

A Littlewood-Richardson rule for evaluation representations of $U_q(\widehat{\mathfrak{sl}}_n)$

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Abstract

We give a combinatorial description of the composition factors of the induction product of two evaluation modules of the affine Iwahori-Hecke algebra of type GL_m . Using quantum affine Schur-Weyl duality, this yields a combinatorial description of the composition factors of the tensor product of two evaluation modules of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$.

1 Introduction

1.1 Let H_m denote the Iwahori-Hecke algebra of type A_{m-1} over $\mathbb{C}(t)$. This is a semisimple associative algebra isomorphic to the group algebra $\mathbb{C}(t)[\mathfrak{S}_m]$ of the symmetric group. Hence its simple modules $S(\lambda)$ are parametrized by the partitions λ of m . Consider a decomposition $m = m_1 + m_2$, and two partitions $\lambda^{(1)}$ and $\lambda^{(2)}$ of m_1 and m_2 , respectively. Then we have a H_{m_1} -module $S(\lambda^{(1)})$ and a H_{m_2} -module $S(\lambda^{(2)})$, and we can form the induced module

$$S(\lambda^{(1)}) \odot S(\lambda^{(2)}) := \text{Ind}_{H_{m_1} \otimes H_{m_2}}^{H_m} \left(S(\lambda^{(1)}) \otimes S(\lambda^{(2)}) \right).$$

Here, $H_{m_1} \otimes H_{m_2}$ is identified to a subalgebra of H_m in the standard way. Using again the isomorphism $H_m \cong \mathbb{C}(t)[\mathfrak{S}_m]$, we see that the multiplicity of a simple H_m -module $S(\mu)$ in $S(\lambda^{(1)}) \odot S(\lambda^{(2)})$ is equal to the classical Littlewood-Richardson coefficient $c_{\lambda^{(1)}\lambda^{(2)}}^\mu$ (see e.g. [Mcd]).

1.2 Let now \widehat{H}_m be the affine Iwahori-Hecke algebra over $\mathbb{C}(t)$ (see 2.1 below). For each invertible $z \in \mathbb{C}(t)$ we have a surjective *evaluation homomorphism* $\tau_z : \widehat{H}_m \rightarrow H_m$. Pulling back the simple H_m -module $S(\lambda)$ via τ_z we obtain a simple \widehat{H}_m -module $S(\lambda; z)$ called an *evaluation module*. In analogy with 1.1, given two invertible elements z_1 and z_2 of $\mathbb{C}(t)$, we can then form the induced \widehat{H}_m -module

$$S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2) := \text{Ind}_{\widehat{H}_{m_1} \otimes \widehat{H}_{m_2}}^{\widehat{H}_m} \left(S(\lambda^{(1)}; z_1) \otimes S(\lambda^{(2)}; z_2) \right).$$

It turns out that if we fix $\lambda^{(1)}, \lambda^{(2)}$ and vary the spectral parameters z_1, z_2 , this module is generically irreducible, that is, it is simple except for a finite number of values of the ratio z_1/z_2 . In [LNT, Theorem 36] a combinatorial description of these special values was given.

In this note we shall make this result more precise by describing all the composition factors of $S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$ at these critical values z_1/z_2 . We shall also prove that, in contrast with the classical Littlewood-Richardson rule, all the composition factors appear with multiplicity one. The composition factors occurring in a product will be described using the combinatorics of Lusztig's *symbols*, that is, of certain two-row arrays introduced by Lusztig for parametrizing the irreducible complex representations of the classical reductive groups over finite fields [Lu1, Lu2].

1.3 We will derive our combinatorial formula from some explicit calculations of canonical bases in level 2 representations of the quantum algebra $U_v(\mathfrak{sl}_{n+1})$ performed in [LM]. More precisely, by dualizing [LM, Theorem 3], we get a formula for the expansion of the product of two quantum flag minors on the dual canonical basis of $U_v(\mathfrak{sl}_{n+1})$ (Theorem 4). Using then Ariki's theorem as in [LNT], we obtain immediately the above-mentioned Littlewood-Richardson rule for induction products of two evaluation modules over affine Hecke algebras (Theorem 2).

