xi, \omega]$  is the ordinary cross product  $\xi \times \omega$  of the angular velocity vectors  $\xi$  and  $\omega$  in  $\mathbb{R}^3$ . More generally, if  $G$  is an arbitrary Lie group and  $[\xi, \omega]$  is the Lie bracket on  $\mathfrak{g}$  defined earlier, we have  $ad_\xi \omega = [\xi, \omega]$ .  $\diamond$

**2.4. Coadjoint representation of  $G$ .** Let  $\mathfrak{g}^*$  be the dual vector space to the Lie algebra  $\mathfrak{g}$ . Elements of  $\mathfrak{g}^*$  are linear functionals on  $\mathfrak{g}$ . As we shall see, the leading part in mechanics is not played by the Lie algebra itself but by its dual space  $\mathfrak{g}^*$ . Souriau [25] pointed out the importance of this space in physics and called the elements of  $\mathfrak{g}^*$  *torsors* of the group  $G$ . This definition is justified by the fact that torsors of the usual group of affine Euclidean isometries of  $\mathbb{R}^3$  represent the *torsors* or *torques* of mechanicians.

Let  $A : E \rightarrow F$  be a linear mapping between vector spaces. The dual (or adjoint) operator  $A^*$ , acting in the reverse direction between the corresponding dual spaces,  $A^* : F^* \rightarrow E^*$ , is defined by

$$(A^* \alpha)(x) = \alpha(Ax)$$

for every  $x \in E$ ,  $\alpha \in F^*$ .

The *coadjoint representation* of a Lie group  $G$  in the space  $\mathfrak{g}^*$  is the representation that associates to each group element  $g$  the linear transformation

$$Ad_g^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$$

given by  $Ad_g^* = (Ad_{g^{-1}})^*$ . In other words,

$$(Ad_g^* m)(\omega) = m(Ad_g \omega)$$

for every  $g \in G$ ,  $m \in \mathfrak{g}^*$  and  $\omega \in \mathfrak{g}$ . The choice of  $g^{-1}$  in the definition of  $Ad_g^*$  is to ensure that  $Ad^*$  is a *left representation*, that is  $Ad_{gh}^* = Ad_g^* Ad_h^*$  and not the converse (or *right representation*). The *orbit* of a point  $m$  of  $\mathfrak{g}^*$  under the action of the coadjoint representation is called a *coadjoint orbit*.

The *Killing form* on  $\mathfrak{g}$  is defined by

$$k(\xi, \omega) = \text{tr}(ad_\xi ad_\omega).$$

Notice that  $k$  is invariant under the adjoint representation of  $G$ . The Lie group  $G$  is *semi-simple* if  $k$  is non-degenerate. In that case,  $k$  induces an isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}^*$  which permutes the adjoint and coadjoint representation. The adjoint and coadjoint representation of a semi-simple Lie group are *equivalent*.

*Example.* For the group  $SO(3)$  the coadjoint orbits are the sphere centered at the origin of the 3-dimensional space  $\mathfrak{so}(3)^*$ . They are similar to the adjoint orbits of this group, which are spheres in the space  $\mathfrak{so}(3)$ .  $\diamond$

*Example.* For the group  $SO(n)$  ( $n \geq 3$ ), the adjoint representation and coadjoint representations are equivalent due to the non-degeneracy of the Killing form<sup>3</sup>

$$k(\Xi, \Omega) = \frac{1}{2} \text{tr}(\Xi \Omega^*),$$

where  $\Omega^*$  is the transpose of  $\Omega$  relative to the corresponding inner product of  $\mathbb{R}^n$ . Therefore

$$Ad_g^* M = gMg^{-1},$$

for  $M \in \mathfrak{so}(n)^*$  and  $g \in SO(n)$ .  $\diamond$

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<sup>3</sup>This formula is exact up to a scaling factor since a precise computation for  $\mathfrak{so}(n)$  gives  $k(X, Y) = (n-2) \text{tr}(XY)$ .

Despite the previous two examples, in general the coadjoint and the adjoint representations are not alike. For example, this is the case for the *Poincaré group* (the non-homogenous *Lorentz group*) cf. [13].

**2.5. Coadjoint representation of  $\mathfrak{g}$ .** Similar to the adjoint representation of  $\mathfrak{g}$ , there is the coadjoint representation of  $\mathfrak{g}$ . This later is defined as the dual of the adjoint representation of  $\mathfrak{g}$ , that is,

$$ad^* : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*), \quad ad_\xi^* m = (ad_\xi)^*(m) = -\frac{d}{dt}\Big|_{t=0} Ad_{g(t)}^* m,$$

where  $g(t)$  is a curve on the group  $G$  such that  $g(0) = e$  and  $\dot{g}(0) = \xi$ .

