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ON THE CENTRAL AND LOCAL LIMIT THEOREM FOR MARTINGALE DIFFERENCE SEQUENCES

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Abstract

Let $(\Omega, \mathcal{A}, \mu)$ be a Lebesgue space and $T : \Omega \to \Omega$ an ergodic measure preserving automorphism with positive entropy. We show that there is a bounded and strictly stationary martingale difference sequence defined on Ω with a common non-degenerate lattice distribution satisfying the central limit theorem with an arbitrarily slow rate of convergence and not satisfying the local limit theorem. A similar result is established for martingale difference sequences with densities provided the entropy is infinite. In addition, the martingale difference sequence may be chosen to be strongly mixing.

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1 Introduction

In this note we consider strictly stationary martingale difference sequences $f \circ T^k$ defined on a Lebesgue probability space $(\Omega, \mathcal{A}, \mu)$, where $T : \Omega \to \Omega$ is an invertible measure preserving transformation and $f : \Omega \to \mathbb{R}$ is measurable.

Setting

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$$

the central limit theorem we are concerned with has a formulation that $\mu(S_n(f) \leq t\sigma\sqrt{n})$ converges to $\Phi(t)$, the distribution function of the standard normal distribution at $t \in \mathbb{R}$. This result was first obtained by de Moivre, Laplace and Gauss, the first rigorous proof for independent, identically distributed (i.i.d.) random variables with finite second moment was discovered by Lyapunov around 1901. Later Berry (1941) and Esseen (1942) showed that the rate of convergence in this central limit theorem is of order $n^{-1/2}$ in case of existing third moments (this rate is also best possible). It is also well known that the central limit theorem does not imply a local limit theorem for densities (if they exist). In 1954 Gnedenko (see [11], [15] or [22]) solved the convergence problem of densities with the following result.

Theorem A (Gnedenko, 1954) *Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. random variables such that X_0 has zero mean and unit variance and denote by f_n and φ respectively the density function of the random variable $n^{-1/2}(X_0 + X_1 + ... + X_{n-1})$ and the density function of the standard normal law. In order that*

$$\sup_x |f_n(x) - \varphi(x)| \xrightarrow[n\to+\infty]{} 0$$

it is necessary and sufficient that the density function f_{n_0} be bounded for some integer n_0.

A similar result is valid for sums of lattice-valued random variables. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. random variables having a non-degenerate distribution concentrated on $\{b + Nh : n \in \mathbb{Z}\}$ where $h > 0$ and $b \in \mathbb{R}$. h is called a maximal step if it is the largest value with this property. Suppose the distribution has finite variance σ^2. Let $m = EX_0$ denote the expectation, $S_n = X_0 + ... + X_{n-1}$ and

$$P_n(N) = \mathbb{P}(S_n = nb + Nh).$$

The following result has been proved by Gnedenko as well.

Theorem B (Gnedenko, 1948) *In order that*

$$\sup_N \left| \frac{\sigma\sqrt{n}}{h} P_n(N) - \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{nb + Nh - nm}{\sigma\sqrt{n}} \right)^2 \right) \right| \xrightarrow[n\to+\infty]{} 0$$

it is necessary and sufficient that the step h should be maximal.

Such limit theorems which deal with the local rather than with the cumulative behaviour of the random variables are called local limit theorems (LLT). Local limit theorems have been intensively studied for sums of independent random variables and vectors together with estimates of the rate of convergence in these theorems. In case of independent random variables the local limit theorems can be found e.g. in [22], chapter 7 (in the normal case), or in [15] (in the stable case). In case of interval transformations the central limit theorem and local limit theorems have been established by Rousseau-Egele [25] and Broise [6] using spectral theory of the Perron-Frobenius operator. The stable local limit theorem for general transformations can be found in Aaronson and Denker [1].

Many theorems which hold true in case of independent random variables have been extended to martingale difference sequences, like the central limit theorem, the law of iterated logarithm or the invariance principle in the sense of Donsker (see [13]). However, there exist some results which cannot be extended. For example, Lesigne and Volný (see [17]) have proved that the classical estimations in large deviations inequalities for partial sums of i.i.d. random variables cannot be attained by martingale difference sequences even in the restricted class of ergodic and stationary sequences. Moreover, El Machkouri and Volný (see [10]) showed that the invariance principle in the sense of Dudley may not be valid for martingale difference random fields.
In this work, we consider local and central limit theorems for martingale difference sequences in the following setup. Let $T : \Omega \to \Omega$ be an ergodic, measure preserving automorphism of the Lebesgue probability space $(\Omega, \mathcal{A}, \mu)$ with positive entropy. We show (cf. Theorem 1) that one can define a stationary, bounded martingale difference sequence with non-degenerate lattice distribution which does not satisfy the local limit theorem as formulated in Theorem B, but satisfies the central limit theorem as formulated above with an arbitrarily slow rate of convergence. In [20], Peligrad and Utev showed that even a very mild mixing condition imposed on a martingale difference sequence has a substantial impact on the limit behaviour of linear processes generated by the sequence. In Theorem 3 we show that our examples can be constructed with the additional property of strong (i.e. α) mixing. We also

give a result for martingale difference sequences with densities (cf. Theorem 2). The rate of convergence in the central limit theorem as in the Berry-Esseen Theorem has been investigated for martingale difference sequences as well (see [13]). Ibragimov (see [14]) established a Berry-Esseen type theorem for stopped partial sums $(S_{\nu(n)}(X))_{n\geq 1}$ of bounded martingale difference random variables $(X_k)_{k\geq 0}$ with the rate of convergence $n^{-1/4}$ and his estimation holds for the whole partial sum process $(S_n(X))_{n\geq 1}$ if the conditional variances of the random variables are almost surely constant. L. Ouchti [19] showed that the bounded condition in Ibragimov's result can be weakened. In 1982, Bolthausen (see [5]) has obtained the better convergence rate of $n^{-1/2}\log n$ under conditions related to the behaviour of the conditional variances. For more recent results, see for example Haeusler [12] and Jan [16]).
Our results show that on the other hand, without assumptions on the conditional variances, there is no rate of convergence for general stationary and bounded martingale difference sequences.

2 Main results

Let $(X_k)_{k\geq 0}$ be a sequence of real random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that for any integer $k \geq 0$,

$$E(X_k|\sigma(X_j\,;\,j<k)) = 0 \quad \text{a.s.}$$

Such a process is called a martingale difference sequence. A more strict definition of the martingale difference sequence was introduced by Nahapetian and Petrosian (see [18]): $(X_k)_{k\geq 0}$ is a strong martingale difference sequence if for any integer $k \geq 0$,

$$E(X_k|\sigma(X_j\,;\,j\neq k)) = 0 \quad \text{a.s.}$$

Assume that the sequence $(X_k)_{k\geq 0}$ is stationary and ergodic, X_0 has unit variance and denote by F the common distribution function of the random variables X_k, $k \geq 0$. Recall that the Billingsley-Ibragimov central limit theorem (see [3], [14] or [13]) ensures the convergence of the distribution functions F_n of the normalized partial sum process $n^{-1/2}(X_0+X_1+\ldots+X_{n-1})$ to the standard normal distribution Φ. Throughout the paper we consider the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ where Ω is a Lebesgue space, μ a probability measure and $T : \Omega \to \Omega$ a bijective and bimeasurable transformation which preserves the measure μ. We denote by $\mathcal{I}$ the σ-algebra of all sets A in $\mathcal{F}$

with $TA = A$ a.s. (recall that if $\mathcal{I}$ is trivial then μ is said to be ergodic). For any integer $n \geq 1$ and any zero mean random variable f we denote

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^k \tag{1}$$

and

$$F_n(f, x) = \mu(S_n(f) \leq x\sigma\sqrt{n}), \quad x \in \mathbb{R}, \tag{2}$$

where $\sigma^2 = E(f^2)$.