Finally, by means of the quantum affine analogue of the Schur-Weyl duality developed by Cherednik, Chari-Pressley and Ginzburg-Reshetikhin-Vasserot, we can deduce from this rule a similar one for the tensor product of two evaluation modules over the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_N)$.

2 Composition factors of induced \widehat{H}_m -modules

2.1 Let \widehat{H}_m be the affine Hecke algebra of type GL_m over $\mathbb{C}(t)$. It has invertible generators $T_1, \dots, T_{m-1}, y_1, \dots, y_m$ subject to the relations

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & (1 \leq i \leq m-2), \\ T_i T_j &= T_j T_i, & (|i-j| > 1), \\ (T_i - t)(T_i + 1) &= 0, & (1 \leq i \leq m-1), \\ y_i y_j &= y_j y_i, & (1 \leq i, j \leq m), \\ y_j T_i &= T_i y_j, & (j \neq i, i+1), \\ T_i y_i T_i &= t y_{i+1}, & (1 \leq i \leq m-1). \end{aligned}$$

The subalgebra H_m generated by the T_i 's is the Iwahori-Hecke algebra of type A_{m-1} .

For any invertible $z \in \mathbb{C}(t)$ we have a unique algebra homomorphism $\tau_z : \widehat{H}_m \rightarrow H_m$ such that

$$\tau_z(T_i) = T_i, \quad \tau_z(y_1) = z, \quad (i = 1, \dots, m-1).$$

This is called the *evaluation at z*.

We also have an algebra automorphism $\sigma_z : \widehat{H}_m \rightarrow \widehat{H}_m$ such that

$$\sigma_z(T_i) = T_i, \quad \sigma_z(y_i) = zy_i, \quad (i = 1, \dots, m-1).$$

This is called the *shift by z* .

2.2 As mentioned in the introduction, given two partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, the structure of the induced \widehat{H}_m -module

$$S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$$

depends essentially on the ratio z_1/z_2 . Indeed, by twisting this module with the shift automorphism σ_z we obtain the induced module

$$S(\lambda^{(1)}; zz_1) \odot S(\lambda^{(2)}; zz_2).$$

For example, it is known that if $z_1/z_2 \notin t^{\mathbb{Z}}$ then $S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$ is irreducible. Therefore, we can assume without loss of generality that

$$z_i = t^{a_i}, \quad a_i \in \mathbb{Z}, \quad a_i \geq \ell(\lambda^{(i)}), \quad (i = 1, 2), \quad (1)$$

where as usual $\ell(\lambda)$ denotes the length of the partition λ . Since $S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$ and $S(\lambda^{(2)}; z_2) \odot S(\lambda^{(1)}; z_1)$ have the same composition factors with the same multiplicities, we can also assume that $a_1 \leq a_2$.

2.3 It will be convenient to write partitions in weakly *increasing* order. Given a partition λ and an integer $a \geq \ell(\lambda)$ we can make λ into a non-decreasing sequence $(\lambda_1, \dots, \lambda_a)$ of length a by setting $\lambda_j = 0$ for $j = 1, \dots, a - \ell(\lambda)$. We can then associate to (λ, a) the increasing sequence

$$\beta = (\beta_1, \dots, \beta_a), \quad \beta_j = j + \lambda_j. \quad (2)$$

In this way, given $(\lambda^{(i)}, a_i)$ ($i = 1, 2$) as in 2.2, we obtain a *symbol*

$$S = \begin{pmatrix} \beta^{(2)} \\ \beta^{(1)} \end{pmatrix} = \begin{pmatrix} \beta_1^{(2)}, \dots, \beta_{a_2}^{(2)} \\ \beta_1^{(1)}, \dots, \beta_{a_1}^{(1)} \end{pmatrix}. \quad (3)$$

For example, the symbol attached to the pairs $((1, 1, 2), 3)$ and $((2, 3), 5)$ is

$$S = \begin{pmatrix} 1 & 2 & 3 & 6 & 8 \\ 2 & 3 & 5 & & \end{pmatrix}.$$

Conversely, given a symbol S , *i.e.* a two-row array as in Eq. (3) with

$$1 \leq \beta_1^{(i)} < \dots < \beta_{a_i}^{(i)} \quad (i = 1, 2),$$

there is a unique pair $(\lambda^{(i)}, a_i)$ ($i = 1, 2$) whose symbol is S .

