ANALYTIC TORSION OF HIRZEBRUCH SURFACES

CHRISTOPHE MOUROUGANE

Abstract. *Using different forms of the arithmetic Riemann-Roch theorem and the computations of Bott-Chern secondary classes, we compute the analytic torsion and the height of Hirzebruch surfaces.* [1]

1. Introduction

In [R-S-73], Ray and Singer defined the analytic torsion for hermitian complex manifolds, which can be seen as a regularized determinant of the Laplacian for the $\overline{\partial}$ operator acting on the spaces of forms. They computed it for curves as functions on the moduli spaces and related it with modular forms. Yoshikawa generalized this relation for Theta divisors on Abelian varieties [Y-99]. Another kind of result is about Hermitian symmetric manifolds, where the spectrum of the Laplacian (and hence the analytic torsion) can be explicitly derived from the spectrum of the Casimir operator [K-95].

We compute the analytic torsion $\tau(S_n) = -\log T_0(S_n, 1)$ of (the trivial flat Hermitian line bundle on) the Hirzebruch surfaces S_n the only ruled surfaces over $\mathbb{P}^1$, endowed with some canonically defined hermitian metric. We prove

Main theorem

$$\tau(S_n) - \log Vol(S_n) = \frac{n\log(n+1)}{24} - \frac{n}{6} + 2\tau(\mathbb{P}^1).$$

For the analytic structure of S_n varies with n, results on adiabatic limits (as in [B-B-94] for example) do not apply at once. There is nevertheless a striking resemblance with the asymptotic of the analytic torsion associated with the positive line bundles $\mathcal{O}_{\mathbb{P}^1}(n) \to \mathbb{P}^1$ computed in [B-V-89] Theorem 8.

Our method consists in applying different forms of the arithmetic Riemann-Roch theorem, reducing the computation of the analytic torsion to the computations of distinguished Bott-Chern classes. For we have a description of the arithmetic Chow group of the natural $\mathbb{Z}$-model $\mathcal{S}_n$ of the Hirzebruch surface S_n, we can also compute its height.

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2. Preliminaries

2.1. **With the relative Euler sequence.** We will recall some facts about the arithmetic Chow ring of the projective line $\mathbb{P}^1_{\mathbb{Z}}$ and of the Hirzebruch surfaces $\mathcal{S}_n$. We refer to the book [S-92] for basics about the arithmetic constructions.

Let $\mathcal{V}$ be a free $\mathbb{Z}$-module of rank 2 and $\mathbb{P}^1_{\mathbb{Z}} = \mathbb{P}(\mathcal{V}) \to Spec\,\mathbb{Z}$ the projective space of rank one quotients of $\mathcal{V}$. The choice of a Hermitian scalar product on $V := \mathcal{V} \otimes_{\mathbb{Z}} \mathbb{C}$ determines canonically a Hermitian metric h on the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$, as the quotient metric through $\mathbb{P}^1 \times V \to \mathcal{O}_{\mathbb{P}^1}(1)$. Its curvature form gives a Kähler form on $\mathbb{P}^1$ and will be denoted by $x = \omega_{\mathbb{P}^1}$. The arithmetic first Chern class of $\overline{(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1), h)}$ will be denoted by $\widehat{x}$. Considering the exact sequence $\overline{\mathcal{S}}$ with natural metrics

$$0 \to \overline{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(-1)} \to \overline{\mathcal{V}^\star} \to \overline{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)} \to 0$$

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we derive the relation $(1-\widehat{x})(1+\widehat{x})-1 = a(\widetilde{c}(\overline{\mathcal{S}}))$ that is, using [G-S-90] proposition 5.3, $\widehat{x}^2 = -a(\widetilde{c}_2(\overline{\mathcal{S}})) = a(x)$.

Consider the ample vector bundle $\mathcal{E}_n := \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n+1)$ on $\mathbb{P}^1_{\mathbb{Z}}$. Then,

$$\mathcal{S}_n = \mathbb{P}(\mathcal{E}_n) \xrightarrow{\pi} \mathbb{P}^1_{\mathbb{Z}} \xrightarrow{f} Spec\,\mathbb{Z}$$

is a $\mathbb{Z}$-model of the Hirzebruch surface S_n. The tautological ample line bundle on $\mathcal{S}_n$ will be denoted by $\mathcal{O}_{\mathcal{E}_n}(1)$. From the metric h, we construct an orthogonal sum Hermitian metric h_n on $E_n := \mathcal{E}_n(\mathbb{C})$. The arithmetic Chern class of $\overline{\mathcal{E}_n}$ is hence $\widehat{c}(\overline{\mathcal{E}_n}) = (1+\widehat{x})(1+(n+1)\widehat{x})$. We also get a Hermitian metric (still denoted by h_n) of positive curvature $\alpha_n := \Theta(\mathcal{O}_{E_n}(1), h_n)$ on $\mathcal{O}_{E_n}(1) := \mathcal{O}_{\mathcal{E}_n}(1)(\mathbb{C})$ as quotient metric through $\pi^\star E_n \to \mathcal{O}_{E_n}(1)$. The arithmetic first Chern class of $\overline{\mathcal{O}_{\mathcal{E}_n}(1)}$ will be denoted by $\widehat{\alpha_n}$. Considering the relative Euler sequence $\overline{\Sigma(1)}$ metrized with induced and quotient metric from the metric h_n on E_n

$$0 \to \overline{\mathcal{O}_{\mathbb{P}(\mathcal{E}_n)}} \to \pi^\star \overline{\mathcal{E}_n^\star} \otimes \overline{\mathcal{O}_{\mathcal{E}_n}(1)} \xrightarrow{q} \overline{T_{\mathbb{P}(\mathcal{E}_n)/\mathbb{P}^1_{\mathbb{Z}}}} \to 0$$

we derive the relations

$$\begin{aligned} 0 &= \widehat{c}_2(\pi^\star \overline{\mathcal{E}_n^\star} \otimes \overline{\mathcal{O}_{\mathcal{E}_n}(1)}) + a(\widetilde{c}_2(\overline{\Sigma(1)})) \\ &= \widehat{\alpha_n}^2 - (n+2)\pi^\star \widehat{x}\widehat{\alpha_n} + (n+1)\pi^\star \widehat{x}^2 + a(\widetilde{c}_2(\overline{\Sigma(1)})). \qquad (2.1.1) \end{aligned}$$

$$\widehat{c}_1(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}}, \omega_q) = \widehat{c}_1(\pi^\star \overline{\mathcal{E}_n^\star} \otimes \overline{\mathcal{O}_{\mathcal{E}_n}(1)}) + a(\widetilde{c}_1(\overline{\Sigma(1)})) = 2\widehat{\alpha_n} - (n+2)\pi^\star \widehat{x}. \qquad (2.1.2)$$

Recall from our computations [M-03], that

$$\widetilde{c}_2(\overline{\Sigma(1)}) = \widetilde{c}_2(\overline{\Sigma}) = -\Omega = -\alpha_n - \frac{\langle \Theta(E_n^\star, h) a^\star, a^\star \rangle_h}{\langle a^\star, a^\star \rangle_h}$$

where Ω denotes the relative Fubini-Study form. We have proved the following arithmetic analogs of relations in Chow groups.