Theorem 1 *Assume that the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is ergodic and has positive entropy (cf. [21] for a definition of the entropy). Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers which decreases to zero. There exists an integer valued function f in $L^\infty(\Omega)$ with a non-degenerate distribution such that the following assumptions hold:*

- *the process $(f \circ T^k)_{k\geq 0}$ is a strong martingale difference sequence which takes only the values -1, 0 or 1.*
- *there exists an increasing sequence $(n_k)_{k\geq 0}$ of integers such that for any $k \geq 0$,*

$$\mu(S_{n_k}(f) = 0) \geq a_{n_k} \tag{3}$$

and

$$\sup_{x\in\mathbb{R}} |F_{n_k}(f, x) - \Phi(x)| \geq \frac{a_{n_k}}{2}. \tag{4}$$

Remark 1 By the Billingsley-Ibragimov central limit theorem for martingale difference random variables (see [13]), the sequences

$$(\mu(S_n(f) = 0))_{n\geq 1} \quad \textit{and} \quad \left(\sup_{x\in\mathbb{R}} |F_n(f, x) - \Phi(x)|\right)_{n\geq 1}$$

converge to zero. The inequalities (3) and (4) obtained in Theorem 1 show that this convergence can be arbitrarily slow. In particular, the stationary process $(f \circ T^k)_{k\geq 0}$ does not satisfy the local limit theorem for lattice distributions (cf. Theorem B in section 1).

Remark 2 Let $(X_k)_{k\geq 1}$ be a sequence of bounded martingale difference random variables. T. de la Rue [7] proved the following result: if there exists a positive constant β such that for any integer $k \geq 1$,

$$E(X_{k+1}^2 | \sigma(X_j \,;\, j \leq k)) \geq \beta > 0 \quad a.s. \tag{5}$$

then the martingale $S_n = X_1 + ... + X_n$ has a polynomial speed of dispersion. More precisely, there exist two positive universal constants C and λ such that for any integer $n \geq 1$,

$$\sup_t \mathbb{P}(S_n \in I_t) \leq Cn^{-\lambda}$$

where $I_t = [t-1, t+1]$ for any t in $\mathbb{R}$. Our counter-example (Theorem 1) shows that if the condition (5) is not satisfied then the speed of dispersion of the martingale S_n can be arbitrary slow.

The following result is an analogue of Theorem 1 for sequences of martingale difference random variables with densities.

Theorem 2 *Assume that the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is ergodic and has infinite entropy. Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers which decreases to zero. For an arbitrarily large positive constant L, there exists a function f in $L^\infty(\Omega)$ such that*

- *the random variables $\{S_n(f)\,;\, n \geq 1\}$ have densities and the density of f is bounded,*
- *the process $(f \circ T^k)_{k\geq 0}$ is a strong martingale difference sequence,*
- *there exist an increasing sequence $(n_k)_{k\geq 0}$ of integers, a sequence $(\rho_k)_{k\geq 0}$ of positive real numbers which converges to zero such that for any integer $k \geq 0$,*

$$\frac{1}{\rho_k}\mu\left(\frac{S_{n_k}(f)}{\sigma(f)\sqrt{n_k}} \in [-\rho_k, \rho_k]\right) \geq L \tag{6}$$

  and

$$\sup_{x\in\mathbb{R}} |F_{n_k}(f, x) - \Phi(x)| \geq a_{n_k}. \tag{7}$$

Remark 3 If the LLT holds for a stationary sequence $(g \circ T^k)_{k\geq 0}$ of zero mean random variables then for any sequence $(d_k)_{k\geq 0}$ which converges to zero, we have

$$\frac{1}{d_k}\mu\left(\frac{S_{n_k}(g)}{\sigma(g)\sqrt{n_k}} \in [-d_k, d_k]\right) \xrightarrow[k\to+\infty]{} 2\varphi(0)$$

where φ is the density of the standard normal law. So if L is sufficiently large, Inequality (6) implies that the LLT does not hold for the strong martingale difference sequence $(f \circ T^k)_{k\geq 0}$.

Now, let $(\Omega, \mathcal{F}, \mu)$ be a non-atomic probability space and let $\mathcal{U}$ and $\mathcal{V}$ be two σ-algebras of $\mathcal{F}$. To evaluate their dependence Rosenblatt [24] introduced the α-mixing coefficient defined by

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mu(U \cap V) - \mu(U)\mu(V)|,\, U \in \mathcal{U}, V \in \mathcal{V}\}.$$

For any sequence $(X_k)_{k\geq 0}$ of real random variables and for any nonnegative integers s and t we denote by $\mathcal{F}_\infty^s$ and $\mathcal{F}_t^\infty$ the σ-algebra generated by $..., X_{s-1}, X_s$ and $X_t, X_{t+1}, ...$ respectively. We shall use the following α-mixing coefficients defined for any positive integer n by

$$\alpha(n) = \sup_{k\geq 0} \alpha(\mathcal{F}_\infty^k, \mathcal{F}_{k+n}^\infty). \tag{8}$$

We say that the sequence $(X_k)_{k\geq 0}$ is strongly mixing (or α-mixing) if $\alpha(n)$ converge to zero for n going to infinity. For more about mixing coefficients we can refer to Doukhan [9] or Rio [23]. Our last result is the following counter-example in the topic of strongly mixing processes.

Theorem 3 *Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers which decreases to zero. There exists an endomorphism T of Ω and an integer-valued function f in $L^\infty(\Omega)$ with a non-degenerate distribution such that*

- *the process $(f \circ T^k)_{k\geq 0}$ is a strongly mixing martingale difference sequence which takes only the values -1, 0 and 1.*
- *there exists an increasing sequence $(n_k)_{k\geq 0}$ of integers such that for any integer $k \geq 0$,*

$$\mu(S_{n_k}(f) = 0) \geq a_{n_k} \tag{9}$$

  and

$$\sup_{x\in\mathbb{R}} |F_{n_k}(f, x) - \Phi(x)| \geq \frac{a_{n_k}}{2}. \tag{10}$$

3 Proofs

3.1 Proof of Theorem 1

In order to construct the function f, we need the following lemma (cf. [17]).