*Example.* For  $\Omega \in \mathfrak{so}(n)$  and  $M \in \mathfrak{so}(n)^*$ , we have  $ad_\Omega^* M = -[\Omega, M]$ .  $\diamond$

Given  $m \in \mathfrak{g}^*$ , the vectors  $ad_\xi^* m$ , with various  $\xi \in \mathfrak{g}$ , constitute the tangent space to the coadjoint orbit of the point  $m$ .

**2.6. Left invariant metric on  $G$ .** A *Riemannian* or *pseudo-Riemannian* metric on a Lie group  $G$  is left invariant if it is preserved under every left shift  $L_g$ , that is,

$$\langle X_g, Y_g \rangle_g = \langle L_h X_g, L_h Y_g \rangle_{hg}, \quad g, h \in G.$$

A left-invariant metric is uniquely defined by its restriction to the tangent space to the group at the unity, hence by a quadratic form on  $\mathfrak{g}$ . To such a quadratic form on  $\mathfrak{g}$ , a symmetric operator  $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$  defined by

$$\langle \xi, \omega \rangle = (A\xi, \omega) = (A\omega, \xi), \quad \xi, \omega \in \mathfrak{g},$$

is naturally associated, and conversely<sup>4</sup>. The operator  $A$  is called the *inertia operator*.  $A$  can be extended to a left-invariant tensor  $A_g : T_g G \rightarrow T_g G^*$  defined by  $A_g = L_{g^{-1}}^* A L_g$ . More precisely, we have

$$\langle X, Y \rangle_g = (A_g X, Y)_g = (A_g Y, X)_g, \quad X, Y \in T_g G.$$

The *Levi-Civita* connection of a left-invariant metric is itself left-invariant: if  $L_a$  and  $L_b$  are left-invariant vector fields, so is  $\nabla_{L_a} L_b$ . We can write down an expression for this connection using the operator  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  defined by

$$(11) \quad \langle [a, b], c \rangle = \langle B(c, a), b \rangle$$

for every  $a, b, c$  in  $\mathfrak{g}$ . An exact expression for  $B$  is

$$B(a, b) = A^{-1} ad_b^*(Aa).$$

With these definitions, we get

$$(12) \quad (\nabla_{L_a} L_b)(e) = \frac{1}{2}[a, b] - \frac{1}{2}\{B(a, b) + B(b, a)\}$$

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<sup>4</sup>The round brackets correspond to the natural pairing between elements of  $\mathfrak{g}$  and  $\mathfrak{g}^*$ .

**2.7. Geodesics.** Geodesics are defined as extremals of the *Lagrangian*

$$(13) \quad \mathcal{L}(g) = \int K(g(t), \dot{g}(t)) dt$$

where

$$(14) \quad K(X) = \frac{1}{2} \langle X_g, X_g \rangle_g = \frac{1}{2} (A_g X_g, X_g)_g$$

is called the *kinetic energy* or *energy functional*.

If  $g(t)$  is a geodesic, the velocity  $\dot{g}(t)$  can be translated to the identity via left or right shifts and we obtain two elements of the Lie algebra  $\mathfrak{g}$ ,

$$\omega_L = L_{g^{-1}} \dot{g}, \quad \omega_R = R_{g^{-1}} \dot{g},$$

called the *left angular velocity*, respectively the *right angular velocity*. Letting  $m = A_g \dot{g} \in T_g G^*$ , we define the *left angular momentum*  $m_L$  and the *right angular momentum*  $m_R$  by

$$m_L = L_g^* m \in \mathfrak{g}^*, \quad m_R = R_g^* m \in \mathfrak{g}^*.$$

Between these four elements, we have the relations

$$(15) \quad \omega_R = Ad_g \omega_L, \quad m_R = Ad_g^* m_L, \quad m_L = A \omega_L.$$

Note that the *kinetic energy* is given by the formula

$$(16) \quad K = \frac{1}{2} \langle \dot{g}, \dot{g} \rangle_g = \frac{1}{2} \langle \omega_L, \omega_L \rangle = \frac{1}{2} (m_L, \omega_L) = \frac{1}{2} (A_g \dot{g}, \dot{g})_g.$$

*Example.* The kinetic energy of an  $n$ -dimensional rigid body, defined by

$$(17) \quad K(t) = \frac{1}{2} \int_{\Sigma} \|\dot{g} r\|^2 d\mu = -\frac{1}{2} \text{tr}(\Omega J \Omega)$$

is clearly a left-invariant Riemannian metric on  $SO(n)$ . In this example, we have  $\Omega = \omega_L$  and  $M = m_L$ . Physically, the left-invariance is justified by the fact that the physics of the problem must not depend on a particular choice of reference frame used to describe it. It is a special case of Galilean invariance.  $\diamond$