2.4 The symbol S of Eq. (3) is said to be *standard* if $\beta_k^{(2)} \leq \beta_k^{(1)}$ for $k \leq a_1$. In [LM, §2.5] we have defined the *pairs* of a standard symbol S , and the set $\mathcal{C}(S)$ of all symbols Σ obtained from S by permuting some of its pairs. As shown in [LM, Lemma 9], these notions are equivalent to the notion of admissible involution of Lusztig [Lu4].

For the convenience of the reader we shall recall these definitions. Let $S = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ be a standard symbol. We define an injection $\psi : \gamma \rightarrow \beta$ such that $\psi(j) \leq j$ for all $j \in \gamma$. To do so it is enough to describe the subsets

$$\gamma^l = \{j \in \gamma \mid \psi(j) = j - l\}, \quad (0 \leq l \leq n).$$

We set $\gamma^0 = \gamma \cap \beta$ and for $l \geq 1$ we put

$$\gamma^l = \{j \in \gamma - (\gamma^0 \cup \dots \cup \gamma^{l-1}) \mid j - l \in \beta - \psi(\gamma^0 \cup \dots \cup \gamma^{l-1})\}.$$

Observe that the standardness of S implies that ψ is well-defined.

Example 1 Take

$$S = \begin{pmatrix} 1 & 3 & 5 & 8 & 9 \\ 3 & 6 & 7 & 10 & \end{pmatrix}.$$

Then

$$\gamma^0 = \{3\}, \gamma^1 = \{6, 10\}, \gamma^2 = \dots = \gamma^5 = \emptyset, \gamma^6 = \{7\}.$$

Hence

$$\psi(3) = 3, \psi(6) = 5, \psi(7) = 1, \psi(10) = 9.$$

The pairs $(j, \psi(j))$ with $\psi(j) \neq j$ (that is, $j \notin \beta \cap \gamma$) will be called the *pairs* of S . Given a standard symbol S with p pairs, we denote by $\mathcal{C}(S)$ the set of all symbols obtained from S by permuting some pairs in S and reordering the rows. We consider S itself as an element of $\mathcal{C}(S)$, hence $\mathcal{C}(S)$ has cardinality 2^p .

2.5 Given a partition λ and an integer a we call *Young diagram of (λ, a)* the Young diagram of λ in which each cell (i, j) is filled with the integer $i - j + a$. For instance, if $\lambda = (2, 3)$ and $a = 5$ then the Young diagram of (λ, a) is

4	5	
5	6	7

The rows of the Young diagram of (λ, a) yield a *multisegment*

$$\mathbf{m}(\lambda, a) := \sum_{1 \leq k \leq a} [k, k + \lambda_k - 1].$$

This is a formal sum (or multiset) of intervals in \mathbb{Z} , in which we discard the empty intervals corresponding to the k 's with $\lambda_k = 0$. Thus, continuing with the same example, we have

$$\mathbf{m}((2, 3), 5) = [4, 5] + [5, 7].$$

Similarly, we attach to a pair $(\lambda^{(i)}, a_i)$ ($i = 1, 2$) or to its symbol S the multisegment

$$\mathbf{m}(S) = \mathbf{m}(\lambda^{(1)}, a_1) + \mathbf{m}(\lambda^{(2)}, a_2).$$

2.6 To each multisegment

$$\mathbf{m} := \sum_k [\alpha_k, \beta_k]$$

is attached an irreducible \widehat{H}_m -module $L_{\mathbf{m}}$, where $m = \sum_k (\beta_k + 1 - \alpha_k)$ (see e.g. [LNT, §2.1]).

2.7 Let us assume that the pair $(\lambda^{(i)}, a_i)$ ($i = 1, 2$) satisfies the conditions of 2.2. Let Σ denote the symbol attached to this pair. We can now state:

Theorem 2 *The composition factors of $S(\lambda^{(1)}; t^{a_1}) \odot S(\lambda^{(2)}; t^{a_2})$ are the modules $L_{\mathbf{m}(S)}$ where S runs through the set of standard symbols such that $\Sigma \in \mathcal{C}(S)$. Each of them occurs with multiplicity one.*

Theorem 2 will be deduced from Theorem 4 below.