Lemma 1. *In the arithmetic Chow ring of $\mathbb{P}^1_{\mathbb{Z}}$, we have*

$$\widehat{x}^2 = a(x).$$

In the arithmetic Chow ring of $\mathcal{S}_n$, we have

$$\widehat{\alpha_n}^2 - (n+2)\widehat{x}\widehat{\alpha_n} = a\left(\alpha_n - (n+1)x + \frac{\langle \Theta(E^\star, h) a^\star, a^\star \rangle_h}{\langle a^\star, a^\star \rangle_h}\right).$$

This exact sequence also enables to compute the height of Hirzebruch surfaces with respect to the polarization $\overline{\mathcal{O}_{\mathcal{E}_n}(1)}$. The arithmetic height of an arithmetic variety $\mathcal{X} \xrightarrow{p} Spec\,\mathbb{Z}$ of relative dimension n with respect to a polarization $\overline{\mathcal{L}} \to \mathcal{X}$ is defined to be $\widehat{h}_{\overline{\mathcal{L}}}(\mathcal{X}) := \widehat{deg}\, \widehat{p}_\star \left(\widehat{c}_1(\overline{\mathcal{L}})^{n+1}\right)$ where $\widehat{deg} : \widehat{CH}^1(\mathbb{Z}) \to \mathbb{R}$ sends $(\sum n_{\mathcal{P}} \mathcal{P}, \lambda)$ to $\sum n_{\mathcal{P}} \log \mathcal{P} + \lambda/2$. The arithmetic height of $\mathcal{S}_n$ is hence given by

$$\widehat{h}_{\overline{\mathcal{O}_{\mathcal{E}_n}(1)}}(\mathcal{S}_n) = \widehat{deg}\; \widehat{f}_\star \widehat{\pi}_\star \left(\widehat{\alpha_n}^3\right) = \widehat{deg}\; \widehat{f}_\star \widehat{s'_2}(\overline{\mathcal{E}_n})$$

where $\widehat{s'_m}(\overline{\mathcal{E}_n}) := \widehat{\pi}_\star \left(\widehat{\alpha_n}^{1+m}\right)$ is the m-th geometric Segre class of $\overline{\mathcal{E}_n}$ in the arithmetic Chow group $\widehat{CH}^m(\mathbb{P}^1_{\mathbb{Z}})$ of $\mathbb{P}^1_{\mathbb{Z}}$. Define the secondary form

$$S_{m+1}(E_n, h_n) = \pi_\star(\widetilde{c}_2(\overline{\Sigma}(1))\alpha_n^m) \in \widetilde{A^{m,m}(\mathbb{P}^1)}.$$

From the degree two relation (2.1.1) on arithmetic Chern classes, taking product with $\widehat{\alpha_n}^m$ ($m = 0\,;\,1$) pushing-forward through $\widehat{\pi}_\star$ and applying usual rules of calculus on arithmetic Chow groups we are led to relations in $\widehat{CH}(\mathbb{P}^1_{\mathbb{Z}})$

$$\begin{aligned} \widehat{s'_1}(\overline{\mathcal{E}_n}) - (n+2)\widehat{x} + a(S_1(E_n, h_n)) &= 0 \\ \widehat{s'_2}(\overline{\mathcal{E}_n}) - (n+2)\widehat{x}\widehat{s'_1}(\overline{\mathcal{E}_n}) + (n+1)\widehat{x}^2 + a(S_2(E_n, h_n)) &= 0 \end{aligned}$$

and then to

$$\widehat{s_2'}(\overline{\mathcal{E}_n}) = (n^2+3n+3)\widehat{x}^2 - (n+2)a(xS_1(E_n,h_n)) - a(S_2(E_n,h_n)).$$

Using, $\widetilde{c}_2(\overline{\Sigma}(1)) = -\Omega$ and $\displaystyle\int_{\mathbb{C}} \frac{idz\wedge d\overline{z}}{2\pi(1+|z|^2)^3} = \frac{1}{2}$ we infer,

$$\begin{aligned} S_1(E_n,h_n) &= \pi_\star(\widetilde{c}_2(\overline{\Sigma}(1))) = -\pi_\star(\Omega) = -1 \\ S_2(E_n,h_n) &= -\pi_\star(\Omega\alpha_n) = \pi_\star(\Omega\frac{\langle\Theta(E_n^\star,h)a^\star,a^\star\rangle_h}{\langle a^\star,a^\star\rangle_h}) = \frac{1}{2}c_1(E_n^\star,h_n). \end{aligned}$$

Hence,

$$\widehat{s_2'}(\overline{\mathcal{E}_d}) = (n^2+3n+3)\widehat{x}^2 + \frac{3(n+2)}{2}a(x) = \frac{2n^2+9n+12}{2}a(x).$$

Finally, computing the arithmetic degree we get

Theorem 1. *The arithmetic height of the Hirzebruch surface $\mathcal{S}_n$ is*

$$\widehat{h}_{\overline{\mathcal{O}_{\mathcal{E}_n}(1)}}(\mathcal{S}_n) = \frac{2n^2+9n+12}{4}.$$

2.2. **With the sequence associated to the fibration.** To complete the picture of the arithmetic datas of the Hirzebruch surfaces, we need to compute the Bott-Chern secondary classes of the short exact sequence

$$0 \to (T_{S_n/\mathbb{P}^1},\alpha_n) \xrightarrow{\iota} (TS_n,\alpha_n) \xrightarrow{d\pi} (\pi^\star T\mathbb{P}^1,\alpha_n) \to 0$$

of hermitian vector bundles on S_n.

We need to make explicit the metric $\alpha_n = \Theta(\mathcal{O}_{E_n}(1),h_n)$. We choose a local holomorphic frame e for $\mathcal{O}_{E_n}(1)$. This provides a frame $(e^\star, e^{\star n+1})$ for $E_n^\star$. We denote by (a_1,a_2) the corresponding coordinates on $E_n^\star$ and by z the holomorphic coordinate $z := a_1/a_2$ on an appropriate open set of $\mathbb{P}(E_n)$. Then, α_n is computed by $dd^c\log(\|e^\star + ze^{\star n+1}\|^2_{h_n})$. We find

$$\begin{aligned} \alpha_n &= \frac{1+(n+1)|z|^2\|e^\star\|^{2n}}{1+|z|^2\|e^\star\|^{2n}} dd^c\log\|e^\star\|^2 \\ &\quad + \frac{\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2}\frac{i}{2\pi}\Big|dz + nzd'\log\|e^\star\|^2\Big|^2. \end{aligned} \tag{2.2.1}$$

The choice of the frame e naturally induces a choice for a local holomorphic splitting $TS_n \simeq \pi^\star T\mathbb{P}^1 \oplus T_{S_n/\mathbb{P}^1}$ so that for a vector X in $T\mathbb{P}^1$, $d\pi(X,0) = X$ and $i\left(\frac{\partial}{\partial z}\right) = (0,\frac{\partial}{\partial z})$. We find

$$\begin{aligned} \|(X,0)\|^2_{\alpha_n} &= \frac{1+(n+1)|z|^2\|e^\star\|^{2n}}{1+|z|^2\|e^\star\|^{2n}}\|X\|^2_{\mathbb{P}^1} + \frac{1}{2\pi}\frac{\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2}|nzd'_X\log\|e^\star\|^2|^2 \\ \langle(X,0),(0,\frac{\partial}{\partial z})\rangle_{\alpha_n} &= \frac{1}{2\pi}\frac{\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2}nzd'_X\log\|e^\star\|^2 \\ \|(0,\frac{\partial}{\partial z})\|^2_{\alpha_n} &= \frac{1}{2\pi}\frac{\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2} = \frac{1}{2\pi}\frac{\|e^\star\|^{2(n+2)}}{\|e^\star+ze^{\star n+1}\|^4}. \end{aligned}$$

Let X be a vector in $T\mathbb{P}^1$. Its image $(d\pi)^\star\pi^\star(X)$ can be written as $(d\pi)^\star\pi^\star(X) = (X,0) + V_X(0,\frac{\partial}{\partial z})$ subject to the condition $\langle X + V_X\frac{\partial}{\partial z},\frac{\partial}{\partial z}\rangle_{\alpha_n} = 0$. This leads to $V_X = -nzd'_X\log\|e^\star\|^2$. We can then derive the expression for the quotient metric on $\pi^\star T\mathbb{P}^1$

$$\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}} = \frac{\|(d\pi)^\star X\|^2_{\alpha_n}}{\|X\|^2_{\mathbb{P}^1}} = \frac{1+(n+1)|z|^2\|e^\star\|^{2n}}{1+|z|^2\|e^\star\|^{2n}} = \frac{\langle\pi^\star\Theta(E_n^\star,h_n)a^\star,a^\star\rangle}{\pi^\star\omega_{\mathbb{P}^1}\langle a^\star,a^\star\rangle}.$$

We can now compute the Bott-Chern class for the short exact sequence constructed from $d\pi$.