**2.8. Euler-Arnold equation.** The invariance of the energy with respect to left translations leads to the existence of a *momentum map*  $\mu : TG \rightarrow \mathfrak{g}^*$  defined by

$$\mu((g, \dot{g}))(\xi) = \frac{\partial K}{\partial \dot{g}} Z_{\xi} = \langle \dot{g}, R_g \xi \rangle_g = (m, R_g \xi) = (R_g^* m, \xi) = m_R(\xi),$$

where  $Z_{\xi}$  is the right-invariant vector field generated by  $\xi \in \mathfrak{g}$ . According to Noether's theorem [25], this map is constant along a geodesic, that is

$$(18) \quad \frac{dm_R}{dt} = 0.$$

As we did in the special case of the group  $SO(n)$ , using the relation  $m_R = Ad_g^* m_L$  and computing the time derivative, we obtain

$$(19) \quad \frac{dm_L}{dt} = ad_{\omega_L}^* m_L.$$

This equation is known as the *Arnold-Euler equation*. Using  $\omega_L = A^{-1} m_L$ , it can be rewritten as an evolution equation on the Lie algebra

$$(20) \quad \frac{d\omega_L}{dt} = B(\omega_L, \omega_L).$$

*Remark.* The *Euler-Lagrange* equations of problem (17) are given by

$$(21) \quad \begin{cases} \dot{g} &= L_g \omega_L, \\ \dot{\omega}_L &= B(\omega_L, \omega_L). \end{cases}$$

If the metric is bi-invariant, then  $B(a, b) = 0$  for all  $a, b \in \mathfrak{g}$  and  $\omega_L$  is constant. In that special case, geodesics are one-parameter subgroups, as expected.  $\diamond$

**2.9. Lie-Poisson structure on  $\mathfrak{g}^*$ .** A *Poisson structure* on a manifold  $M$  is a skew-symmetric bilinear function  $\{, \}$  that associates to a pair of smooth functions on the manifold a third function, and which satisfies the *Jacobi identity*

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

as well as the *Leibniz identity*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

On the torsor space  $\mathfrak{g}^*$  of a Lie group  $G$ , there is a natural Poisson structure defined by

$$(22) \quad \{f, g\}(m) = (m, [d_m f, d_m g])$$

for  $m \in \mathfrak{g}^*$  and  $f, g \in C^\infty(\mathfrak{g}^*)$ . Note that the differential of  $f$  at each point  $m \in \mathfrak{g}^*$  is an element of the Lie algebra  $\mathfrak{g}$  itself. Hence, the commutator  $[d_m f, d_m g]$  is also a vector of this Lie algebra. The operation defined above is called the *natural Lie-Poisson structure* on the dual space to a Lie algebra. For more materials on Poisson structures, we refer to [20, 26].

*Remark.* A Poisson structure on a vector space  $E$  is *linear* if the Poisson bracket of two linear functions is itself a linear function. This property is satisfied by the Lie-Poisson bracket on the torsors space  $\mathfrak{g}^*$  of a Lie group  $G$ .  $\diamond$

To each function  $H$  on a Poisson manifold  $M$  one can associate a vector field  $\xi_H$  defined by

$$L_{\xi_H} f = \{H, f\}$$

and called the *Hamiltonian field* of  $H$ . Notice that

$$[\xi_F, \xi_H] = \xi_{\{F, H\}}.$$

Conversely, a vector field  $v$  on a Poisson manifold is said to be *Hamiltonian* if there exists a function  $H$  such that  $v = \xi_H$ .

*Example.* On the torsors space  $\mathfrak{g}^*$  of a Lie group  $G$ , the Hamiltonian field of a function  $H$  for the natural Lie-Poisson structure is given by  $\xi_H(m) = ad_{d_m H}^* m$ .

Let  $A$  be the inertia operator associated to a left-invariant metric on  $G$ . Then equation (19) on  $\mathfrak{g}^*$  is Hamiltonian with quadratic Hamiltonian

$$H(m) = \frac{1}{2} (A^{-1} m, M), \quad m \in \mathfrak{g}^*,$$

which is nothing else but the kinetic energy expressed in terms of  $m = A\omega$ . Notice that since  $m_L(t) = Ad_{g(t)}^* m_R$  where  $m_R \in \mathfrak{g}^*$  is a constant, each integral curve  $m_L(t)$  of this equation stays on a coadjoint orbit.  $\diamond$

A Poisson structure on a manifold  $M$  is non-degenerate if it derives from a *symplectic structure* on  $M$ . That is

$$\{f, g\} = \omega(\xi_f, \xi_g),$$

where  $\omega$  is a non-degenerate closed two form on  $M$ . Unfortunately, the Lie-Poisson structure on  $\mathfrak{g}^*$  is degenerate in general. However, the restriction of this structure on each coadjoint orbit is non-degenerate. The symplectic structure on each coadjoint orbit is known as the Kirillov<sup>5</sup> form. It is given by

$$\omega(ad_a^* m, ad_b^* m) = (m, [a, b])$$

where  $a, b \in \mathfrak{g}$  and  $m \in \mathfrak{g}^*$ . Recall that the tangent space to the coadjoint orbit of  $m \in \mathfrak{g}^*$  is spanned by the vectors  $ad_\xi^* m$  where  $\xi$  describes  $\mathfrak{g}$ .

