

Invariant multidimensional matrices

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“On se persuade mieux, pour l'ordinaire , par les raisons qu'on a soi-même trouvées que par celles qui sont venues dans l'esprit des autres”(Pascal).

Abstract : In [AO] the authors study Steiner bundles via their unstable hyperplanes and proved that (see [AO], Tmm 5.9) :

A rank n Steiner bundle on \mathbb{P}^n which is $SL(2, \mathbb{C})$ invariant is a Schwarzenberger bundle.

In this note we give a very short proof of this result based on Clebsch-Gordon problem for $SL(2, \mathbb{C})$ -modules.

1 Introduction

Let A , B , and C three vector spaces over \mathbb{C} and $\phi : A \otimes B \rightarrow C^*$ a linear surjective map. We consider the sheaf \mathcal{S}_ϕ on $\mathbb{P}(A) \times \mathbb{P}(B)$ defined by,

$$0 \longrightarrow \mathcal{S}_\phi \longrightarrow C \otimes \mathcal{O}_{\mathbb{P}(A) \times \mathbb{P}(B)} \xrightarrow{t_\phi} \mathcal{O}_{\mathbb{P}(A) \times \mathbb{P}(B)}(1, 1)$$

with fibers $\mathcal{S}_\phi(a \otimes b) = \{c \in C \mid \phi(a \otimes b)(c) = 0\}$.

Remark 1. If $\dim_{\mathbb{C}} C < \dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B - 1$ then $\ker(\phi)$ meets the set of decomposable tensors, so there exist $a \in A$, $a \neq 0$ and $b \in B$, $b \neq 0$ such that $\phi(a \otimes b) = 0$.

Remark 2. When \mathcal{S}_ϕ is a vector bundle it gives two “associated” Steiner bundles S_A on $\mathbb{P}(A)$ and S_B on $\mathbb{P}(B)$ after projections (see [DK], prop. 3.20).

We denote by $(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ the variety of hyperplanes tangent to the Segre $\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ and by $(\{a\} \times \{b\} \times \mathbb{P}(C))^\vee$ the set of hyperplanes in $\mathbb{P}(A \otimes B \otimes C)$ containing $\{a\} \times \{b\} \times \mathbb{P}(C)$.

The next proposition is a reformulation of many results from, [GKZ] (see for instance Thm 3.1' page 458, prop 1.1 page 445), [AO] (see thm page 1) and [DO] (see cor.3.3).

Proposition 1.1. *Let A , B , and C three vector spaces over \mathbb{C} with $\dim_{\mathbb{C}} A = n + 1$, $\dim_{\mathbb{C}} B = m + 1$ and $\dim_{\mathbb{C}} C \geq n + m + 1$ and $\phi : A \otimes B \rightarrow C^*$ a surjective linear map. Then the following propositions are equivalent :*

- 1) \mathcal{S}_ϕ is a vector bundle over $\mathbb{P}(A) \times \mathbb{P}(B)$.
- 2) $\phi(a \otimes b) \neq 0$ for all $a \in A$, $a \neq 0$ and $b \in B$, $b \neq 0$.
- 3) $\phi \notin (\{a\} \times \{b\} \times \mathbb{P}(C))^\vee$ for all $a \in A$, $a \neq 0$ and $b \in B$, $b \neq 0$.
- 4) $\phi \notin (\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$

Remark 3. When $\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B - 1$ the variety $(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ is an hypersurface in $\mathbb{P}((A \otimes B \otimes C)^\vee)$. This hypersurface is defined by the vanishing of the

hyperdeterminant, say $Det(\Phi)$ where Φ is the generic tridimensional matrix (see [GKZ], chapter 1 and 14).

Proof. It is clear that 1) 2) and 3) are equivalent. It remains to show that 3) and 4) are equivalent too.

Since $\partial(abc) = (\partial(a))bc + a(\partial(b))c + ab(\partial(c))$ an hyperplane H is tangent to the Segre in a point (a, b, c) if and only if it contains $\mathbb{P}(A) \times \{b\} \times \{c\}$ and $\{a\} \times \mathbb{P}(B) \times \{c\}$ and $\{a\} \times \{b\} \times \mathbb{P}(C)$. We prove here that the third condition implies the two others. Let H an hyperplane containing $\{a\} \times \{b\} \times \mathbb{P}(C)$, we show that there exists $c \in C$ such that H contains $\mathbb{P}(A) \times \{b\} \times \{c\}$ and $\{a\} \times \mathbb{P}(B) \times \{c\}$. Let ϕ the trilinear application corresponding to H . Since $\phi(a \otimes b)(C) = 0$ we have a $\dim_{\mathbb{C}} C$ -dimensional family of bilinear forms vanishing on (a, b) . Now finding a bilinear form of the above family (i.e. finding $c \in C$) which verify $\phi(A \otimes b)(c) = 0$ and $\phi(a \otimes B)(c) = 0$ imposes at most $n+m$ conditions. Since $\dim_{\mathbb{C}} C \geq n+m+1$, this point c exists. \square

2 Invariant tridimensional matrix under $SL(2, \mathbb{C})$ -action

In the second part of this note we will consider the boundary case

$$\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B - 1$$

Then, instead of writing ϕ induces a vector bundle or $\phi \notin (\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^{\vee}$ we will write equivalently $Det(\phi) \neq 0$.