Lemma 2. *The Bott-Chern class for Chern classes in the short exact sequence*

$$0 \to (T_{S_n/\mathbb{P}^1}, \alpha_n) \xrightarrow{\iota} (TS_n, \alpha_n) \xrightarrow{d\pi} (\pi^\star T\mathbb{P}^1, \alpha_n) \to 0$$

is

$$\widetilde{c}(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) = \left(\frac{n}{1+(n+1)|z|^2\|e^\star\|^{2n}} - \frac{n}{1+|z|^2\|e^\star\|^{2n}} \right) \pi^\star x.$$

Proof. As a consequence of [G-S-90] prop 1.2.5, we derive $\widetilde{c}_1(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) = 0$. Using the original computations of Bott and Chern,

$$\widetilde{c}_2(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) = \int_0^1 \frac{\Phi(u)-\Phi(0)}{u} du$$

with

$$\begin{aligned} \Phi(u) &= Trace\left((1-u)\,\Theta(\pi^\star T\mathbb{P}^1, \alpha_n) + u(d\pi)\Theta(TS_n, \alpha_n)(d\pi)^\star\right) \\ &= (d\pi)\Theta(TS_n, \alpha_n)(d\pi)^\star \\ &\quad -(1-u)\,\frac{i}{2\pi}(d\pi)(\nabla'_{Hom(T_{S_n/\mathbb{P}^1}, TS_n)}\iota) \wedge \iota^\star \nabla''_{Hom(\pi^\star T\mathbb{P}^1, TS_n)}(d\pi)^\star, \end{aligned}$$

follows

$$\widetilde{c}_2(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) = \frac{i}{2\pi} d\pi(\nabla'_{Hom(T_{S_n/\mathbb{P}^1}, TS_n)}\iota) \wedge \iota^\star \nabla''_{Hom(\pi^\star T\mathbb{P}^1, TS_n)}(d\pi)^\star.$$

Now, remark that

$$\iota^\star(\nabla''(d\pi)^\star)X = \iota^\star\nabla''((d\pi)^\star X) = d''V_X\,\iota^\star(0, \frac{\partial}{\partial z}) = d''V_X\frac{\partial}{\partial z}.$$

Also,

$$\begin{aligned} d\pi(\nabla'\iota)(\frac{\partial}{\partial z}) &= d\pi\nabla'\frac{\partial}{\partial z} = \frac{\{\nabla'\frac{\partial}{\partial z}, X+V_X\frac{\partial}{\partial z}\}_{\alpha_n}}{\|X+V_X\frac{\partial}{\partial z}\|^2_{\alpha_n}} d\pi\left((X,0)+V_X(0,\frac{\partial}{\partial z})\right) \\ &= \frac{d'\langle\frac{\partial}{\partial z}, X+V_X\frac{\partial}{\partial z}\rangle_{\alpha_n} - \{\frac{\partial}{\partial z}, d''V_X\frac{\partial}{\partial z}\}_{\alpha_n}}{\|X+V_X\frac{\partial}{\partial z}\|^2_{\alpha_n}} X = -d'\overline{V_X}\frac{\|\frac{\partial}{\partial z}\|^2_{\alpha_n}}{\|X+V_X\frac{\partial}{\partial z}\|^2_{\alpha_n}} X \\ &= -\frac{1}{2\pi}\frac{\|e^\star\|^{2(n+2)}}{(1+|z|^2\|e^\star\|^{2n})(1+(n+1)|z|^2\|e^\star\|^{2n})} d'\overline{V_X}. \end{aligned}$$

Summing up

$$\begin{aligned} \widetilde{c}_2(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) &= -\frac{i}{4\pi^2}\frac{\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})(1+(n+1)|z|^2\|e^\star\|^{2n})}\frac{d'\overline{V_X}\wedge d''V_X}{\|X\|^2_{\mathbb{P}^1}} \\ &= -\frac{i}{4\pi^2}\frac{n^2|z|^2\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})(1+(n+1)|z|^2\|e^\star\|^{2n})}\frac{d'd''_X\log\|e^\star\|^2 \wedge d''d'_X\log\|e^\star\|^2}{\|X\|^2_{\mathbb{P}^1}} \\ &= -\frac{n^2|z|^2\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})(1+(n+1)|z|^2\|e^\star\|^{2n})}\pi^\star\omega_{\mathbb{P}^1}. \end{aligned}$$

The last equality is derived from the fact that those $(1,1)$-forms take the same values on $\pi^\star X$ which generates $\pi^\star T\mathbb{P}^1$. □

2.3. **Some characteristic classes.** In the previous two exact sequences the bundle $T_{S_n/\mathbb{P}^1}$ appeared equipped with two different metrics, the one ω_q gotten by quotient of that on $\pi^\star E_n^\star \otimes \mathcal{O}_{E_n}(1)$ and the one induced by α_n. We will compare those metrics and derive expressions for some characteristic classes.

Choose a point x_0 in $\mathbb{P}^1$ and a normal frame e for $\mathcal{O}_{\mathbb{P}^1}(1)$ at x_0. Then, on the fiber of x_0

$$\alpha_{n|T_{S_n/\mathbb{P}^1}} = d_z d_z^c \log(\|e^\star + ze^{\star n+1}\|^2) = d_z d_z^c \log(1+|z|^2) = \frac{i}{2\pi}\frac{dz\wedge d\overline{z}}{(1+|z|^2)^2}.$$

On the other end, the map q is built from the differential of the quotient map $E_n^\star - \mathbb{P}^1 \times \{0\} \to \mathbb{P}(E_n), (x, a_1 e^\star + a_2 e^{\star n+1}) \mapsto (x, [a_1 : a_2]) \simeq (x, z := \frac{a_2}{a_1})$

$$\begin{array}{cccc} q: & E_n^\star \otimes \mathcal{O}_{E_n}(1) & \to & T_{\mathbb{P}(E_n)/\mathbb{P}^1} \\ & (b_1 e^\star + b_2 e^{\star n+1}) \otimes (a_1 e^\star + a_2 e^{\star n+1})^\star & \mapsto & \frac{a_1 b_2 - a_2 b_1}{a_1^2} \frac{\partial}{\partial z}. \end{array}$$

Follows the expression for its adjoint map

$$\begin{aligned} q^\star\left(\frac{1}{a_1}\frac{\partial}{\partial z}\right) &= \left(e^{\star n+1} - \frac{\langle e^{\star n+1}, a_1 e^\star + a_2 e^{\star n+1}\rangle}{\|a_1 e^\star + a_2 e^{\star n+1}\|^2}(a_1 e^\star + a_2 e^{\star n+1})\right) \otimes (a_1 e^\star + a_2 e^{\star n+1})^\star \\ q^\star\left(\frac{\partial}{\partial z}\right) &= \frac{-\overline{z} e^\star + e^{\star n+1}}{1+|z|^2}(e^\star + z e^{\star n+1})^\star. \end{aligned}$$

We derive that the quotient metric is given by $\omega_q = i\dfrac{dz \wedge d\overline{z}}{(1+|z|^2)^2} = 2\pi\alpha_{n|T_{S_n/\mathbb{P}^1}}$ on the fiber of x_0. This enables to express the arithmetic first Chern class of $\overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}}$ with the metric $\alpha_{n|T_{S_n/\mathbb{P}^1}}$

$$\begin{aligned} \widehat{c}_1(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}}, \alpha_n) &= \widehat{c}_1(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}}, \omega_q) + a(\log 2\pi) \\ &= 2\widehat{\alpha_n} - (n+2)\widehat{x} + a(\log 2\pi) \end{aligned}$$

where we have used the relation (2.1.2). Similarly, the map $(T\mathbb{P}^1, 2\pi\omega_{\mathbb{P}^1}) \to (\mathcal{O}_{\mathbb{P}^1}(2), h)$ is an isometry. Just recall

$$\widehat{c}(\overline{T\mathcal{S}_n}, \alpha_n) = \widehat{c}(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}}, \alpha_n)\widehat{c}(\overline{T_{\mathbb{P}^1_{\mathbb{Z}}}}, \alpha_n) - a(\widetilde{c}(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n))$$

to end the proof of

Lemma 3.