### 3. RIGHT-INVARIANT METRIC ON THE DIFFEOMORPHISM GROUP

In [1], Arnold showed that Euler equations of an incompressible fluid may be viewed as the geodesic flow of a right-invariant metric on the group of volume-preserving diffeomorphism of a 3-dimensional Riemannian manifold  $M$  (filled by the fluid). More precisely, let  $G = Diff_\mu(M)$  be the group of diffeomorphisms preserving a volume form  $\mu$  on some closed Riemannian manifold  $M$ . According to the Action Principle, motions of an ideal (incompressible and inviscid) fluid in  $M$  are geodesics of a right-invariant metric on  $Diff_\mu(M)$ . Such a metric is defined by a quadratic form  $K$  (the kinetic energy) on the Lie algebra  $\mathcal{X}_\mu(M)$  of divergence-free vector fields

$$K = \frac{1}{2} \int_M \|v\|^2 d\mu$$

where  $\|v\|^2$  is the square of the Riemannian length of a vector field  $v \in \mathcal{X}(M)$ . An operator  $B$  on  $\mathcal{X}_\mu(M) \times \mathcal{X}_\mu(M)$  defined by the relation

$$\langle [u, v], w \rangle = \langle B(w, u), v \rangle$$

exists. It is given by the formula

$$B(u, v) = \text{curl } u \times v + \text{grad } p,$$

where  $\times$  is the cross product and  $p$  a function on  $M$  defined uniquely (modulo an additive constant) by the condition  $\text{div } B = 0$  and the tangency of  $B(u, v)$  to  $\partial M$ . The Euler equation for ideal hydrodynamics is the evolution equation

$$(23) \quad \frac{\partial u}{\partial t} = u \times \text{curl } u - \text{grad } p.$$

If at least formally, the theory works as well in infinite dimension and the unifying concepts it brings form a beautiful piece of mathematics, the details of the theory are far from being as clear as in finite dimension. The main reason of these difficulties is the fact that the diffeomorphism group is just a

<sup>5</sup>Jean-Marie Souriau has generalized this construction for other natural  $G$ -actions on  $\mathfrak{g}^*$  when the group  $G$  has non null symplectic cohomology [25].

*Fréchet Lie group*, where the main theorems of differential geometry like the Cauchy-Lipschitz theorem and the Inverse function theorem are no longer valid.

In this section, we are going to apply the results of Section 2 to study the geodesic flow of a  $H^k$  right-invariant metrics on the diffeomorphism group of the circle  $\mathbb{S}^1$ . This may appear to be less ambitious than to study the 3-dimensional diffeomorphism group. However, we will be able to understand in that example some phenomena which may lead to understand why the 3-dimensional ideal hydrodynamics is so difficult to handle. Moreover, we shall give an example where things happen to work well, the Camassa-Holm equation.

**3.1. The diffeomorphism group of the circle.** The group  $Diff(\mathbb{S}^1)$  is an open subset of  $C^\infty(\mathbb{S}^1, \mathbb{S}^1)$  which is itself a closed subset of  $C^\infty(\mathbb{S}^1, \mathbb{C})$ . We define a local chart  $(U_0, \Psi_0)$  around a point  $\varphi_0 \in Diff(\mathbb{S}^1)$  by the neighborhood

$$U_0 = \{ \|\varphi - \varphi_0\|_{C^0(\mathbb{S}^1)} < 1/2 \}$$

of  $\varphi_0$  and the map

$$\Psi_0(\varphi) = \frac{1}{2\pi i} \log(\overline{\varphi_0(x)}\varphi(x)) = u(x), \quad x \in \mathbb{S}^1.$$

The structure described above endows  $Diff(\mathbb{S}^1)$  with a smooth manifold structure based on the Fréchet space  $C^\infty(\mathbb{S}^1)$ . The composition and the inverse are both smooth maps  $Diff(\mathbb{S}^1) \times Diff(\mathbb{S}^1) \rightarrow Diff(\mathbb{S}^1)$ , respectively  $Diff(\mathbb{S}^1) \rightarrow Diff(\mathbb{S}^1)$ , so that  $Diff(\mathbb{S}^1)$  is a Lie group.