Example 3 Let $(\lambda^{(1)}, a_1) = ((1, 4), 2)$ and $(\lambda^{(2)}, a_2) = ((1, 2, 3), 4)$. The corresponding symbol is

$$\Sigma = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \end{pmatrix}.$$

The standard symbols S such that $\Sigma \in \mathcal{C}(S)$ are

$$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 5 & 7 \\ 3 & 6 & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 & 6 \\ 2 & 7 & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \end{pmatrix}.$$

It follows that the composition factors of $S((1, 4); t^2) \odot S((1, 2, 3); t^4)$ are the $L_{\mathbf{m}}$ where \mathbf{m} is one the following multisegments:

$$\mathbf{n}_1 = [1, 2] + [2, 6] + [3, 4] + [4, 5], \quad \mathbf{n}_2 = [1, 2] + [2, 5] + [3, 4] + [4, 6],$$

$$\mathbf{n}_3 = [1, 1] + [2, 2] + [2, 6] + [3, 4] + [4, 5], \quad \mathbf{n}_4 = [1, 1] + [2, 2] + [2, 5] + [3, 4] + [4, 6].$$

3 Canonical bases

3.1 Fix $n \geq 2$ and let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We consider the quantum enveloping algebra $U_v(\mathfrak{g})$ over $\mathbb{Q}(v)$ with Chevalley generators e_j, f_j, t_j ($1 \leq j \leq n$). The simple roots and the fundamental weights are denoted by α_k and Λ_k ($1 \leq k \leq n$) respectively. The irreducible representation of $U_v(\mathfrak{g})$ with highest weight Λ is denoted by $V(\Lambda)$. We denote by $U_v(\mathfrak{n})$ the subalgebra of $U_v(\mathfrak{g})$ generated by e_j ($1 \leq j \leq n$).

3.2 Let \mathbf{B} (resp. \mathbf{B}^*) denote the canonical basis (resp. the dual canonical basis) of $U_v(\mathfrak{n})$ ([Lu3], [BZ]; see also [LNT, §3]). The elements of \mathbf{B} and \mathbf{B}^* are naturally labelled by the multisegments \mathbf{m} supported on $[1, n]$. We shall denote them by $b_{\mathbf{m}}$ and $b_{\mathbf{m}}^*$ respectively.

The vectors $b_{\mathbf{m}}^*$ for which \mathbf{m} is of the form

$$\mathbf{m} = \mathbf{m}(\lambda, a)$$

for some partition λ and some integer a are called *quantum flag minors*. Indeed, by [BZ], they can be expressed as quantum minors of a triangular matrix whose entries are iterated brackets of the e_i 's (see [LNT, §5.2]).

3.3 Let $(\lambda^{(i)}, a_i)$ ($i = 1, 2$) be as in 2.2. We also assume that the multisegments

$$\mathbf{m}_i = \mathbf{m}(\lambda^{(i)}, a_i) \quad (i = 1, 2)$$

are supported on $[1, n]$. Let Σ be the symbol attached to the pair $(\lambda^{(i)}, a_i)$ ($i = 1, 2$). For a standard symbol S such that $\Sigma \in \mathcal{C}(S)$ we denote by $n(S, \Sigma)$ the number of pairs of S which are permuted to get Σ . Finally, we denote by $N_j(\lambda, a)$ the number of cells of the Young diagram of (λ, a) containing the integer j .

Theorem 4 We have

$$b_{\mathbf{m}_1}^* b_{\mathbf{m}_2}^* = v^{-N_{a_1}(\lambda^{(2)}, a_2)} \sum_S v^{n(S, \Sigma)} b_{\mathbf{m}(S)}^*$$

where the sum runs through all standard symbols S such that $\Sigma \in \mathcal{C}(S)$.