$$\begin{aligned} \widehat{c}_1(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}}, \alpha_n) &= 2\widehat{\alpha_n} - (n+2)\widehat{x} + a(\log 2\pi) \\ \widehat{c}_1(\overline{T\mathbb{P}^1_{\mathbb{Z}}}, \alpha_n) &= 2\widehat{x} - a\left(\log\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{2\pi\pi^\star\omega_{\mathbb{P}^1}}\right) \\ \widehat{c}_1(\overline{T\mathcal{S}_n}, \alpha_n) &= 2\widehat{\alpha_n} - n\widehat{x} - a\left(\log\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{4\pi^2\pi^\star\omega_{\mathbb{P}^1}}\right) \\ \widehat{c}_2(\overline{T\mathcal{S}_n}, \alpha_n) &= 4\widehat{\alpha_n}\widehat{x} - 2(n+2)\widehat{x}^2 + a\Big(2\log 2\pi x \\ &\quad - \log\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{2\pi\pi^\star\omega_{\mathbb{P}^1}} c_1(T_{S_n/\mathbb{P}^1}) - \widetilde{c}_2(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n)\Big). \end{aligned}$$

3. USING THE ARITHMETIC RIEMANN-ROCH THEOREM

We will apply the degree one arithmetic Riemann-Roch theorem [G-S-92] for the map $F : \mathcal{S}_n \to Spec\,\mathbb{Z}$ to compute the analytic torsion of the bundles of holomorphic differential forms $\Omega^p_{S_n}$ on S_n.

3.1. **On the analytic torsion.** We recall the definition of the analytic torsion of a Hermitian vector bundle (E, h) on a compact Kähler manifold (X, ω). We endow the space of differential forms with values in E with its L^2 metric (constructed with ω and h) in order to construct the adjoint $\overline{\partial_q}^\star$ of the Dolbeault operator $\overline{\partial_q} : A^{0,q}(X,E) \to A^{0,q+1}(X,E)$. The associated Laplace operator Δ''_q has a discrete spectrum

$$0 = 0 = \cdots = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq \cdots.$$

Its spectral function $\zeta_q(s) := \sum_1^{+\infty} \lambda_N^{-s}$ extends to a meromorphic function on $\mathbb{C}$, holomorphic at 0. We define the regularized determinant of the restriction of the Laplace operator to the orthogonal complement of its kernel to be $det'\Delta''_q := \exp\left(-\frac{d}{ds}\zeta_q(0)\right)$. The analytic torsion of (X, ω, E, h) is then given by

$$T_0(X, \omega, E, h) := \exp(-\tau((X, \omega, E, h)) := \prod_{q \geq 0} (det'\Delta''_q)^{(-1)^q q}.$$

3.2. **On arithmetic first Chern class of determinant bundles.** On one hand, from the very definition of the arithmetic first Chern class of a Hermitian line bundle and from the definition of the Quillen metric on the determinant of the direct image of a Hermitian vector bundle, identifying $CH^1(Spec\,\mathbb{Z})$ with $\mathbb{R}$ through the degree map $\widehat{deg}$, we get (see [G-S-92] 4.1.5)

$$\begin{aligned}
&\widehat{deg}\ \widehat{c}_1(\lambda_F(\overline{\Omega^p_{S_n}}), Quillen) \\
&= \sum_{q=0}^{2} (-1)^q \left(\log \sharp H^q(S_n, \Omega^p_{S_n})_{torsion} - \log Vol_{L^2} \frac{H^q(S_n, \Omega^p_{S_n})}{H^q(S_n, \Omega^p_{S_n})_{\mathbb{Z}}} \right) + \frac{1}{2}\tau(S_n, \Omega^p_{S_n}).
\end{aligned}$$

From Leray-Hirsch theorem, the cohomology ring $H^\bullet(S_n, \mathbb{Z})$ of $S_n = \mathbb{P}(E_n)$ is

$$\frac{H^\bullet(\mathbb{P}^1, \mathbb{Z})[\{\alpha_n\}]}{\{\alpha_n\}^2 - (n+2)\{\pi^\star\omega_{\mathbb{P}^1}\} \cup \{\alpha_n\}} = \frac{\mathbb{Z}[\{\pi^\star\omega_{\mathbb{P}^1}\}, \{\alpha_n\}]}{\{\pi^\star\omega_{\mathbb{P}^1}\}^2, \{\alpha_n\}^2 - (n+2)\{\pi^\star\omega_{\mathbb{P}^1}\} \cup \{\alpha_n\}}.$$

We hence get, applying Hodge decomposition to the De Rham cohomology,

for $p = 0$, $H^\bullet(S_n, \mathcal{O}_{S_n})_{\mathbb{Z}} = H^0(S_n, \mathcal{O}_{S_n})_{\mathbb{Z}} = \mathbb{Z}1$
for $p = 1$, $H^\bullet(S_n, \Omega^1_{S_n})_{\mathbb{Z}} = H^1(S_n, \Omega^1_{S_n})_{\mathbb{Z}} = \mathbb{Z}\{\pi^\star\omega_{\mathbb{P}^1}\} + \mathbb{Z}\{\alpha_n\}$
for $p = 2$, $H^\bullet(S_n, \Omega^2_{S_n})_{\mathbb{Z}} = H^2(S_n, \Omega^2_{S_n})_{\mathbb{Z}} = \mathbb{Z}\{\pi^\star\omega_{\mathbb{P}^1}\} \cup \{\alpha_n\} = \mathbb{Z}\frac{\{\alpha_n\}^2}{n+2}$

We now intend to find the harmonic representatives of those generators, with respect to the metric α_n. The forms 1, α_n and $\frac{\alpha_n^2}{n+2}$ are easily seen to be harmonic of norm $\sqrt{\frac{n+2}{2}}$, $\sqrt{n+2}$ and $\sqrt{\frac{2}{n+2}}$ respectively. Computation of the harmonic representative of $\{\pi^\star\omega_{\mathbb{P}^1}\}$ is quite involved and will not actually be used. Details will be given in the appendix. The form $\omega_H := \pi^\star\omega_{\mathbb{P}^1} - \frac{1}{n+2}dd^c \log \frac{\langle\Theta(E^\star, h)a^\star, a^\star\rangle_h}{\pi^\star\omega_{\mathbb{P}^1}\langle a^\star, a^\star\rangle_h}$ is harmonic. To compute its L^2-norm, we need to introduce the Hodge $\star$-operator. It is defined in order to fulfill the relation $u \wedge \star v = \langle u, v\rangle_{\alpha_n} dV_{\alpha_n}$. Recall that $dV_{\alpha_n} = \alpha_n^2/2$ and that $|\alpha_n|^2_{\alpha_n} = 2$. Local computation of the Hodge $\star$-operator leads to $\star\alpha_n = \alpha_n$. For $\star\omega_H$ is also harmonic, it can be written as $a\alpha_n + b\omega_H$. Writing the relation $\star\star\omega_H = \omega_H$ leads to either $(b = 1; a = 0)$ or $b = -1$. Noting that $b = 1$ would lead to $\|\omega_H\|^2 = \int \omega_H \wedge \star\omega_H = \int \omega_H \wedge \omega_H = \int \pi^\star\omega_{\mathbb{P}^1} \wedge \pi^\star\omega_{\mathbb{P}^1} = 0$, we derive $b = -1$. Equating $\omega_H \wedge \star\alpha_n$ and $\star\omega_H \wedge \alpha_n$ we get $a\alpha_n^2 = 2\alpha_n \wedge \omega_H$. Integrating, we find $a = \frac{2}{n+2}$. We can now conclude

$$\|\omega_H\|^2 = \int_{S_n} \omega_H \wedge (\frac{2}{n+2}\alpha_n - \omega_H) = \frac{2}{n+2}.$$