A tangent vector  $V$  at a point  $\varphi \in Diff(\mathbb{S}^1)$  is a function  $V : \mathbb{S}^1 \rightarrow T\mathbb{S}^1$  such that  $\pi(V(x)) = \varphi(x)$ . It is represented by a pair  $(\varphi, v) \in Diff(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$ . Left and right translations are smooth maps and their derivatives at a point  $\varphi \in Diff(\mathbb{S}^1)$  are given by

$$L_\psi V = (\psi(\varphi), \psi_x(\varphi) v)$$

$$R_\psi V = (\varphi(\psi), v(\psi))$$

The adjoint action on  $\mathfrak{g} = Vect(\mathbb{S}^1) \equiv C^\infty(\mathbb{S}^1)$  is

$$Ad_\psi u = \psi_x(\psi^{-1})u(\psi^{-1}),$$

whereas the Lie bracket on the Lie algebra  $T_{Id} Diff(\mathbb{S}^1) = Vect(\mathbb{S}^1) \equiv C^\infty(\mathbb{S}^1)$  of  $Diff(\mathbb{S}^1)$  is given by

$$[u, v] = -(u_x v - u v_x), \quad u, v \in C^\infty(\mathbb{S}^1)$$

Each  $v \in Vect(\mathbb{S}^1)$  gives rise to a one-parameter subgroup of diffeomorphisms  $\{\eta(t, \cdot)\}$  obtained by solving

$$(24) \quad \eta_t = v(\eta) \quad \text{in} \quad C^\infty(\mathbb{S}^1)$$

with initial data  $\eta(0) = Id \in Diff(\mathbb{S}^1)$ . Conversely, each one-parameter subgroup  $t \mapsto \eta(t) \in Diff(\mathbb{S}^1)$  is determined by its infinitesimal generator

$$v = \left. \frac{\partial}{\partial t} \eta(t) \right|_{t=0} \in Vect(\mathbb{S}^1).$$

Evaluating the flow  $t \mapsto \eta(t, \cdot)$  of (24) at  $t = 1$  we obtain an element  $\exp_L(v)$  of  $Diff(\mathbb{S}^1)$ . The Lie-group exponential map  $v \rightarrow \exp_L(v)$  is a smooth map

of the Lie algebra to the Lie group [23]. Although the derivative of  $\exp_L$  at  $0 \in C^\infty(\mathbb{S}^1)$  is the identity,  $\exp_L$  is not locally surjective [23]. This failure, in contrast with the case of Hilbert Lie groups [18], is due to the fact that the inverse function theorem does not necessarily hold in Fréchet spaces [15].

**3.2.  $H^k$  metrics on  $Diff(\mathbb{S}^1)$ .** For  $k \geq 0$  and  $u, v \in Vect(\mathbb{S}^1) \equiv C^\infty(\mathbb{S}^1)$ , we define

$$(25) \quad \langle u, v \rangle_k = \int_{\mathbb{S}^1} \sum_{i=0}^k (\partial_x^i u) (\partial_x^i v) dx = \int_{\mathbb{S}^1} A_k(u) v dx,$$

where

$$(26) \quad A_k = 1 - \frac{d^2}{dx^2} + \dots + (-1)^k \frac{d^{2k}}{dx^{2k}}$$

is a continuous linear isomorphism of  $C^\infty(\mathbb{S}^1)$ . Note that  $A_k$  is a symmetric operator for the  $L^2$  inner product

$$\int_{\mathbb{S}^1} A_k(u) v dx = \int_{\mathbb{S}^1} u A_k(v) dx.$$

*Remark.* What should be  $\mathfrak{g}^*$  for  $G = Diff(\mathbb{S}^1)$  and  $\mathfrak{g} = vect(\mathbb{S}^1)$ ? If we let  $\mathfrak{g}^*$  be the space of distributions,  $A_k$  is no longer an isomorphism. This is the reason why we restrict  $\mathfrak{g}^*$  to the range of  $A_k$

$$Im(A_k) = C^\infty(\mathbb{S}^1).$$

The pairing between  $\mathfrak{g}$  and  $\mathfrak{g}^*$  is then given by the  $L^2$  inner product

$$(m, u) = \int_{\mathbb{S}^1} mu dx.$$

With these definitions, the coadjoint action of  $Diff(\mathbb{S}^1)$  on  $\mathfrak{g}^* = C^\infty(\mathbb{S}^1)$  is given by

$$Ad_\varphi^* m = \frac{1}{(\varphi_x(\varphi^{-1}))^2} m(\varphi^{-1}).$$

Notice that this formula corresponds exactly to the action of the diffeomorphism group  $Diff(\mathbb{S}^1)$  on *quadratic differentials* of the circle (expressions of the form  $m(x) dx^2$ ). This is the reason why one generally speaks of the torsor space of the group  $Diff(\mathbb{S}^1)$  as the space of quadratic differentials.  $\diamond$