Example 5 We take $(\lambda^{(1)}, a_1)$ and $(\lambda^{(2)}, a_2)$ as in Example 3. Hence

$$\mathbf{m}_1 = [1, 1] + [2, 5], \quad \mathbf{m}_2 = [2, 2] + [3, 4] + [4, 6].$$

Then $N_{a_1}(\lambda^{(2)}, a_2) = N_2((1, 2, 3), 4) = 1$, and we obtain, using the notation of Example 3,

$$b_{\mathbf{m}_1}^* b_{\mathbf{m}_2}^* = v^{-1}(v^2 b_{\mathbf{n}_1}^* + v b_{\mathbf{n}_2}^* + v b_{\mathbf{n}_3}^* + b_{\mathbf{n}_4}^*).$$

3.4 Proof of Theorem 4. Following [LNT, §7.2], we will replace calculations of products of elements of \mathbf{B}^* by calculations of dual canonical bases of finite-dimensional representations of $U_v(\mathfrak{g})$.

3.4.1 Let $U_v(\mathfrak{n}^-)$ denote the subalgebra of $U_v(\mathfrak{g})$ generated by the f_i 's, and let $x \mapsto x^\sharp$ denote the algebra isomorphism from $U_v(\mathfrak{n})$ to $U_v(\mathfrak{n}^-)$ defined by $e_i^\sharp = f_i$ ($i = 1, \dots, n$). Let Λ be a dominant integral weight and let u_Λ be a highest weight vector of the irreducible module $V(\Lambda)$. Then the map $\pi_\Lambda : x \mapsto x^\sharp u_\Lambda$ projects the canonical basis \mathbf{B} of $U_v(\mathfrak{n})$ to the union of the canonical basis $\mathbf{B}(\Lambda)$ of $V(\Lambda)$ with the set $\{0\}$. The dual map π_Λ^* gives an embedding of the dual canonical basis $\mathbf{B}^*(\Lambda)$ of $V(\Lambda) \simeq V(\Lambda)^*$ into the dual canonical basis \mathbf{B}^* of $U_v(\mathfrak{n}) \simeq U_v(\mathfrak{n})^*$.

3.4.2 In particular the subset of \mathbf{B}^* obtained by embedding the bases $\mathbf{B}^*(\Lambda_a)$ ($1 \leq a \leq n$) of the fundamental representations is precisely the subset of quantum flag minors. It is well known that $V(\Lambda_a)$ is a minuscule representation whose bases $\mathbf{B}^*(\Lambda_a)$ and $\mathbf{B}(\Lambda_a)$ coincide. Moreover the elements of these bases are naturally labelled by the pairs (λ, a) whose Young diagram (as defined in 2.5) contains only cells numbered by integers between 1 and n . Denoting them by $b_{(\lambda, a)}^*$ we have

$$\pi_{\Lambda_a}^*(b_{(\lambda, a)}^*) = b_{\mathbf{m}(\lambda, a)}^*$$

Equivalently, we can also label the elements of $\mathbf{B}^*(\Lambda_a)$ by one-row symbols β as in Eq. (2) with $\beta_i \leq n + 1$.

3.4.3 Similarly, the basis $\mathbf{B}^*(\Lambda_{a_1}) \otimes \mathbf{B}^*(\Lambda_{a_2})$ is naturally labelled by the set of symbols S as in Eq. (3) with $\beta_{a_i}^{(i)} \leq n + 1$ ($i = 1, 2$). Using the theory of crystal bases [K1, K2] one can see that the basis $\mathbf{B}^*(\Lambda_{a_1} + \Lambda_{a_2})$ has a natural labelling by the subset of standard symbols [LM, §2.3]. Moreover, denoting by b_S^* the element of $\mathbf{B}^*(\Lambda_{a_1} + \Lambda_{a_2})$ labelled by the standard symbol S we have, using also the notation of 2.5,