We hence find an orthonormal basis of $H^\bullet(S_n, \Omega^1_{S_n}) = \mathbb{C}\{\alpha_n\} \oplus H^\bullet_{prim}(S_n, \Omega^1_{S_n})$

$$\frac{\{\alpha_n\}}{\sqrt{n+2}}\ ;\ \sqrt{n+2}\left(\{\pi^\star\omega_{\mathbb{P}^1}\} - \frac{\{\alpha_n\}}{n+2}\right)$$

The L^2-volume of $\frac{H^1(S_n,\Omega^1_{S_n})}{H^1(S_n,\Omega^1_{S_n})_{\mathbb{Z}}}$, which is the norm of the generator $\{\pi^\star\omega_{\mathbb{P}^1}\}\wedge\{\alpha_n\}$ of $\Lambda^2 H^1(S_n,\Omega^1_{S_n})_{\mathbb{Z}}$ is 1. Concluding this first step, we get

$$\begin{aligned}\widehat{c}_1(\lambda_F(\overline{\Omega^0_{\mathcal{S}_n}}),Quillen) &= -\log\sqrt{\frac{n+2}{2}}+\frac{1}{2}\tau(S_n)\\ \widehat{c}_1(\lambda_F(\overline{\Omega^1_{\mathcal{S}_n}}),Quillen) &= -\frac{1}{2}\tau(S_n,\Omega^1_{S_n})\\ \widehat{c}_1(\lambda_F(\overline{\Omega^2_{\mathcal{S}_n}}),Quillen) &= \log\sqrt{\frac{n+2}{2}}+\frac{1}{2}\tau(S_n,\Omega^2_{S_n})\end{aligned}$$

3.3. **The arithmetic Riemann-Roch theorem.** On the other hand, the arithmetic Riemann-Roch theorem reads

$$\begin{aligned}\widehat{c}_1(\lambda_F(\overline{\Omega^p_{\mathcal{S}_n}},Quillen)) &= \widehat{F}_\star\left(\widehat{Td}^R(\overline{T\mathcal{S}_n},\alpha_n)\widehat{ch}(\overline{\Omega^p_{\mathcal{S}_n}},\alpha_n)\right)\\ &= \widehat{F}_\star\left(\widehat{Td}(\overline{T\mathcal{S}_n},\alpha_n)\widehat{ch}(\overline{\Omega^p_{\mathcal{S}_n}},\alpha_n)\right)-a(F_\star\left(Td(TS_n)R(TS_n)ch(\Omega^p_{S_n})\right)\end{aligned}$$

Recall

$$\widehat{Td}(\overline{T\mathcal{S}_n})=1+\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})}{2}+\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})^2+\widehat{c}_2(\overline{T\mathcal{S}_n})}{12}+\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})\widehat{c}_2(\overline{T\mathcal{S}_n})}{24}$$

and

$$\begin{aligned}\widehat{ch}(\overline{\Omega^1_{\mathcal{S}_n}},\alpha_n) &= 2-\widehat{c}_1(\overline{T\mathcal{S}_n})+\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})^2-2\widehat{c}_2(\overline{T\mathcal{S}_n})}{2}-\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})^3-3\widehat{c}_1(\overline{T\mathcal{S}_n})\widehat{c}_2(\overline{T\mathcal{S}_n})}{6}\\ \widehat{ch}(\overline{\Omega^2_{\mathcal{S}_n}},\alpha_n) &= 1-\widehat{c}_1(\overline{T\mathcal{S}_n})+\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})^2}{2}-\frac{\widehat{c}_1(\overline{T\mathcal{S}_n})^3}{6}\end{aligned}$$

so that

$$\begin{aligned}\left[\widehat{Td}(\overline{T\mathcal{S}_n})\widehat{ch}(\overline{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}},1)\right]_3=-\left[\widehat{Td}(\overline{T\mathcal{S}_n})\widehat{ch}(\overline{\Omega^2_{\mathcal{S}_n}},\alpha_n)\right]_3 &= \frac{\widehat{c}_1(\overline{T\mathcal{S}_n})\widehat{c}_2(\overline{T\mathcal{S}_n})}{24}\\ \left[\widehat{Td}(\overline{T\mathcal{S}_n})\widehat{ch}(\overline{\Omega^1_{\mathcal{S}_n}},\alpha_n)\right]_3 &= 0.\end{aligned}$$

From formulas for the arithmetic Chern classes of $(\overline{T\mathcal{S}_n},\alpha_n)$ in lemma 3 we get

$$\begin{aligned}\widehat{c}_1(\overline{T\mathcal{S}_n},\alpha_n)\widehat{c}_2(\overline{T\mathcal{S}_n},\alpha_n) &= 8\widehat{\alpha_n}^2\widehat{x}-8(n+1)\widehat{\alpha_n}\widehat{x}^2+a\Bigg(-4\log\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{4\pi^2\pi^\star\omega_{\mathbb{P}^1}}\alpha_n x+2\log 2\pi x c_1(TS_n)\\ &\quad -\log\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{2\pi\pi^\star\omega_{\mathbb{P}^1}}c_1(TS_n,\alpha_n)c_1(T_{S_n/\mathbb{P}^1})-c_1(TS_n,\alpha_n)\widetilde{c}_2(TS_n,T\mathbb{P}^1,\alpha_n,\alpha_n\Bigg).\end{aligned}$$

To compute the integral, recall from lemma 1 that

$$\begin{aligned}\widehat{F}_\star(\widehat{\alpha_n}\widehat{x}^2) &= a(F_\star(\alpha_n x))=a(1)\\ \widehat{F}_\star(\widehat{\alpha_n}^2\widehat{x}) &= (n+2)\widehat{F}_\star(\widehat{\alpha_n}\widehat{x}^2)+a(F_\star(\alpha_n x))=a(n+3)\end{aligned}$$

Applying the morphism ω in lemma 3, we can compute the curvature form of (TS_n,α_n).

$$\begin{aligned}c_1(TS_n,\alpha_n) &= 2\alpha_n-nx-dd^c\log\frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}}\\ &= \left(n+2-\frac{3n}{1+|z|^2\|e^\star\|^{2n}}+\frac{n}{1+(n+1)|z|^2\|e^\star\|^{2n}}\right)\pi^\star x\\ &\quad+\left(\frac{3\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2}-\frac{(n+1)\|e^\star\|^{2n}}{(1+(n+1)|z|^2\|e^\star\|^{2n})^2}\right)\frac{i}{2\pi}\left|dz+nzd'\log\|e^\star\|^2\right|^2\end{aligned}$$

where we used the expression (2.2.1) for α_n and the expression in the appendix for $dd^c \log \frac{\alpha_{n\,|\pi^\star T\mathbb{P}^1}}{\pi^\star \omega_{\mathbb{P}^1}}$.
Then note that from lemma 3 we find

$$\begin{aligned} c_1(T_{S_n/\mathbb{P}^1}, \alpha_n) &= 2\alpha_n - (n+2)\pi^\star x \\ &= (n - \frac{2n}{1+|z|^2\|e^\star\|^{2n}})\pi^\star x + \frac{2\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2}\left|dz + nzd'\log\|e^\star\|^2\right|^2. \end{aligned}$$

so that

$$\frac{c_1(TS_n,\alpha_n)c_1(T_{S_n/\mathbb{P}^1},\alpha_n)}{\pi^\star x \wedge \left|dz + nzd'\log\|e^\star\|^2\right|^2} = \frac{(5n+2)\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^2} + \frac{(n+1)(n+2)\|e^\star\|^{2n}}{(1+(n+1)|z|^2\|e^\star\|^{2n})^2} - \frac{12n\|e^\star\|^{2n}}{(1+|z|^2\|e^\star\|^{2n})^3}.$$