We obtain a smooth right-invariant metric on  $Diff(\mathbb{S}^1)$  by extending the inner product (25) to each tangent space  $T_\varphi Diff(\mathbb{S}^1)$ ,  $\varphi \in Diff(\mathbb{S}^1)$ , by right-translations i.e.

$$\langle V, W \rangle_\varphi = \langle R_{\varphi^{-1}} V, R_{\varphi^{-1}} W \rangle_k, \quad V, W \in T_\varphi Diff(\mathbb{S}^1).$$

The existence of a connection compatible with the metric is ensured (see [10]) by the existence of a bilinear operator  $B : C^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  such that

$$\langle B(u, v), w \rangle = \langle u, [v, w] \rangle, \quad u, v, w \in Vect(\mathbb{S}^1) = C^\infty(\mathbb{S}^1).$$

For the  $H^k$  metric, this operator is given by (see [11])

$$(27) \quad B_k(u, v) = -A_k^{-1} \left( 2v_x A_k(u) + v A_k(u_x) \right), \quad u, v \in C^\infty(\mathbb{S}^1).$$

**3.3. Geodesics.** The existence of the connection  $\nabla^k$  enables us to define the geodesic flow. A  $C^2$ -curve  $\varphi : I \rightarrow \text{Diff}(\mathbb{S}^1)$  such that  $\nabla_{\dot{\varphi}} \dot{\varphi} = 0$ , where  $\dot{\varphi}$  denotes the time derivative  $\varphi_t$  of  $\varphi$ , is called a *geodesic*. As we did in Section 2, in the case of a left-invariant metric, we let

$$u(t) = R_{\varphi^{-1}} \dot{\varphi} = \varphi_t \circ \varphi^{-1}$$

which is the right angular velocity on the group  $\text{Diff}(\mathbb{S}^1)$ . Therefore, a curve  $\varphi \in C^2(I, \text{Diff}(\mathbb{S}^1))$  with  $\varphi(0) = \text{Id}$  is a geodesic if and only if

$$(28) \quad u_t = B_k(u, u), \quad t \in I.$$

Equation (28) is the *Euler-Arnold equation* associated to the right-invariant metric (25). Here are two examples of problems of type (28) on  $\text{Diff}(\mathbb{S}^1)$  which arise in mechanics.

*Example.* For  $k = 0$ , that is for the  $L^2$  right-invariant metric, equation (28) becomes the inviscid Burgers equation

$$(29) \quad u_t + 3uu_x = 0.$$

All solutions of (29) but the constant functions have a finite life span and (29) is a simplified model for the occurrence of shock waves in gas dynamics (see [16]).  $\diamond$

*Example.* For  $k = 1$ , that is for the  $H^1$  right-invariant metric, equation (28) becomes the Camassa-Holm equation (cf. [24])

$$(30) \quad u_t + uu_x + \partial_x (1 - \partial_{x^2})^{-1} \left( u^2 + \frac{1}{2} u_{x^2} \right) = 0.$$

Equation (30) is a model for the unidirectional propagation of shallow water waves [3, 17]. It has a bi-Hamiltonian structure [14] and is completely integrable [12]. Some solutions of (30) exist globally in time [5, 6], whereas others develop singularities in finite time [6, 7, 8, 22]. The blowup phenomenon can be interpreted as a simplified model for wave breaking – the solution (representing the water’s surface) stays bounded while its slope becomes unbounded [8].  $\diamond$

**3.4. The momentum.** As a consequence of the right-invariance of the metric by the action of the group on itself, we obtain the conservation of the *left angular momentum*  $m_L$  along a geodesic  $\varphi$ . Since  $m_L = \text{Ad}_{\varphi^{-1}}^* m_R$  and  $m_R = A_k(u)$ , we get that

$$(31) \quad m_k(\varphi, t) = A_k(u) \circ \varphi \cdot \varphi_x^2,$$

satisfies  $m_k(t) = m_k(0)$  as long as  $m_k(t)$  is defined.