Lemma 1 *There exist two T-invariant sub-σ-algebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ and a $\mathcal{B}$-measurable function g defined on Ω such that*

- *the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent,*

- *the process* $\left(g \circ T^k\right)_{k \geq 0}$ *is a sequence of i.i.d. zero mean random variables which take only the values -1, 0 or 1,*
- *the dynamical system* $(\Omega, \mathcal{C}, \mu_{\mathcal{C}}, T)$ *is aperiodic (that is, for any* k *in* $\mathbb{Z} \setminus \{0\}$ *and* $\mu_{\mathcal{C}}$*-almost all* ω *in* Ω*, we have* $T^k\omega \neq \omega$*).*

Moreover, there exists $0 < a \leq 1$ *depending only on the entropy of* $(\Omega, \mathcal{F}, \mu, T)$ *such that* $\mu(g = \pm 1) = a/2$ *and* $\mu(g = 0) = 1 - a$*.*

The following lemma is a particular case of a result established by del Junco and Rosenblatt ([8], Theorem 2.2).

Lemma 2 *Consider the dynamical system* $(\Sigma, \mathcal{S}, \nu, S)$ *where* Σ *is a Lebesgue space and let us fix* $\varepsilon > 0$*,* N *in* $\mathbb{N}$ *and* x *in* $]0,1[$*. There exists an* $\mathcal{S}$*-measurable set* A *such that* $\nu(A) = x$ *and* $\nu(A \Delta S^{-n} A) < \varepsilon \nu(A)$ *for any integer* $0 \leq n \leq N$*.*

For any $k \geq 0$, we fix $\varepsilon_k > 0$, $N_k \in \mathbb{N}$ and $d_k > 0$ such that

- the sequence $(\varepsilon_k)_{k \geq 0}$ decreases to zero,
- the sequence $(N_k)_{k \geq 0}$ increases to $+\infty$,
- the sequence $(d_k)_{k \geq 0}$ satisfies $\sum_{k=0}^{+\infty} d_k < 1$.

Consider the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ and the function g given by Lemma 1. By Lemma 2, for any integer $k \geq 0$, there exists A_k in $\mathcal{C}$ such that

$$\mu(A_k) = d_k \quad \text{and} \quad \forall 0 \leq n \leq N_k \quad \mu(A_k \Delta T^{-n} A_k) < \varepsilon_k d_k. \tag{11}$$

Let

$$A = \bigcup_{k=0}^{+\infty} A_k \in \mathcal{C}$$

and

$$f = g \ \mathbb{1}_{A^c} \in L^\infty(\Omega).$$

One can notice that $0 < \mu(A) < 1$. Moreover, the distribution of f is not degenerate since

$$\mu(f = \pm 1) = \mu(A^c)\mu(g = \pm 1) > 0.$$

Moreover, we have for any integer $k \geq 0$,

$$E(f \circ T^k | \mathcal{F}_k) = 0 \quad \text{a.s.}$$

where

$$\mathcal{F}_k = \sigma(g \circ T^j \, ; \, j \neq k) \vee \mathcal{C}. \tag{12}$$

In particular, the stationary process $(f \circ T^k)_{k \geq 0}$ is a strong martingale difference sequence which takes only the values -1, 0 or 1.

The Local Limit Theorem

Let $k \geq 0$ be a fixed integer. For any integer $1 \leq n \leq N_k$, we have

$$\bigcap_{i=0}^{n-1} T^{-i} A_k \subset \{S_n(f) = 0\} \tag{13}$$

Using the condition (11), we derive

$$\mu\left(A_k \backslash \bigcap_{i=0}^{n-1} T^{-i} A_k\right) = \mu\left(\bigcup_{i=0}^{n-1} (A_k \backslash T^{-i} A_k)\right) \leq N_k \, \varepsilon_k \, d_k,$$

hence,

$$\mu\left(\bigcap_{i=0}^{n-1} T^{-i} A_k\right) \geq \mu(A_k) - N_k \, \varepsilon_k \, d_k.$$

Combining the last inequality with the assertion (13), we deduce

$$\mu(S_n(f) = 0) \geq d_k(1 - N_k \, \varepsilon_k).$$

Let $(\rho_k)_{k \geq 0}$ be a sequence of positive numbers which converges to zero. For any $k \geq 0$ and for ε_k sufficiently small, we can suppose that $N_k \, \varepsilon_k \leq \rho_k$. Consequently, for any integer $k \geq 0$ and any $1 \leq n \leq N_k$,

$$\mu(S_n(f) = 0) \geq d_k(1 - \rho_k). \tag{14}$$

Let $(n_k)_{k \geq 0}$ be an increasing sequence of integers such that $\sum_{k=0}^{+\infty} a_{n_k} < 1/2$. For any integer $k \geq 0$ we make the following choice:

$$N_k = n_k, \quad \rho_k = 2^{-k-1} \quad \text{and} \quad d_k = 2a_{n_k}.$$

From (14) we deduce for any integer $k \geq 0$,

$$\mu(S_{n_k}(f) = 0) \geq 2(1 - 2^{-k-1}) a_{n_k} \geq a_{n_k}.$$

We have

$$a_{n_k} \leq \mu(S_{n_k}(f) = 0) \leq |F_{n_k}(f, y) - F_{n_k}(f, z)|, \quad z < 0 < y.$$

From the triangular inequality it follows

$$a_{n_k} \leq 2 \sup_x |F_{n_k}(f, x) - \Phi(x)| + |\Phi(y) - \Phi(z)|.$$

We have $\Phi(y) - \Phi(z) \to 0$ for $y - z$ converging to zero, hence for any integer $k \geq 0$,

$$\sup_x |F_{n_k}(f, x) - \Phi(x)| \geq \frac{a_{n_k}}{2}.$$

The proof of Theorem 1 is complete. □

3.2 Proof of Theorem 2

Since the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ has infinite entropy, one has the following version of Lemma 1.

Lemma 3 *There exist two T-invariant sub-σ-algebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ and a $\mathcal{B}$-measurable function g defined on Ω such that*

- *the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent,*
- *the process $\left(g \circ T^k\right)_{k \geq 0}$ is a sequence of i.i.d. random variables with common density $\mathbb{1}_{[-1,-1/2]} + \mathbb{1}_{[1/2,1]}$,*
- *the dynamical system $(\Omega, \mathcal{C}, \mu_{\mathcal{C}}, T)$ is aperiodic.*

Proof of Lemma 3. Let $(\Omega, \mathcal{F}, \mu)$ be a Lebesgue space and T be an ergodic automorphism of Ω. W'll use the relative Sinai theorem which is contained in the proposition $2'$ in the article [28] by Thouvenot. For any finite partition $\mathcal{M} = (M_1, ..., M_k)$ of Ω, we denote by $d(\mathcal{M})$ the vector $(\mu(M_1), ..., \mu(M_k))$ and by $H(M)$ the entropy of the partition.