Integration leads to

$$\begin{aligned} & F_\star\left(c_1(TS_n,\alpha_n)c_1(T_{S_n/\mathbb{P}^1},\alpha_n)\log\frac{\alpha_{n\,|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}}\right) \\ = \; & (5n+2)\left[\frac{n+1}{n}\log(n+1) - 1\right] + (n+2)\left[1 - \frac{1}{n}\log(n+1)\right] \\ & -6(n+1)\left[\frac{n+1}{n}\log(n+1) - 1\right] + 3n = 5n+6 - \left[n+6+\frac{6}{n}\right]\log(n+1) \end{aligned}$$

Thanks to lemma 2 we get

$$\begin{aligned} & F_\star\left(c_1(TS_n,\alpha_n)\widetilde{c_2}(TS_n, T\mathbb{P}^1,\alpha_n,\alpha_n)\right) \\ = \; & \int_{\mathbb{C}}\left(\frac{3}{(1+|z|^2)^2} - \frac{n+1}{(1+(n+1)|z|^2)^2}\right)\left(\frac{n}{1+(n+1)|z|^2} - \frac{n}{1+|z|^2}\right)dz\wedge d\overline{z} \\ = \; & \int_{\mathbb{C}} -\frac{3n}{(1+|z|^2)^3} - n\frac{n+1}{(1+(n+1)|z|^2)^3} + (n+1)\frac{n+1}{(1+(n+1)|z|^2)^2} \\ & -\frac{3}{(1+|z|^2)^2} + 2\frac{n+1}{n}\left(\frac{n+1}{1+(n+1)|z|^2} - \frac{1}{1+|z|^2}\right)dz\wedge d\overline{z} \\ = \; & \frac{-4n}{2} + (n+1) - 3 + 2\frac{n+1}{n}\log(n+1) = -n-2+\left[2+\frac{2}{n}\right]\log(n+1). \end{aligned}$$

We are now able to compute

$$\begin{aligned} \widehat{F}_\star\left(\widehat{c}_1(\overline{T\mathcal{S}_n},\alpha_n)\widehat{c}_2(\overline{T\mathcal{S}_n},\alpha_n)\right) &= a\left(16 + 8\log 2\pi - 4\left(-1+\left[1+\frac{1}{n}\right]\log(n+1)\right) + 4\log 2\pi\right. \\ &\quad \left. +4\log 2\pi - \left(4n+4-\left[n+4+\frac{4}{n}\right]\log(n+1)\right)\right) \\ &= a\left(n\log(n+1) - 4n + 16 + 16\log 2\pi\right). \end{aligned}$$

For analytic terms,

$$\begin{aligned} F_\star\left[Td(TS_n)R(TS_n)ch(\mathcal{O})\right] &= \frac{2\zeta'(-1)+\zeta(-1)}{2}F_\star c_1(TS_n)^2 \\ &= 8\zeta'(-1) + 4\zeta(-1)) \\ F_\star\left[Td(TS_n)R(TS_n)ch(\Omega^1_{S_n})\right] &= 0 \\ F_\star\left[Td(TS_n)R(TS_n)ch(\Omega^2_{S_n})\right] &= -8\zeta'(-1) - 4\zeta(-1)). \end{aligned}$$

3.4. **Conclusion.** We are now about to end our computations. We just have to notice from application of the arithmetic Riemann-Roch theorem to the map $f : \mathbb{P}^1_{\mathbb{Z}} \to Spec\,\mathbb{Z}$ using $\widehat{c}_1(\overline{T\mathbb{P}^1_{\mathbb{Z}}},\omega_{\mathbb{P}^1}) = 2\widehat{x} + a(\log 2\pi)$ that the analytic torsion of $\mathbb{P}^1$ is

$$\tau(\mathbb{P}^1,\omega_{\mathbb{P}^1}) = \frac{1+\log 2\pi}{3} - 4\zeta'(-1) - 2\zeta(-1).$$

We find

Theorem 2.

$$\begin{aligned}\tau(S_n) &= \frac{n\log(n+1)}{24} - \frac{n}{6} + \log\frac{n+2}{2} + 2\tau(\mathbb{P}^1, \omega_{\mathbb{P}^1}).\\ \tau(S_n, \Omega^1_{S_n}) &= 0\\ \tau(S_n, \Omega^2_{S_n}) &= -\tau(S_n).\end{aligned}$$

Theses results are compatible with theorem 3.1 in [R-S-73].

4. Using Berthomieu-Bismut formula

The second idea for the computation of the analytic torsion $\tau(S_n)$ is to apply Berthomieu-Bismut formula [B-B-94] for the composition of submersions :

$$\mathcal{S}_n = \mathbb{P}(\mathcal{E}_n) \xrightarrow{\pi} \mathbb{P}^1_{\mathbb{Z}} \xrightarrow{f} Spec\,\mathbb{Z}.$$

We follow their construction in our particular setting.

4.1. **On Leray spectral sequence.** Leray spectral sequence of π gives a canonical isomorphism σ between the determinant line bundles.

$$\lambda_f(R\pi_\star\mathcal{O}_{S_n}) \to \lambda_F(\mathcal{O}_{S_n})$$

where we have set $F := f \circ \pi$. Note that we do not consider their dual as in [B-B-94]. Here, for π is a locally trivial family of projective lines, $R\pi_\star\mathcal{O}_{S_n} = \pi_\star\mathcal{O}_{S_n} = \mathcal{O}_{\mathbb{P}^1}$. The spectral sequence hence degenerates in E_2 and the isomorphism σ is

$$\begin{array}{ccc} detH^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) & \to & detH^0(S_n, \mathcal{O}_{S_n})\\ 1' & \mapsto & 1'' = \pi^\star 1'\end{array}$$

4.2. **On Quillen metrics.** We now describe the Quillen metrics on determinant line bundles. We choose the trivial metric on $\mathcal{O}_{S_n}$. The fibers of π are endowed with the metric induced by $\alpha_n := \Theta(\mathcal{O}_{E_n}(1), h_n)$. The bundle $R\pi_\star\mathcal{O}_{S_n} = \mathcal{O}_{\mathbb{P}^1}$ is endowed with its L^2 metric which is the trivial one for α_n is of volume 1 on every fiber of π. Choose $\omega_{\mathbb{P}^1} := x = \Theta(\mathcal{O}_{\mathbb{P}^1}(1), h)$ the Fubini-Study metric of volume 1 as metric on $\mathbb{P}^1$.

Hence for $1' \in Rf_\star(R\pi_\star\mathcal{O}_{S_n}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$

$$\|1'\|^2_{L^2} = \int_{\mathbb{P}^1} 1\omega_{\mathbb{P}^1} = 1 \quad ; \quad \|1'\|^2_Q = e^{-\tau(\mathbb{P}^1)}.$$

For $1'' \in RF_\star\mathcal{O}_{S_n} = H^0(S_n, \mathcal{O}_{S_n})$

$$\|1''\|^2_{L^2} = \int_{S_n} 1\frac{\alpha_n^2}{2} = \frac{n+2}{2} \quad ; \quad \|1''\|^2_Q = \frac{n+2}{2}e^{-\tau(S_n)}.$$

We can compute the Quillen norm of the isomorphism σ

$$\log\|\sigma\|^2_{\lambda_f^{-1}(R\pi_\star\mathcal{O}_{S_n})\otimes\lambda_F(\mathcal{O}_{S_n})} = \log\frac{\|1''\|^2_Q}{\|1'\|^2_Q} = \tau(\mathbb{P}^1) - \tau(S_n) + \log\frac{n+2}{2}.$$

4.3. **Berthomieu-Bismut formula.** On the other hand this norm is computed by the Berthomieu-Bismut formula

$$\begin{aligned}\log\|\sigma\|^2_{\lambda_f^{-1}(R\pi_\star\mathcal{O}_{S_n})\otimes\lambda_F(\mathcal{O}_{S_n})} &= -\int_{\mathbb{P}^1} Td(T\mathbb{P}^1, \omega_{\mathbb{P}^1})Tors(\alpha_n, 1)\\ &\quad + \int_{S_n}\widetilde{Td}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})ch(\mathcal{O}_{S_n}, 1).\end{aligned}$$