**3.5. Existence of the geodesics.** In a local chart the geodesic equation (28) can be expressed as the Cauchy problem

$$(32) \quad \begin{cases} \varphi_t = v, \\ v_t = P_k(\varphi, v), \end{cases}$$

with  $\varphi(0) = \text{Id}$ ,  $v(0) = u(0)$ . However, the local existence theorem for differential equations with smooth right-hand side, valid for Hilbert spaces [18], does not hold in  $C^\infty(\mathbb{S}^1)$  (see [15]) and we cannot conclude at this stage. However, in [11], we proved

**Theorem 3.1.** *Let  $k \geq 1$ . For every  $u_0 \in C^\infty(\mathbb{S}^1)$ , there exists a unique geodesic  $\varphi \in C^\infty([0, T], \text{Diff}(\mathbb{S}^1))$  for the metric (25), starting at  $\varphi(0) = \text{Id} \in \text{Diff}(\mathbb{S}^1)$  in the direction  $u_0 = \varphi_t(0) \in \text{Vect}(\mathbb{S}^1)$ . Moreover, the solution depends smoothly on the initial data  $u_0 \in C^\infty(\mathbb{S}^1)$ .*

*Sketch of proof.* The operator  $P_k$  in (32) is specified by

$$P_k(\varphi, v) = \left[ Q_k(v \circ \varphi^{-1}) \right] \circ \varphi,$$

where  $Q_k : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  is defined by  $Q_k(w) = B_k(w, w) + ww_x$ . Since

$$C^\infty(\mathbb{S}^1) = \bigcap_{r \geq n} H^r(\mathbb{S}^1)$$

for all  $n \geq 0$ , we may consider the problem (32) on each Hilbert space  $H^n(\mathbb{S}^1)$ . If  $k \geq 1$  and  $n \geq 3$ , then  $P_k$  is a smooth map from  $U^n \times H^n(\mathbb{S}^1)$  to  $H^n(\mathbb{S}^1)$ , where  $U^n \subset H^n(\mathbb{S}^1)$  is the open subset of all functions having a strictly positive derivative. The classical Cauchy-Lipschitz theorem in Hilbert spaces [18] yields the existence of a unique solution  $\varphi_n(t) \in U^n$  of (32) for all  $t \in [0, T_n)$  for some maximal  $T_n > 0$ . Relation (31) can then be used to prove that  $T_n = T_{n+1}$  for all  $n \geq 3$ .  $\square$

*Remark.* For  $k = 0$ , in problem (32), we obtain

$$P_0(\varphi, v) = -2 \frac{v \cdot v_x}{\varphi_x}$$

which is not an operator from  $U^n \times H^n(\mathbb{S}^1)$  into  $H^n(\mathbb{S}^1)$  and the proof of Theorem 3.1 is no longer valid. However, in that case, the method of characteristics can be used to show that even for  $k = 0$  the geodesics exist and are smooth (see [10]).  $\diamond$

**3.6. The exponential map.** The previous results enable us to define the Riemannian exponential map  $\mathbf{exp}$  for the  $H^k$  right-invariant metric ( $k \geq 0$ ). In fact, there exists  $\delta > 0$  and  $T > 0$  so that for all  $u_0 \in \text{Diff}(\mathbb{S}^1)$  with  $\|u_0\|_{2k+1} < \delta$  the geodesic  $\varphi(t; u_0)$  is defined on  $[0, T]$  and we can define  $\mathbf{exp}(u_0) = \varphi(1; u_0)$  on the open set

$$\mathcal{U} = \left\{ u_0 \in \text{Diff}(\mathbb{S}^1) : \|u_0\|_{2k+1} < \frac{2\delta}{T} \right\}$$

of  $\text{Diff}(\mathbb{S}^1)$ . The map  $u_0 \mapsto \mathbf{exp}(u_0)$  is smooth and its Fréchet derivative at zero,  $D\mathbf{exp}_0$ , is the identity operator. On a Fréchet manifold, these facts alone do not necessarily ensure that  $\mathbf{exp}$  is a smooth local diffeomorphism [15]. However, in [11], we proved

**Theorem 3.2.** *The Riemannian exponential map for the  $H^k$  right-invariant metric on  $\text{Diff}(\mathbb{S}^1)$ ,  $k \geq 1$ , is a smooth local diffeomorphism from a neighborhood of zero on  $\text{Vect}(\mathbb{S}^1)$  to a neighborhood of  $\text{Id}$  on  $\text{Diff}(\mathbb{S}^1)$ .*

*Sketch of proof.* Working in  $H^{k+3}(\mathbb{S}^1)$ , we deduce from the inverse function theorem in Hilbert spaces that  $\mathbf{exp}$  is a smooth diffeomorphism from an open neighborhood  $\mathcal{O}_{k+3}$  of  $0 \in H^{k+3}(\mathbb{S}^1)$  to an open neighborhood  $\Theta_{k+3}$  of  $\text{Id} \in U^{k+3}$ .