Proposition (relative Sinai theorem) *Let Q be a partition of Ω and let $\mathcal{P}$ be a virtual finite partition such that $H(\mathcal{P}) + H(\mathcal{Q}, T) \leq H(T)$. Then there exists a partition $\mathcal{R}$ of Ω which satisfies*

- *the sequence $\{T^i\mathcal{R}\}_{i \in \mathbb{Z}}$ is independent,*
- *$d(\mathcal{R}) = d(\mathcal{P})$,*

- $\bigvee_{-\infty}^{+\infty} T^i\mathcal{R}$ *and* $\bigvee_{-\infty}^{+\infty} T^i\mathcal{Q}$ *are independent.*

Recall that the entropy of the ergodic dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is assumed to be infinite. Using the relative Sinai theorem we construct by induction a sequence $\{\mathcal{Q}_s\}_{s\geq 0}$ of finite partitions such that for any integer $s \geq 0$,

- the sequence $\{T^i\mathcal{Q}_s\}_{i\in\mathbb{Z}}$ is independent,
- $\mathcal{Q}_s = \{A_s, A_s^c\}$ with $\mu(A_s) = 1/2$, hence, $H(\mathcal{Q}_s) = 1$

and such that the σ-algebras $\{\bigvee_{-\infty}^{+\infty} T^i\mathcal{Q}_s\}_{s\geq 0}$ are mutually independent. Denote by $\mathcal{B}_0$ the σ-algebra generated by the partitions $\{\mathcal{Q}_s,\ s \geq 1\}$ and remark that the sequence $\{T^i\mathcal{B}_0\}_{i\in\mathbb{Z}}$ is independent. Now consider the following two invariant sub-σ-algebras of $\mathcal{F}$

$$\mathcal{C} = \bigvee_{-\infty}^{+\infty} T^i\mathcal{Q}_0 \quad \text{and} \quad \mathcal{B} = \bigvee_{-\infty}^{+\infty} T^i\mathcal{B}_0.$$

By construction, the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent. Moreover, there exists a $\mathcal{B}_0$-measurable function ψ_1 defined on Ω to $[0,1]$ such that for any integer $s \geq 1$,

$$A_s = \psi_1^{-1}\left(\{x \in [0,1]\,;\, 2^{s-1}x - [2^{s-1}x] < 1/2\}\right)$$

where $[\,.\,]$ denotes the integer part function. Let ψ_2 be the function defined on $[0,1]$ to $[-1,-1/2[\cup[1/2,1]$ by

$$\psi_2(x) = \begin{cases} x-1 & \text{if} \quad 0 \leq x < 1/2 \\ x & \text{if} \quad 1/2 \leq x \leq 1. \end{cases}$$

Finally the function $g = \psi_2 \circ \psi_1$ satisfies the required properties. The proof of Lemma 3 is complete. $\square$

Let $(p_k)_{k\geq 0}$ be a sequence of positive real numbers such that

$$\sum_{k=0}^{+\infty} p_k = 1 \tag{15}$$

and let $(N_k)_{k\geq 0}$ be an increasing sequence of positive integers such that the greatest common divisor of all N_k be 1.

By the multiple Rokhlin tower theorem of Alpern (see [2]), there exist $\mathcal{C}$-measurables sets $(F_k)_{k\geq 0}$ with $\mu(F_k) = p_k/N_k$ for any integer $k \geq 0$ such that $\{T^i F_k, 0 \leq i \leq N_k - 1, k \in \mathbb{N}\}$ is a partition of Ω. Let $(d_k)_{k\geq 0}$ be a sequence of positive real numbers which converges to zero and let us denote

$$G_k = \bigcup_{i=0}^{N_k-1} T^i F_k, \tag{16}$$

$$h = \sum_{k=0}^{+\infty} d_k \, \mathbb{1}_{G_k}, \tag{17}$$

and

$$f = g\, h \in L^\infty(\Omega) \tag{18}$$

where g is the function given by Lemma 3. Considering the σ-algebras defined by (12) we derive (as in the proof of Theorem 1) that the stationary process $(f \circ T^k)_{k\geq 0}$ is a strong martingale difference sequence.

The densities of the partial sums $\{S_n(f),\, n \geq 1\}$

Let us show that the random variables $\{S_n(f)\,;\, n \geq 1\}$ defined by (1) have densities. Let $\mathcal{P}$ be the partition $\{G_0, G_1, ...\}$ of Ω and let n be a fixed positive integer. Consider the following partition of Ω

$$\mathcal{P}_n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{P},$$

let A in $\mathcal{P}_n$ be fixed and let $r_0(A), r_1(A), ..., r_{n-1}(A)$ be nonnegative integers such that

$$A = \bigcap_{j=0}^{n-1} T^{-j} G_{r_j(A)}.$$

For any real x, define

$$F_A(S_n, x) = \mu(A \cap \{S_n(f) \leq x\}) \quad \text{and} \quad F_n(x) = \mu(S_n(f) \leq x).$$

Let $0 \leq j < n$ be fixed and let $d_j(A)$ denote $d_{r_j(A)}$. For any ω in A we can check that $f(T^j\omega) = d_j(A)\, g(T^j\omega)$, hence for any real x,

$$A \cap \{S_n(f) \leq x\} = A \cap \left\{ \sum_{j=0}^{n-1} d_j(A)\, g \circ T^j \leq x \right\}.$$

By the independence of the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ we thus get that for any real number x,

$$F_A(S_n, x) = \mu(A)\,\mu\left(\sum_{j=0}^{n-1} d_j(A)\, g \circ T^j \leq x\right).$$

The distribution function of g is differentiable at all $x \notin \{-1, -1/2, 1/2, 1\}$, hence, the function $F_A(S_n, .)$ is differentiable at all real x except a finite set. Let $x < 0$ be a fixed real number. Since the sequence $(d_k)_{k\geq 0}$ converges to zero, there is only a finite number of sets A in $\mathcal{P}_n$ such that $\mu(A) > 0$ and $\sum_{j=0}^{n-1} d_j(A) > -x$. Consequently, the sum

$$F_n(x) = \sum_{A \in \mathcal{P}_n} F_A(S_n, x)$$

contains only finitely many nonzero terms. So the distribution function F_n of $S_n(f)$ is differentiable on $\mathbb{R}_-^*$ except a countable set. By the symmetry of the density of g and the independence of the process $(g \circ T^k)_{k\geq 0}$, we deduce the differentiability of F_n at all points x in $\mathbb{R}$ except a countable set. Hence for any positive integer n, the random variable $S_n(f)$ has a density. Now we will show that the density of the random variable f (or $S_1(f)$) is bounded. Consider the distribution function F_1 of the random variable f. Using the independence of the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ we have for any real x

$$\begin{aligned} F_1(x) &= \sum_{k=0}^{+\infty} \mu(G_k \cap \{d_k g \leq x\}) \\ &= \sum_{k=0}^{+\infty} \mu(G_k)\,\mu(d_k g \leq x) \\ &= \sum_{k=0}^{+\infty} p_k\,\mu(d_k g \leq x). \end{aligned}$$