We denote by S_i the irreducible $SL(2, \mathbb{C})$ -representations of degree i and by $(x^{i-k}y^k)_{k=0, \dots, i}$ a basis of S_i .

Theorem 2.1. *Let A, B and C be three non trivial $SL(2, \mathbb{C})$ -modules with dimension $n+1, m+1$ and $n+m+1$ and $\phi \in \mathbb{P}(A \otimes B \otimes C)$ an invariant hyperplane under $SL(2, \mathbb{C})$. Then,*

$$Det(\phi) \neq 0 \Leftrightarrow \phi \text{ is the multiplication } S_n \otimes S_m \rightarrow S_{n+m}$$

Proof. When $\phi \in \mathbb{P}(S_n \otimes S_m \otimes S_{n+m})$ is just the multiplication $S_n \otimes S_m \rightarrow S_{n+m}$ it is well known that it corresponds to Schwarzenberger bundles (see [DK], prop 6.3).

Conversely, let $A = \bigoplus_{i \in I} S_i \otimes U_i$, $B = \bigoplus_{j \in J} S_j \otimes V_j$ where U_i, V_j are trivial $SL(2, \mathbb{C})$ -representations of dimension n_i and m_j . Let $x^i \in S_i, x^j \in S_j$ be two highest weight vectors and $u \in U_i, v \in V_j$. Since $Det(\phi) \neq 0$, $\phi((x^i \otimes u) \otimes (x^j \otimes v)) \neq 0$ and by $SL(2, \mathbb{C})$ -invariance $\phi((x^i \otimes u) \otimes (x^j \otimes v)) = x^{i+j} \phi(u \otimes v) \in S_{i+j} \otimes W_{i+j}$. By hypothesis $\phi(u \otimes v) \neq 0$ for all $u \in U_i$ and $v \in V_j$ so, by the Remark 1, it implies that $\dim W_{i+j} \geq n_i + m_j - 1$, and $S_{i+j}^{n_i+m_j-1} \subset C^*$.

Assume now that B contains at least two distinct irreducible representations. Let i_0 and j_0 the greatest integers in I and J . We consider the submodule B_1 such that $B_1 \oplus S_{j_0}^{m_{j_0}} = B$. Then the restricted map $A \otimes B_1 \rightarrow C^*$ is not surjective because the image is concentrated in the submodule C_1^* of C^* defined by $C_1^* \oplus S_{i_0+j_0}^{n_{i_0}+m_{j_0}-1} = C^*$. Now since

$$\dim_{\mathbb{C}}(C_1) < \dim_{\mathbb{C}}(A) + \dim_{\mathbb{C}}(B_1) - 1$$

there exist $a \in A, b \in B_1 \subset B$ such that $\phi(a \otimes b) = 0$. A contradiction with the hypothesis $Det(\phi) \neq 0$.

So $A = S_i^{n_i}$, $B = S_j^{m_j}$ and $S_{i+j}^{n_i+m_j-1} \subset C^*$. Since $\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B - 1$, we have $(i+1)n_i + (j+1)m_j - 1 = \dim_{\mathbb{C}} C \geq (i+j+1)(n_i+m_j-1)$ which is possible if and only if $n_i = m_j = 1$ and $C = S_{i+j}$. \square

Corollary 2.2. *A rank n Steiner bundle on \mathbb{P}^n which is $SL(2, \mathbb{C})$ invariant is a Schwarzenberger bundle.*

Proof. Let S a rank n Steiner bundle on \mathbb{P}^n , i.e S appears in an exact sequence

$$0 \longrightarrow S \longrightarrow C \otimes O_{\mathbb{P}(A)} \longrightarrow B^* \otimes O_{\mathbb{P}(A)}(1) \longrightarrow 0$$

where $\mathbb{P}(A) = \mathbb{P}^n$, $\mathbb{P}(B) = \mathbb{P}^m$ and $\mathbb{P}(C) = \mathbb{P}^{n+m}$. If $SL(2, \mathbb{C})$ acts on S the vector spaces A , B and C are $SL(2, \mathbb{C})$ -modules since A is the basis, $B^* = H^1 S(-1)$ and $C^* = H^0(S^*)$. If S is $SL(2, \mathbb{C})$ -invariant the linear surjective map

$$A \otimes (H^1 S(-1))^* \rightarrow H^0(S^*)$$

is $SL(2, \mathbb{C})$ -invariant too. \square

Remark. The proofs of the theorem and the proposition, given in this paper, are still valid for more than three vector spaces when the format is the boundary format.

I would like to thank L.Gruson, M.Meulien and N.Perrin for their help.

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