Two kinds of secondary objects are used in this result. On one hand, the form $Tors(\alpha_n, 1)$ is the analytic torsion form of Bismut-Köhler [B-K-92] which fulfills the relation

(4.3.1) $$dd^c Tors(\alpha_n, 1) = \pi_\star(Td(T_{S_n/\mathbb{P}^1}, \alpha_n)ch(\mathcal{O}_{S_n}, 1)) - ch(R\pi_\star\mathcal{O}_{S_n}, 1).$$

On the other hand, the class $\widetilde{Td}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})$ is the Bott-Chern class for the Todd characteristic form [G-S-90] of the following metrized exact sequence

$$0 \to (T_{S_n/\mathbb{P}^1}, \alpha_n) \to (TS_n, \alpha_n) \xrightarrow{d\pi} (\pi^\star T\mathbb{P}^1, \pi^\star\omega_{\mathbb{P}^1}) \to 0$$

which fulfills the relation

$$dd^c\widetilde{Td}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1}) = Td(T_{S_n/\mathbb{P}^1}, \alpha_n)Td(\pi^\star T\mathbb{P}^1, \pi^\star\omega_{\mathbb{P}^1}) - Td(TS_n, \alpha_n).$$

Note that our sign conventions differ from the ones in [B-B-94] both in the definition of σ and of the secondary objects.

4.4. **On the analytic torsion term.** The tool for the computation of the torsion form $Tors(\alpha_n, 1)$ is the arithmetic Riemann-Roch theorem gotten from equation (4.3.1) by double transgression (see for example [B-98] Theorem 4.4)

$$\begin{aligned} a(Tors(\alpha_n,1)) &= \widehat{\pi}_\star\left(\widehat{Td}^R(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}})\widehat{ch}(\overline{\mathcal{O}_{\mathcal{S}_n}},1)\right) - \widehat{ch}(\overline{R\pi_\star\mathcal{O}_{\mathcal{S}_n}}) \\ &= \widehat{\pi}_\star\left(\widehat{Td}(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}})\right) - \widehat{\pi}_\star\left(\widehat{Td}(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}})a(R(T_{S_n/\mathbb{P}^1}))\right) - 1. \end{aligned} \tag{4.4.1}$$

We compute the arithmetic Todd class of $\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}}$ using $\widehat{Td} = 1 + \frac{\widehat{c}_1}{2} + \frac{\widehat{c}_1^2}{12}$ for a line bundle on an arithmetic surface. From lemma 3, we get

$$\begin{aligned} \widehat{Td}(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}}, \alpha_n) &= 1 + \widehat{\alpha_n} - \frac{n+2}{2}\widehat{x} + a(\frac{\log 2\pi}{2}) \\ &\quad + \frac{1}{3}\widehat{\alpha_n}^2 - \frac{n+2}{3}\widehat{\alpha_n}\widehat{x} + \frac{(n+2)^2}{12}\widehat{x}^2 + \frac{\log 2\pi}{6}a(2\alpha_n - (n+2)x) \\ &= 1 + \widehat{\alpha_n} - \frac{n+2}{2}\widehat{x} + a(\frac{\log 2\pi}{2}) \\ &\quad + a\left(\frac{1+\log 2\pi}{3}\alpha_n + \frac{n^2 - 2(n+2)\log 2\pi}{12}x + \frac{1}{3}\frac{\langle\Theta(E^\star,h)a^\star, a^\star\rangle_h}{\langle a^\star, a^\star\rangle_h}\right). \end{aligned}$$

We compute the direct image,

$$\widehat{\pi}_\star(\widehat{Td}(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}})) = 1 + a(\frac{1+\log 2\pi}{3}).$$

The contribution of the R class is purely analytic. First recall that $c_1(T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}, \alpha_n) = 2\alpha_n - (n+2)\pi^\star x$ and that $\pi_\star\alpha_n^2 = c_1(E_n, h_n) = (n+2)x$ so that $\pi_\star c_1(T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}, \alpha_n)^2 = 0$. We find

$$\begin{aligned} &\widehat{\pi}_\star(\widehat{Td}(\overline{T_{\mathcal{S}_n/\mathbb{P}^1_\mathbb{Z}}})a(R(T_{S_n/\mathbb{P}^1_\mathbb{Z}}))) \\ &\quad= a\left(\pi_\star\left(Td(T_{S_n/\mathbb{P}^1}, \alpha_n)R(T_{S_n/\mathbb{P}^1}, \alpha_n)\right)\right) \\ &\quad= a\left(\pi_\star\left((1 + \frac{1}{2}c_1(T_{S_n/\mathbb{P}^1}))(2\zeta'(-1) + \zeta(-1))c_1(T_{S_n/\mathbb{P}^1})\right)\right) \\ &\quad= a\left(4\zeta'(-1) + 2\zeta(-1)\right). \end{aligned}$$

Back to formula (4.4.1), we find

$$Tors(\alpha_n, 1) = \frac{1+\log 2\pi}{3} - 4\zeta'(-1) - 2\zeta(-1) = \tau(\mathbb{P}^1, \omega_{\mathbb{P}^1}).$$

We remark that the degree two part of the torsion vanishes. We conclude this step

$$\begin{aligned} \int_{\mathbb{P}^1} Td(T\mathbb{P}^1, \omega_{\mathbb{P}^1})Tors(\alpha_n, 1) &= Tors(\alpha_n, 1)\int_{\mathbb{P}^1} Td(T\mathbb{P}^1) \\ &= Tors(\alpha_n, 1) = \tau(\mathbb{P}^1). \end{aligned}$$

4.5. **On the Bott-Chern term.** We now turn to the computation of $\widetilde{Td}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})$.

For $ch(\mathcal{O}_{S_n}, 1) = 1$, we only need to know $\widetilde{Td_3}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})$. For $Td_3 = \frac{1}{24}c_1c_2$, we derive from [G-S-90] (prop.1.3.1.2) that

$$\begin{aligned} 24\widetilde{Td_3}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1}) &= \widetilde{c_1}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})c_2(T_{S_n/\mathbb{P}^1} \oplus \pi^\star T\mathbb{P}^1) \\ &\quad + c_1(TS_n, \alpha_n)\widetilde{c_2}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1}). \end{aligned} \tag{4.5.1}$$

The arithmetic relations between Chern classes in the following two exact sequences

$$\begin{array}{ccccccccc} 0 & \to & (T_{S_n/\mathbb{P}^1}, \alpha_n) & \to & (TS_n, \alpha_n) & \to & (\pi^\star T\mathbb{P}^1, \alpha_n) & \to & 0 \\ 0 & \to & (T_{S_n/\mathbb{P}^1}, \alpha_n) & \to & (TS_n, \alpha_n) & \to & (\pi^\star T\mathbb{P}^1, \pi^\star\omega_{\mathbb{P}^1}) & \to & 0 \end{array}$$

and the relation

$$\widehat{c_1}(\overline{\pi^\star T\mathbb{P}^1_{\mathbb{Z}}}, \alpha_n) = \widehat{c_1}(\overline{\pi^\star T\mathbb{P}^1_{\mathbb{Z}}}, \pi^\star\omega_{\mathbb{P}^1}) - a(\log \frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}})$$

enables us to infer

$$\widetilde{c}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1}) = \widetilde{c}(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) + c(T_{S_n/\mathbb{P}^1}, \alpha_n) \log \frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}}.$$