Moreover, for any integer $k \geq 0$ and any real x,

$$\mu(d_k g \leq x) = \begin{cases} 1 & \text{if } \ x \geq d_k \\ \frac{x}{d_k} & \text{if } \ \frac{d_k}{2} \leq x < d_k \\ \frac{1}{2} & \text{if } \ -\frac{d_k}{2} \leq x < \frac{d_k}{2} \\ \frac{x+d_k}{d_k} & \text{if } \ -d_k \leq x < -\frac{d_k}{2} \\ 0 & \text{if } \ x \leq -d_k. \end{cases}$$

Let L_1 and L_2 be two positive constants such that $L_2 \gg L_1$. In the sequel, for any integer $k \geq 0$ we put

$$p_k = \frac{\sqrt{2}-1}{\sqrt{2}} 2^{-k/2}$$

and

$$d_k = \begin{cases} p_k/L_1 & \text{if } k \text{ is even} \\ p_k/L_2 & \text{if } k \text{ is odd.} \end{cases}$$

Thus, the condition (15) still holds and the function f defined by (18) depends on the constants L_1 and L_2. If we put

$$c_1 = \sum_{j\geq 0} p_{2j}^3 \quad \text{and} \quad c_2 = \sum_{j\geq 0} p_{2j+1}^3$$

then the variance of the function f is given by

$$\sigma^2(f) = \frac{7}{12}\left(\frac{c_1}{L_1^2} + \frac{c_2}{L_2^2}\right). \tag{19}$$

If $x \geq d_0$ then $x \geq d_k$ for any integer $k \geq 0$, hence

$$F_1(x) = \sum_{j=0}^{+\infty} p_j = 1.$$

If $0 < x < d_0$ then there exists a unique odd integer $k = k(x)$ such that $d_{k+2} = d_k/2 \leq x < d_k$ and there exists a unique even integer $l = l(x)$ such that $d_{l+2} = d_l/2 \leq x < d_l$. So we have

$$\sum_{j=0}^{+\infty} p_{2j+1}\, \mu(d_{2j+1}g \leq x) = \sum_{\substack{j\leq k-1 \\ j \text{ odd}}} \frac{p_j}{2} + x\frac{p_k}{d_k} + \sum_{\substack{j\geq k+1 \\ j \text{ odd}}} p_j$$

and

$$\sum_{j=0}^{+\infty} p_{2j}\, \mu(d_{2j}g \leq x) = \sum_{\substack{j\leq l-1 \\ j \text{ even}}} \frac{p_j}{2} + x\frac{p_l}{d_l} + \sum_{\substack{j\geq l+1 \\ j \text{ even}}} p_j.$$

Consequently the distribution function F_1 of f is differentiable at all points $x > 0$ which do not belong to the set $\{d_k\,;\, k \geq 0\}$. Since F_1 is symmetric we obtain its differentiability at all points of the real line except a countable set and the density $F_1^{'}$ of f is bounded by $L_1 + L_2$.

The Local Limit Theorem

For any integer $n \geq 1$ we denote

$$b_n = \mu\left(n^{-1/2} S_n(g) \in [-1, 1]\right)$$

From the central limit theorem it follows that there exists a positive real number b such that $b_n \geq b > 0$ for all sufficiently large n. Moreover, there exists an increasing sequence $(n_k)_{k \geq 0}$ of integers such that for any $k \geq 0$

$$\rho_k \geq a_{n_k} \tag{20}$$

where $\rho_k = d_k/\sigma(f)$. Let $k \geq 0$ be a fixed integer, choose N_k such that $N_k \geq 2n_k$ and denote

$$\widetilde{G}_k = \bigcup_{i=0}^{N_k - n_k} T^i F_k \subset G_k$$

and

$$E_k = \{\omega \in \widetilde{G}_k \,;\, n_k^{-1/2} \sum_{i=0}^{n_k - 1} f(T^i \omega) \in [-d_k, d_k]\}.$$

Let i in $\{0, ..., n_k - 1\}$ be fixed. Since $T^i \widetilde{G}_k \subset G_k$, we have $h(T^i \omega) = d_k$ for any ω in $\widetilde{G}_k$; using (18) we deduce

$$\begin{aligned} E_k &= \{\omega \in \widetilde{G}_k \,;\, n_k^{-1/2} \sum_{i=0}^{n_k - 1} g(T^i \omega)\, h(T^i \omega) \in [-d_k, d_k]\} \\ &= \{\omega \in \widetilde{G}_k \,;\, n_k^{-1/2} \sum_{i=0}^{n_k - 1} d_k\, g(T^i \omega) \in [-d_k, d_k]\} \\ &= \{\omega \in \widetilde{G}_k \,;\, n_k^{-1/2} \sum_{i=0}^{n_k - 1} g(T^i \omega) \in [-1, 1]\}. \end{aligned}$$

By the independence of $\mathcal{B}$ and $\mathcal{C}$ it follows that

$$\mu\left(n_k^{-1/2} S_{n_k}(f) \in [-d_k, d_k]\right) \geq \mu(E_k) = \mu(\widetilde{G}_k)\, b_{n_k} \geq \mu(\widetilde{G}_k)\, b.$$

Moreover

$$\mu(\widetilde{G}_k) = p_k \left(1 - \frac{n_k}{N_k}\right) \geq \frac{p_k}{2}.$$

Hence, we derive

$$\frac{1}{d_k}\mu\left(n_k^{-1/2}S_{n_k}(f)\in[-d_k,d_k]\right)\geq\frac{bp_k}{2d_k}.$$

From th equality $\rho_k = d_k/\sigma(f)$ and from (19) it follows that

$$\frac{1}{\rho_k}\mu\left(\frac{S_{n_k}(f)}{\sigma(f)\sqrt{n_k}}\in[-\rho_k,\rho_k]\right)\geq\frac{bp_k}{2d_k}\sigma(f)=\frac{\sqrt{7}bp_k}{4\sqrt{3}d_k}\left(\frac{c_1}{L_1^2}+\frac{c_2}{L_2^2}\right)^{1/2}.$$

Consequently, if k is odd then

$$\frac{1}{\rho_k}\mu\left(\frac{S_{n_k}(f)}{\sigma(f)\sqrt{n_k}}\in[-\rho_k,\rho_k]\right)\geq\frac{bp_k}{2d_k}\sigma(f)=\frac{\sqrt{7}b}{4\sqrt{3}}\left(\frac{c_1L_2^2}{L_1^2}+c_2\right)^{1/2}.$$

Finally choosing L_2 sufficiently large we derive that for k odd

$$\frac{1}{\rho_k}\mu\left(\frac{S_{n_k}(f)}{\sigma(f)\sqrt{n_k}}\in[-\rho_k,\rho_k]\right)\geq L. \tag{21}$$

As a consequence the strong martingale difference sequence $(f\circ T^k)_{k\geq 0}$ does not satisfy the local limit theorem for densities.