$$\pi_{\Lambda_{a_1} + \Lambda_{a_2}}^*(b_S^*) = b_{\mathbf{m}(S)}^*.$$

3.4.4 Let $\iota : V(\Lambda_{a_1} + \Lambda_{a_2}) \rightarrow V(\Lambda_{a_1}) \otimes V(\Lambda_{a_2})$ be the $U_v(\mathfrak{g})$ -module embedding which maps $u_{\Lambda_{a_1} + \Lambda_{a_2}}$ to $u_{\Lambda_{a_1}} \otimes u_{\Lambda_{a_2}}$, and let $\iota^* : V(\Lambda_{a_1}) \otimes V(\Lambda_{a_2}) \rightarrow V(\Lambda_{a_1} + \Lambda_{a_2})$ be its dual. Let $b_i^* \in \mathbf{B}^*(\Lambda_{a_i})$ ($i = 1, 2$) and denote by $b_{\mathbf{m}_i}^* = \pi_{\Lambda_{a_i}}^*(b_i^*)$ ($i = 1, 2$) the corresponding quantum flag minors. It is shown in [LNT, §7.2.7] that the image of $b_1^* \otimes b_2^*$ under the composition of maps $\pi_{\Lambda_{a_1} + \Lambda_{a_2}}^* \circ \iota^*$ coincides up to a power of v with the product $b_{\mathbf{m}_1}^* b_{\mathbf{m}_2}^*$. Hence to calculate the \mathbf{B}^* -expansion of $b_{\mathbf{m}_1}^* b_{\mathbf{m}_2}^*$ it is enough to calculate the matrix of the map ι^* with respect to the bases $\mathbf{B}^*(\Lambda_{a_1}) \otimes \mathbf{B}^*(\Lambda_{a_2})$ and $\mathbf{B}^*(\Lambda_{a_1} + \Lambda_{a_2})$.

3.4.5 The matrix of ι with respect to the bases $\mathbf{B}(\Lambda_{a_1} + \Lambda_{a_2})$ and $\mathbf{B}(\Lambda_{a_1}) \otimes \mathbf{B}(\Lambda_{a_1})$ was calculated in [LM, Theorem 3] in terms of Lusztig's symbols. Transposing this matrix we obtain the desired matrix of ι^* . Using 3.4.2 and 3.4.3, we then get the formula of Theorem 4. \square

3.5 *Proof of Theorem 2.* By [LNT, §3.7] the multisegments \mathbf{m} indexing the composition factors $L_{\mathbf{m}}$ of $S(\lambda^{(1)}; t^{a_1}) \odot S(\lambda^{(2)}; t^{a_2})$ are those occurring in the right-hand side of the formula of Theorem 4. Moreover the composition multiplicities are obtained by specializing v to 1 in the coefficients of this formula. Hence they are all equal to 1. \square

4 Tensor products of $U_q(\widehat{\mathfrak{sl}}_N)$ -modules

4.1 Let $U_q(\widehat{\mathfrak{sl}}_N)$ be the quantized affine algebra of type $A_{N-1}^{(1)}$ with parameter q a square root of t (see for example [CP] for the defining relations of $U_q(\widehat{\mathfrak{sl}}_N)$). The quantum affine Schur-Weyl duality between \widehat{H}_m and $U_q(\widehat{\mathfrak{sl}}_N)$ [CP, Ch, GRV] gives a functor $\mathcal{F}_{m,N}$ from the category of finite-dimensional \widehat{H}_m -modules to the category of level 0 finite-dimensional representations of $U_q(\widehat{\mathfrak{sl}}_N)$. If $N \geq m$, $\mathcal{F}_{m,N}$ maps the simple modules of \widehat{H}_m to simple modules of $U_q(\widehat{\mathfrak{sl}}_N)$. However, the image of a non-zero simple \widehat{H}_m -module may be the zero $U_q(\widehat{\mathfrak{sl}}_N)$ -module. More precisely, the simple \widehat{H}_m -module $L_{\mathbf{m}}$ is mapped to a non-zero simple $U_q(\widehat{\mathfrak{sl}}_N)$ -module if and only if all the segments occurring in \mathbf{m} have length $\leq N - 1$. In this case the Drinfeld polynomials of $\mathcal{F}_{m,N}(L_{\mathbf{m}})$ are easily calculated from \mathbf{m} (see [CP]).

The functor $\mathcal{F}_{m,N}$ transforms induction product into tensor product, that is, for M_1 in \mathcal{C}_{m_1} and M_2 in \mathcal{C}_{m_2} one has

$$\mathcal{F}_{m_1+m_2,N}(M_1 \odot M_2) = \mathcal{F}_{m_1,N}(M_1) \otimes \mathcal{F}_{m_2,N}(M_2).$$

4.2 The image under $\mathcal{F}_{m,N}$ of an evaluation module for \widehat{H}_m is an evaluation module for $U_q(\widehat{\mathfrak{sl}}_N)$, and all evaluation modules of $U_q(\widehat{\mathfrak{sl}}_N)$ can be obtained in this way, by varying $m \in \mathbb{N}^*$.