We may choose  $\mathcal{O}_{k+3}$  such that  $D\mathbf{exp}_{u_0}$  is a bijection of  $H^{k+3}(\mathbb{S}^1)$  for every  $u_0 \in \mathcal{O}_{k+3}$ . Given  $n \geq k + 3$ , using (31) and the geodesic equation,

we conclude that there is no  $u_0 \in H^n(\mathbb{S}^1) \setminus H^{n+1}(\mathbb{S}^1)$ , with  $\mathbf{exp}(u_0) \in U^{n+1}$ . We have proved that for every  $n \geq k + 3$ ,

$$\mathbf{exp} : \mathcal{O} = \mathcal{O}_{k+3} \cap C^\infty(\mathbb{S}^1) \rightarrow \Theta = \Theta_{k+3} \cap C^\infty(\mathbb{S}^1)$$

is a bijection. Using similar arguments, (31) and the geodesic equation can be used to prove that there is no  $u_0 \in H^n(\mathbb{S}^1) \setminus H^{n+1}(\mathbb{S}^1)$ , with  $D\mathbf{exp}_{u_0}(v) \in H^{n+1}(\mathbb{S}^1)$  for some  $u_0 \in \mathcal{O}$ . Hence, for every  $u_0 \in \mathcal{O}$  and  $n \geq k + 3$ , the bounded linear operator  $D\mathbf{exp}_{u_0}$  is a bijection from  $H^n(\mathbb{S}^1)$  to  $H^n(\mathbb{S}^1)$ .  $\square$

*Remark.* For  $k = 0$  we have that  $\mathbf{exp}$  is not a  $C^1$  local diffeomorphism from a neighborhood of  $0 \in Vect(\mathbb{S}^1)$  to a neighborhood of  $Id \in Diff(\mathbb{S}^1)$ , as proved in [10]. The crucial difference with the case ( $k \geq 1$ ) lies in the fact that the inverse of the operator  $A_k$ , defined by (26), is not regularizing. This feature makes the previous approach inapplicable but the existence of geodesics can nevertheless be proved by the method of characteristics.  $\diamond$

#### REFERENCES

- [1] Arnold VI, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier (Grenoble)* **16** (1966), 319–361.
- [2] Arnold VI and Khesin BA, *Topological Methods in Hydrodynamics*, Springer-Verlag, New York, 1998.
- [3] Camassa R and Holm DD, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.
- [4] Camassa R, Holm DD and Hyman J, A new integrable shallow water equation, *Adv. Appl. Mech.* **31** (1994), 1–33.
- [5] Constantin A, On the Cauchy problem for the periodic Camassa-Holm equation, *J. Differential Equations* **141** (1997), 218–235.
- [6] Constantin A and Escher J, Well-posedness global existence and blow-up phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.* **51** (1998), 475–504.
- [7] Constantin A and Escher J, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* **181** (1998), 229–243.
- [8] Constantin A and Escher J, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.* **233** (2000), 75–91.
- [9] Constantin A and Kolev B, Least action principle for an integrable shallow water equation, *J. Nonlinear Math. Phys.* **8** (2001), 471–474.
- [10] Constantin A and Kolev B, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A* **35** (2002), R51–R79.
- [11] Constantin A and Kolev B, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** (2003), 787–804.
- [12] Constantin A and McKean HP, A shallow water equation on the circle, *Comm. Pure Appl. Math.* **52** (1999), 949–982.
- [13] Cushman R and van der Kallen W, Adjoint and coadjoint orbits of the Poincaré group, *ArXiv math.RT/0305442* (2003).
- [14] Fokas AS and Fuchssteiner B, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D* **4** (1981), 47–66.
- [15] Hamilton RS, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 65–222.
- [16] Hörmander L, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, Berlin, 1997.

- [17] Johnson RS, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** (2002), 63–82.
- [18] Lang S, Fundamentals of Differential Geometry, Springer-Verlag, New York, 1999.
- [19] Lax PD, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467–490.
- [20] Marsden JE and Ratiu TS, Introduction to mechanics and symmetry, Springer-Verlag, New York, 1999.
- [21] Manakov SV, A remark on the integration of the Eulerian equations of the dynamics of an  $n$ -dimensional rigid body, *Funkcional. Anal. i Priložen.* **10** (1976), 93–94.
- [22] McKean HP, Breakdown of a shallow water equation, *Asian J. Math.* **2** (1998), 867–874.
- [23] Milnor J, Remarks on infinite-dimensional Lie groups, in Relativity, Groups and Topology, North-Holland, Amsterdam, 1984, 1009–1057.
- [24] Misiólek G, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.* **24** (1998), 203–208.
- [25] Souriau JM, Structure of Dynamical Systems, Birkhäuser, Boston, 1997.
- [26] Weinstein A, The local structure of Poisson manifolds, *J. Differential Geom.* **18** (1983), 523–557.

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