The rate of convergence in the Central Limit Theorem

Now we are going to prove the last part of Theorem 2. Recall that Φ and φ are respectively the distribution function and the density function of the standard normal law. Let $k\geq 0$ be a fixed odd integer. Using (21) we obtain that

$$\begin{aligned}L\rho_k &\leq \mu\left(\frac{S_{n_k}(f)}{\sigma(f)\sqrt{n_k}}\in[-\rho_k,\rho_k]\right)\\ &= |F_{n_k}(f,\rho_k)-F_{n_k}(f,-\rho_k)|\\ &\leq 2\sup_x|F_{n_k}(f,x)-\Phi(x)|+|\Phi(\rho_k)-\Phi(-\rho_k)|\\ &\leq 2\sup_x|F_{n_k}(f,x)-\Phi(x)|+2\rho_k\varphi(0),\end{aligned}$$

hence

$$\sup_x|F_{n_k}(f,x)-\Phi(x)|\geq(L/2-\varphi(0))\rho_k.$$

Putting L sufficiently large and using inequality (20) we derive that for any odd integer $k\geq 0$,

$$\sup_x|F_{n_k}(f,x)-\Phi(x)|\geq a_{n_k}.$$

The proof of Theorem 2 is complete. □

3.3 Proof of Theorem 3

Consider the non-atomic probability space $(\Omega, \mathcal{A}, \mu)$. By induction we shall construct a sequence $(p_k)_{k\geq 1}$ of positive real numbers with $\sum_{k\geq 1} p_k = 1$, positive integers $(N_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$, measurable sets $(F_k)_{k\geq 1}$ and a bijective bimeasurable transformation $T : \Omega \to \Omega$ which preserves the measure μ such that the sets $T^j F_k$, $j = 0, ..., N_k - 1$, $k = 1, 2, ...$ form a partition of Ω in the way that

$$\mu\left(\bigcup_{j=0}^{N_k-1} T^j F_k\right) = p_k, \;\; k \geq 1$$

and such that for any $k \geq 1$

$$p_k \geq 4a_{n_k} \quad \text{and} \quad N_k \geq 4n_k^2. \tag{22}$$

Assume that this construction is achieved. Let $k \geq 1$ be a fixed integer and let $\overline{F}_k \subset F_k$ be a measurable set such that $\mu(\overline{F}_k) = \mu(F_k)/2$ and define

$$A_k = \bigcup_{j=0}^{N_k-n_k} T^j \overline{F}_k.$$

Let $(g \circ T^k)_{k\geq 0}$ be a sequence of independent random variables independent of the σ-algebra $\bigvee_{i=-\infty}^{+\infty} T^i \sigma(\overline{F}_k, F_k \backslash \overline{F}_k, k \geq 1)$ and such that $\mu(g = \pm 1) = 1/2$. Denote by

$$f = g\, 1\!\!1_{A^c} \quad \text{where} \quad A = \bigcup_{k\geq 1} A_k. \tag{23}$$

As done in the proof of Theorem 2 we can check that the process $(f \circ T^k)_{k\geq 0}$ is a strong martingale difference sequence. Let $k \geq 1$ be a fixed integer. By construction we have

$$\begin{aligned}
\mu(A_k \backslash \cap_{i=0}^{n_k-1} T^{-i} A_k) &\leq \sum_{i=0}^{n_k-1} \mu(A_k \backslash T^{-i} A_k) \\
&= \sum_{i=0}^{n_k-1} i \times p_k / 2N_k \\
&\leq \frac{n_k^2 p_k}{2N_k}
\end{aligned}$$

So using (22) we derive

$$
\begin{aligned}
\mu(\cap_{i=0}^{n_k-1} T^{-i} A_k) &\geq \mu(A_k) - \frac{n_k^2 p_k}{2N_k} \\
&= \frac{(N_k - n_k + 1)p_k}{2N_k} - \frac{n_k^2 p_k}{2N_k} \\
&\geq \frac{p_k}{2} - \frac{n_k^2 p_k}{N_k} \\
&= p_k \left(\frac{1}{2} - \frac{n_k^2}{N_k} \right) \\
&\geq \frac{p_k}{4} \geq a_{n_k}.
\end{aligned}
$$

Since $\{S_{n_k}(f) = 0\} \supset \cap_{i=0}^{n_k-1} T^{-i} A_k$ we obtain inequality (9). The proof of Inequality (10) is obtained from Inequality (9) as done in section 3.1.2, so it is left to the reader. Now, using induction, we will construct the sequences $(p_k)_k$, $(N_k)_k$, $(n_k)_k$, $(F_k)_k$ and the transformation T such that the condition (22) holds and we have to show that the martingale difference sequence $(f \circ T^k)_{k \geq 0}$ is also strongly mixing.
Let $k \geq 2$ be a fixed integer, let $p_1, ..., p_{k-1} > 0$ be given with $\sum_{i=1}^{k-1} p_i < 1$ and denote $p_k^{'} = 1 - \sum_{i=1}^{k-1} p_i$. Let $n_1 < n_2 < ... < n_{k-1}$ and $N_1 < N_2 < ... < N_{k-1}$ be sufficiently large integers such that the condition (22) holds. There exist k sets F_l, $l = 1, ..., k-1$, $F_k^{'}$ and a bijective bimeasurable measure-preserving transformation $T_k : \Omega \to \Omega$ such that the sets $T_k^j F_l$, $j = 0, ..., N_l - 1$, $l = 1, ..., k-1$, $T_k^j F_k^{'}$, $j = 0, ..., N_k - 1$ form a partition of Ω and

$$
\mu \left(\bigcup_{j=0}^{N_k-1} T_k^j F_k^{'} \right) = p_k^{'}, \quad \mu \left(\bigcup_{j=0}^{N_l-1} T_k^j F_l \right) = p_l, \, l = 1, ..., k-1.
$$

On the sets $T_k^{N_l-1} F_l$, $l = 1, ..., k-1$, $T_k^{N_k-1} F_k^{'}$ the transformation T_k acts in the way that

$$
T_k \left(T_k^{N_k-1} F_k^{'} \cup \bigcup_{l=1}^{k-1} T_k^{N_l-1} F_l \right) = F_k^{'} \cup \bigcup_{l=1}^{k-1} F_l
$$

and for any F in $\{T_k^{N_k-1} F_k^{'}, T_k^{N_1-1} F_1, ..., T_k^{N_{k-1}-1} F_{k-1}\}$ F is mapped into $F_k^{'}$, $F_1, ..., F_{k-1}$ with probabilities

$$
\mu(F_k^{'} | TF) = \frac{p_k^{'}/N_k}{p_k^{'}/N_k + \sum_{j=1}^{k-1} p_j/N_j}
$$

and

$$\mu(F_l|TF) = \frac{p_l/N_l}{p_k^{'}/N_k + \sum_{j=1}^{k-1} p_j/N_j},\ l = 1, ..., k-1.$$

Let us construct the transformation T_{k+1}. We choose an arbitrarily small $\delta_k > 0$ and define $p_k = (1-\delta_k)p_k^{'}$ and $p_{k+1}^{'} = \delta_k p_k^{'}$. We choose also a positive integer N_{k+1} sufficiently large such that the condition (22) holds. Then there exist sets $F_k \subset F_k^{'}$ and $F_{k+1}^{'} \subset F_k^{'}$ and an automorphism $T_{k+1} : \Omega \to \Omega$ such that

- $T_{k+1} = T_k$ on the set $\bigcup_{l=1}^{k} \bigcup_{j=0}^{N_l-2} T_k^j F_l$

- the sets $T_{k+1}^j F_l$, $j = 0, ..., N_l-1$, $l = 1, ..., k$ (which are equal to the sets $T_k^j F_l$, $j = 0, .., N_l - 1$, $l = 1, ..., k$) and $T_{k+1}^j F_{k+1}^{'}$, $j = 0, ..., N_{k+1} - 1$ form a partition of Ω.