We can now evaluate the contribution of the Bott-Chern part. The first term in formula (4.5.1) is

$$\begin{aligned} &\widetilde{c_1}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})c_2(T_{S_n/\mathbb{P}^1} \oplus \pi^\star T\mathbb{P}^1) \\ &= \widetilde{c_1}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})c_1(T_{S_n/\mathbb{P}^1}, \alpha_n)c_1(\pi^\star T\mathbb{P}^1, \pi^\star\omega_{\mathbb{P}^1}) \\ &= \log \frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}} (2\alpha_n - (n+2)x)\, 2x = 4 \log \frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}}\alpha_n x \end{aligned}$$

so that

$$\begin{aligned} &F_\star\left(\widetilde{c_1}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})c_2(T_{S_n/\mathbb{P}^1} \oplus \pi^\star T\mathbb{P}^1)\right) \\ &= 4\pi_\star\left(\log \frac{1 + (n+1)|z|^2\|e^\star\|^{2n}}{1 + |z|^2\|e^\star\|^{2n}}\alpha_n\right) f_\star\omega_{\mathbb{P}^1} \\ &= 4\int_{\mathbb{C}} \log \frac{1 + (n+1)|z|^2}{1 + |z|^2} \frac{i}{2\pi} \frac{dz \wedge d\overline{z}}{(1 + |z|^2)^2} = \left[4 + \frac{4}{n}\right] \log(n+1) - 4. \end{aligned}$$

As for the second Bott-Chern class of $(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})$

$$\widetilde{c_2}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1}) = \widetilde{c_2}(TS_n, T\mathbb{P}^1, \alpha_n, \alpha_n) + c_1(T_{S_n/\mathbb{P}^1}, \alpha_n) \log \frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}}.$$

This in turn cuts the computations into two pieces we already computed in section 3.3. We find

$$F_\star\left(c_1(TS_n, \alpha_n)\widetilde{c_2}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})\right) = 4n + 4 - \left[n + 4 + \frac{4}{n}\right] \log(n+1)$$

Summing up

$$\int_{S_n} \widetilde{Td}(TS_n, T\mathbb{P}^1, \alpha_n, \omega_{\mathbb{P}^1})ch(\mathcal{O}_{S_n}, 1) = \frac{n}{6} - \frac{n\log(n+1)}{24}.$$

This ends the second proof of our main formula.

4.6. **Remark on deformations.** The only deformations of the surface S_n are the surfaces S_m where m is of the same parity as n. Hirzebruch surfaces are pairwise diffeomorphic according to the parity of n, but the natural diffeomorphisms as described in [M-K-71] Part I, Theorem 4.2. are not isometric for the metrics we choose which are constructed algebraically. More precisely, formulas for the deformation families of S_0 specializing to S_{2p}

$$\begin{array}{ccc} S_{0,t} & \overset{t\to 0}{\longrightarrow} & S_{2p} \\ \overset{\Psi_t}{\simeq}\downarrow & & \\ S_0 & & \end{array}$$

lead, on appropriate open sets, to explicit formula for the metric on $S_{0,t}$ transfered from the metric on S_0 :

$$\Psi_t^\star \alpha_0 = \pi^\star \omega_{\mathbb{P}^1} + dd^c \log \left(1 + \left|\frac{x^p}{t} - \frac{1}{z}\right|^2\right)$$

and

$$\Psi_t^\star \alpha_0 = \pi^\star \omega_{\mathbb{P}^1} + dd^c \log \left(1 + \left|\frac{z}{tx^p z + t^2}\right|^2\right).$$

This shows that the metric acquires singularities in the specialization : it vanishes generically on generic fibers and tends to Fubini-Study metric on the fiber over 0.

Using a genuine diffeomorphism between S_{2p} and S_0, the computation of the analytic torsion of S_{2p} would require the anomaly formula [B-G-S-88] Theorem 1.23.

5. Appendix

We will use Hodge identity $\delta'' = [i\Lambda_{\alpha_n}, d']$ to check that the exhibited form is harmonic. Starting from

$$\begin{aligned} & dd^c \log(1 + |z|^2 \|e^\star\|^{2n}) \\ &= (n - \frac{n}{1 + |z|^2\|e^\star\|^{2n}})\pi^\star\omega_{\mathbb{P}^1} + \frac{\|e^\star\|^{2n}}{(1 + |z|^2\|e^\star\|^{2n})^2}\frac{i}{2\pi}\left|dz + nzd'\log\|e^\star\|^2\right|^2 \end{aligned}$$

and

$$\begin{aligned} & dd^c \log(1 + (n+1)|z|^2 \|e^\star\|^{2n}) \\ &= (n - \frac{n}{1 + (n+1)|z|^2\|e^\star\|^{2n}})\pi^\star\omega_{\mathbb{P}^1} + \frac{(n+1)\|e^\star\|^{2n}}{(1 + (n+1)|z|^2\|e^\star\|^{2n})^2}\frac{i}{2\pi}\left|dz + nzd'\log\|e^\star\|^2\right|^2 \end{aligned}$$

we derive

$$\begin{aligned} dd^c \log \frac{\alpha_{n|\pi^\star T\mathbb{P}^1}}{\pi^\star\omega_{\mathbb{P}^1}} &= dd^c \log \frac{1 + (n+1)|z|^2\|e^\star\|^{2n}}{1 + |z|^2\|e^\star\|^2} \\ &= \left(\frac{n}{1 + |z|^2\|e^\star\|^{2n}} - \frac{n}{1 + (n+1)|z|^2\|e^\star\|^{2n}}\right)\pi^\star x \\ &\quad + \left(\frac{(n+1)\|e^\star\|^{2n}}{(1 + (n+1)|z|^2\|e^\star\|^{2n})^2} - \frac{\|e^\star\|^{2n}}{(1 + |z|^2\|e^\star\|^{2n})^2}\right)\frac{i}{2\pi}\left|dz + nzd'\log\|e^\star\|^2\right|^2 \\ &= \frac{n^2|z|^2\|e^\star\|^{2n}}{(1 + |z|^2\|e^\star\|^{2n})(1 + (n+1)|z|^2\|e^\star\|^{2n})}\pi^\star\omega_{\mathbb{P}^1} \\ &\quad + \frac{n\|e^\star\|^{2n}(1 - (n+1)|z|^4\|e^\star\|^{4n})}{(1 + |z|^2\|e^\star\|^{2n})^2(1 + (n+1)|z|^2\|e^\star\|^{2n})^2}\frac{i}{2\pi}\left|dz + nzd'\log\|e^\star\|^2\right|^2. \end{aligned}$$

Recalling,

$$\alpha_n = \frac{1 + (n+1)|z|^2\|e^\star\|^{2n}}{1 + |z|^2\|e^\star\|^{2n}}\pi^\star\omega_{\mathbb{P}^1} + \frac{\|e^\star\|^{2n}}{(1 + |z|^2\|e^\star\|^{2n})^2}\frac{i}{2\pi}\left|dz + nzd'\log\|e^\star\|^2\right|^2.$$

we are led to

$$\Lambda_{\alpha_n}\pi^\star\omega_{\mathbb{P}^1} = \frac{1 + |z|^2\|e^\star\|^{2n}}{1 + (n+1)|z|^2\|e^\star\|^{2n}}$$

and to

$$\Lambda_{\alpha_n} dd^c \log \frac{(1+(n+1)|z|^2\|e^\star\|^{2n})}{1+|z|^2\|e^\star\|^{2n}} \quad = \quad n\frac{1-|z|^2\|e^\star\|^{2n}}{1+(n+1)|z|^2\|e^\star\|^{2n}}.$$

This enables to compute

$$\Lambda_{\alpha_n}\left((n+2)\pi^\star\omega_{\mathbb{P}^1} - dd^c \log \frac{(1+(n+1)|z|^2\|e^\star\|^{2n})}{1+|z|^2\|e^\star\|^{2n}}\right) \quad = \quad 2$$

which proves that $\omega_H := \pi^\star\omega_{\mathbb{P}^1} - \frac{1}{n+2}dd^c \log \frac{\langle\Theta(E^\star,h)a^\star,a^\star\rangle}{\pi^\star\omega_{\mathbb{P}^1}\langle a^\star,a^\star\rangle}$ is harmonic. Remark that ω_H^2 is d-exact and non-zero hence not harmonic even if it is the product of two harmonic forms.

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INSTITUT DE MATHÉMATIQUES DE JUSSIEU / PLATEAU 7D / 175, RUE DU CHEVALERET / 75013 PARIS
E-mail address: christophe.mourougane@math.jussieu.fr