4.3 By application of the Schur functor $\mathcal{F}_{m,N}$ to Theorem 2 we thus obtain a combinatorial description of all composition factors of the tensor product of two evaluation modules of $U_q(\widehat{\mathfrak{sl}}_N)$.

Example 6 We continue Example 3 and Example 5. The image of the \widehat{H}_5 -module $L_{\mathbf{m}_1}$ under $\mathcal{F}_{5,N}$ is the evaluation module $V(\mathbf{m}_1)$ of $U_q(\widehat{\mathfrak{sl}}_N)$ with Drinfeld polynomials

$$\begin{aligned} P_1(u) &= u - q^{-2}, \\ P_2(u) &= P_3(u) = 1, \\ P_4(u) &= u - q^{-7}, \\ P_k(u) &= 1, \quad (5 \leq k \leq N - 1). \end{aligned}$$

This is a non-zero module if and only if $N \geq 5$. Similarly, the image of the \widehat{H}_6 -module $L_{\mathbf{m}_2}$ under $\mathcal{F}_{6,N}$ is the evaluation module $V(\mathbf{m}_2)$ of $U_q(\widehat{\mathfrak{sl}}_N)$ with Drinfeld polynomials

$$\begin{aligned} P_1(u) &= u - q^{-4}, \\ P_2(u) &= u - q^{-7}, \\ P_3(u) &= u - q^{-10}, \\ P_k(u) &= 1, \quad (4 \leq k \leq N - 1). \end{aligned}$$

This is a non-zero module if and only if $N \geq 4$. The images of the \widehat{H}_{11} -modules $L_{\mathbf{m}_1}, L_{\mathbf{m}_2}, L_{\mathbf{m}_3}, L_{\mathbf{m}_4}$ under $\mathcal{F}_{11,N}$ are the modules $V(\mathbf{n}_1), V(\mathbf{n}_2), V(\mathbf{n}_3), V(\mathbf{n}_4)$ with respective Drinfeld polynomials

$$\begin{aligned} P_1(u) &= 1, \\ P_2(u) &= (u - q^{-3})(u - q^{-7})(u - q^{-9}), \\ P_3(u) &= P_4(u) = 1, \\ P_5(u) &= u - q^{-8}, \\ P_k(u) &= 1, \quad (6 \leq k \leq N - 1); \end{aligned}$$

$$\begin{aligned} P_1(u) &= 1, \\ P_2(u) &= (u - q^{-3})(u - q^{-7}), \\ P_3(u) &= u - q^{-10}, \\ P_4(u) &= u - q^{-7}, \\ P_k(u) &= 1, \quad (5 \leq k \leq N - 1); \end{aligned}$$

$$\begin{aligned} P_1(u) &= (u - q^{-2})(u - q^{-4}), \\ P_2(u) &= (u - q^{-7})(u - q^{-9}), \\ P_3(u) &= P_4(u) = 1, \\ P_5(u) &= u - q^{-8}, \\ P_k(u) &= 1, \quad (6 \leq k \leq N - 1); \end{aligned}$$

$$\begin{aligned} P_1(u) &= (u - q^{-2})(u - q^{-4}), \\ P_2(u) &= u - q^{-7}, \\ P_3(u) &= u - q^{-10}, \\ P_4(u) &= u - q^{-7}, \\ P_k(u) &= 1, \quad (5 \leq k \leq N - 1). \end{aligned}$$

The modules $V(\mathbf{n}_1)$ and $V(\mathbf{n}_3)$ are non-zero only if $N \geq 6$. Hence $V(\mathbf{m}_1) \otimes V(\mathbf{m}_2)$ has only two composition factors $V(\mathbf{n}_2)$ and $V(\mathbf{n}_4)$ for $N = 5$, and four composition factors $V(\mathbf{n}_1)$, $V(\mathbf{n}_2)$, $V(\mathbf{n}_3)$, $V(\mathbf{n}_4)$ for $N \geq 6$.

4.4 We note that our result implies the following

Theorem 7 *All composition factors of the tensor product of two evaluation modules of $U_q(\widehat{\mathfrak{sl}}_N)$ occur with multiplicity one.*

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