- $\mu\left(\cup_{j=0}^{N_k-1} T_{k+1}^j F_k\right) = p_k$ and $\mu\left(\cup_{j=0}^{N_{k+1}-1} T_{k+1}^j F_{k+1}^{'}\right) = p_{k+1}^{'}$.

On the sets $T_{k+1}^{N_{k+1}-1} F_{k+1}^{'}$, $T_{k+1}^{N_l-1} F_l$, $l = 1, ..., k$, we define T_{k+1} so that

$$T_{k+1}\left(T_{k+1}^{N_{k+1}-1} F_{k+1}^{'} \cup \bigcup_{l=1}^{k} T_{k+1}^{N_l-1} F_l\right) = F_{k+1}^{'} \cup \bigcup_{l=1}^{k} F_l$$

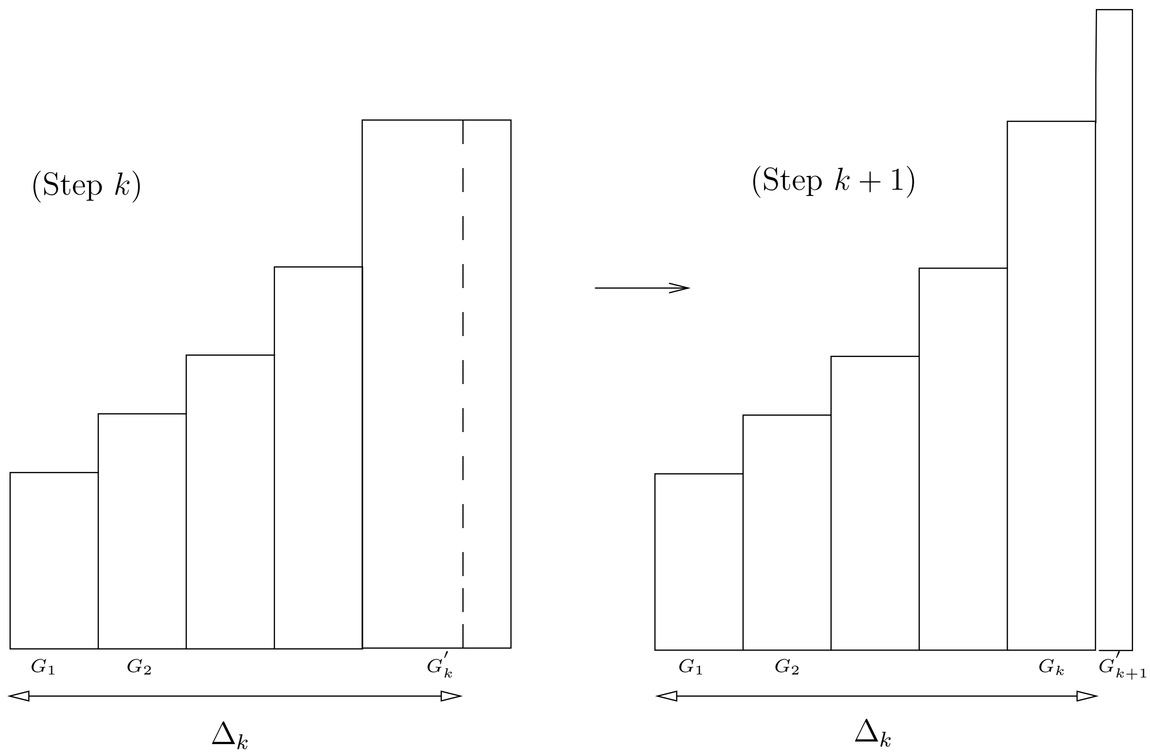
and for any F in $\{T_{k+1}^{N_{k+1}-1} F_{k+1}^{'}, T_{k+1}^{N_1-1} F_1, ..., T_{k+1}^{N_k-1} F_k\}$, F is mapped into $F_{k+1}^{'}$, $F_1, ..., F_k$ with probabilities

$$\mu(F_{k+1}^{'}|TF) = \frac{p_{k+1}^{'}/N_{k+1}}{p_{k+1}^{'}/N_{k+1} + \sum_{j=1}^{k} p_j/N_j}$$

and

$$\mu(F_l|TF) = \frac{p_l/N_l}{p_{k+1}^{'}/N_{k+1} + \sum_{j=1}^{k} p_j/N_j},\ l = 1, ..., k.$$

Consider the sets $G_l = \cup_{j=0}^{N_l-1} T_k^j F_l$, $l = 1, ..., k-1$, $G_k^{'} = \cup_{j=0}^{N_k-1} T_k^j F_k^{'}$ and $G_{k+1}^{'} = \cup_{j=0}^{N_{k+1}-1} T_{k+1}^j F_{k+1}^{'}$ and denote by Δ_k the set $\cup_{l=1}^{k} G_l$. In order to clarify the construction we present the following picture which shows the k-st and $(k+1)$-st partitions of Ω.



Moreover, by choosing δ_k sufficiently small we are able to make the measure of the set $\{\omega \in \Omega\,,\, T_k\omega \neq T_{k+1}\omega\}$ to be as small as we wish. For almost every ω in Ω there then exists $T\omega$ in Ω such that $T_k\omega = T\omega$ for all k sufficiently large. The transformation $T : \Omega \to \Omega$ acts in the way that

$$T\left(\bigcup_{l\geq 1} T^{N_l-1}F_l\right) = \bigcup_{l\geq 1} F_l$$

and for any F in $\{T^{N_1-1}F_1, T^{N_2-1}F_2, ...\}$ F is mapped into $F_1, F_2, ...$ with probabilities

$$\frac{p_l/N_l}{\sum_{j\geq 1} p_j/N_j},\ l \geq 1.$$

By $\mathcal{S}$ we denote the family of the sets $\{T^jF_l,\, j = 0, ..., N_l - 1,\, l = 1, 2, ...\}$ and by $\mathcal{S}_k$ we denote the family $\{T_k^jF_l,\, j = 0, ..., N_l - 1,\, l = 1, ..., k-1\} \cup \{T_k^jF_k^{'},\, j = 0, ..., N_k - 1\}$. The transformation T then defines a Markov chain $(\xi_i)_i$ with the state space $\mathcal{S}$ (in the way that if $T^i\omega \in T^jF_l$ then $\xi_i(\omega) = T^jF_l$) and the transformation T_k defines a Markov chain $(\xi_i^{(k)})_i$ with the state space $\mathcal{S}_k$ (analogically). We choose the numbers $(N_l)_{l\geq 1}$ so that for any $k \geq 2$, the greatest common divisor of $\{N_1, ..., N_k\}$ is 1. Because T and T_k are automorphisms of Ω, the chains $(\xi_i)_i$ and $(\xi_i^{(k)})_i$ are stationary and due to the choice of the numbers $(N_l)_{l\geq 1}$ they are aperiodic and irreducible. For any k the state space $\mathcal{S}_k$ is finite, hence for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ (cf. Billingsley [4], page 363) such that for all $n \geq m$ and all $a, b \in \mathcal{S}_k$

$$|\mu(\xi_n^{(k)} = b | \xi_0^{(k)} = a) - \mu(\xi_n^{(k)} = b)| < \varepsilon.$$

Since $\mu(\xi_n^{(k)} = x) > 0$ for all $x \in \mathcal{S}_k$, we can choose $m = m_k$ so that for a given $\varepsilon_k > 0$

$$\left| \frac{\mu(\xi_0^{(k)} = a, \xi_n^{(k)} = b)}{\mu(\xi_0^{(k)} = a)\mu(\xi_n^{(k)} = b)} - 1 \right| < \varepsilon_k \tag{24}$$

for all $n \geq m_k$ and $a, b \in \mathcal{S}_k$.
Let $A \in \sigma(\xi_i^{(k)}, i \leq 0\}$ and $B \in \sigma(\xi_i^{(k)}, i \geq m_k)$. Suppose first that A and B are elementary cylinders, i.e. $A = \{\xi_{-s}^{(k)} = e_{-s}, ..., \xi_0^{(k)} = e_0\}$ and $B = \{\xi_{m_k}^{(k)} = e_{m_k}, ..., \xi_t^{(k)} = e_t\}$ with $e_0, e_{m_k} \in \mathcal{S}_k$. Then we derive

$$|\mu(A \cap B) - \mu(A)\mu(B)| = \mu(A)\mu(B) \left| \frac{\mu(\xi_{m_k}^{(k)} = e_{m_k}, \xi_0^{(k)} = e_0)}{\mu(\xi_{m_k}^{(k)} = e_{m_k})\mu(\xi_0^{(k)} = e_0)} - 1 \right|.$$

From (24) it follows

$$|\mu(A \cap B) - \mu(A)\mu(B)| \leq \varepsilon_k \mu(A)\mu(B). \tag{25}$$

Let us suppose that $p_1, ..., p_{k-1}, p_k^{'}, N_1, ..., N_k$ have been chosen, we choose the $\varepsilon_k > 0$ and an appropriate m_k (such that $\varepsilon_k \downarrow 0$ and $m_k \uparrow \infty$). Notice that for the Markov chain (ξ_i) the states $T_k^j F_l = T^j F_l$, $j = 0, ..., N_l - 1$, $l = 1, ..., k-1$ remain the same when we turn to the $(k+1)$-st step. If $\delta_k > 0$ is small enough the probabilities of transitions from $T^{N_l - 1} F_l$, $l = 1, ..., k$, will be almost the same (as close as we wish) as the probabilities of transitions from $T_k^{N_l - 1} F_l$, $l = 1, ..., k-1$, $T_k^{N_k - 1} F_k^{'}$ to F_l, $l = 1, ..., k-1$, $F_k^{'}$. Therefore, if we set $\delta_k > 0$ small enough, for each elementary cylinder A in $\sigma(\xi_i, i \leq 0)$ and for each elementary cylinder B in $\sigma(\xi_i, i \geq m_k)$ such that the 0-st coordinate of A and the m_k-coordinate B are in $\Delta_k = \cup_{l=1}^k \cup_{j=0}^{N_l - 1} T^j F_l$, we can deduce (using (25))

$$|\mu(A \cap B) - \mu(A)\mu(B)| \leq \varepsilon_k \mu(A)\mu(B). \tag{26}$$

Now if A and B are finite disjoint unions of such elementary cylinders, i.e. $A = \cup_{i\in I} A_i$ and $B = \cup_{j\in J} B_j$, then

$$\begin{aligned}|\mu(A\cap B) - \mu(A)\mu(B)| &\leq \sum_{(i,j)\in I\times J} |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)| \\ &\leq \varepsilon_k \sum_{(i,j)\in I\times J} \mu(A_i)\mu(B_j) \\ &= \varepsilon_k \mu(A)\mu(B).\end{aligned}$$

Consequently the inequality (26) still holds for any $A \in \sigma(\xi_i,\, i \leq 0)$ with 0-st coordinate in Δ_k and $B \in \sigma(\xi_i,\, i \geq m_k)$ with m_k-st coordinate in Δ_k (in fact, such measurable sets can be approximated by finite disjoint unions of elementary cylinders).
If we choose $\delta_k > 0$ small enough, the measure of Δ_k can be made arbitrarily close to 1, hence we get for any A in $\sigma(\xi_i,\, i \leq 0)$ and any B in $\sigma(\xi_i,\, i \geq m_k)$

$$|\mu(A\cap B) - \mu(A)\mu(B)| < \varepsilon_k \mu(A)\mu(B) + 6\mu(\Delta_k^c) \leq 7\varepsilon_k \xrightarrow[k\to+\infty]{} 0. \qquad (27)$$

Since in the definition (23) of the function f, the i.i.d. sequence $(g \circ T^k)_{k\geq 0}$ is independent of the set A then the process $(f\circ T^k)_{k\geq 0}$ is strongly mixing if and only if the process $(\mathbb{1}_{A^c} \circ T^k)_{k\geq 0}$ is. Actually, we can notice that there exists a measurable function h such that for any integer $i \geq 0$

$$\mathbb{1}_{A^c} \circ T^i = h(\xi_i)$$

where ξ is the Markov chain with state space $\mathcal{S}$ defined above. Using (27) and noting that the σ-algebras $\sigma(\xi_i,\, i \leq 0)$ and $\sigma(\xi_i,\, i \leq m_k)$ contain respectively the σ-algebras $\sigma(h(\xi_i),\, i \leq 0)$ and $\sigma(h(\xi_i),\, i \leq m_k)$ we obtain the strong mixing property. $\square$

Remark 4 The process $(f \circ T^i)$, which gives the counterexample of Theorem 3 can in fact be constructed in any dynamical system of positive entropy. Like in the proof of Theorem 1, it is sufficient to show that it can be constructed in any Bernoulli shift.
In the proof of Theorem 3 we constructed a partition $\mathcal{S} = \{T^j F_k, 0 \leq j \leq N_k - 1, k = 1, 2, \dots\}$ which generates the σ-field $\mathcal{A}$. If we let the sequence of ε_k converge to 0 sufficiently fast we can (using (25)) guarantee that for any $\varepsilon > 0$ there exists N such that for all m the partition $\bigvee_{-m}^{0} T^i\mathcal{S}$ is ε-independent of $\bigvee_{N}^{N+m} T^i\mathcal{S}$ (such dynamical system is called "weakly

Bernoulli"). This implies that the dynamical system is Bernoulli (cf. [26], [27]). Unfortunately, we did not find any article or book where the implication is presented for the case of countably infinite partitions (there are numerous references for the case of finite partitions, e.g. [27]).

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