COEXISTENCE IN TWO-TYPE FIRST-PASSAGE PERCOLATION MODELS

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Abstract. We study the problem of coexistence in a two-type competition model governed by first-passage percolation on $\mathbb{Z}^d$ or on the infinite cluster in Bernoulli percolation. Actually, we prove for a large class of ergodic stationary passage times that for distinct points $x, y \in \mathbb{Z}^d$, there is a strictly positive probability that $\{z \in \mathbb{Z}^d; d(y,z) < d(x,z)\}$ and $\{z \in \mathbb{Z}^d; d(y,z) > d(x,z)\}$ are both infinite sets. We also show that there is a strictly positive probability that the graph of time-minimizing path from the origin in first-passage percolation has at least two topological ends. This generalizes results obtained by Häggström and Pemantle for independent exponential times on the square lattice.

1. Introduction

The two-type Richardson's model was introduced by Häggström and Pemantle in [5] as a simple competition model between two infections: on the cubic grid $\mathbb{Z}^d$, two distinct infections, type 1 and type 2, starting respectively from two distinct sources $s_1, s_2 \in \mathbb{Z}^d$, compete to invade the sites of the grid $\mathbb{Z}^d$. Each one progresses like a first-passage percolation process on $\mathbb{Z}^d$, governed by the same family $(t(e))_{e\in\mathbb{E}^d}$ of i.i.d. exponential random variables, indexed by the set $\mathbb{E}^d$ of edges of $\mathbb{Z}^d$, but the two infections interfere in the following way: once a site is infected by the type i infection, it remains of type i forever and can not transmit the other infection. This leads to two very different possible evolutions of the process:

- either one infection surrounds the other one, stops it and then goes on infecting the remaining healthy sites as if it was alone,
- or the two infections grow mutually unboundedly, which is called *coexistence.*

The probability that, given two distinct sources, coexistence occurs is of course not full, and the relevant question is to determine whether coexistence occurs with positive probability or not. Although this competition problem is interesting in its own right, it is also a powerful tool to study the existence of two semi-infinite geodesics (or topological ends) of the embedded spanning tree in the related first-passage percolation model. Thus, Häggström and Pemantle proved that coexistence for any two initial sources in the two-type Richardson's model on $\mathbb{Z}^2$ occurs with positive probability, and consequently that in first-passage percolation on $\mathbb{Z}^2$ with i.i.d. exponential passage times on the edges, the probability that there exist at least two topological ends in the embedded spanning tree is positive.

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Their results strongly rely on an interacting particle representation of the problem which is typical of the exponential passage times. The aim of this paper is to extend these results to more general passage times, where this representation is not available anymore or at least much less natural. We consider here stationary ergodic first-passage percolation on $\mathbb{Z}^d$, $d \geq 2$ (and also on an infinite cluster of Bernoulli percolation), and prove that, for any two distinct sources, the probability that coexistence occurs is strictly positive. As a consequence, we obtain that in the related first-passage percolation on $\mathbb{Z}^d$, the probability that there exist at least two topological ends in the embedded spanning tree is positive.

The structure of the proof is the following. First, the key step is to prove that there exist two sources such that coexistence occurs, and this is the aim of Section 3. Heuristically, the shape theorem of first-passage percolation, combined with the fact that the two infections have the same speed, gives the intuition that the largest the distance between the two sources is, the hardest it is for one infection to surround the other one. More precisely, Theorem 3.1 says that if $d(x, y)$ denotes the travel time between the sites x and y, then there exists a site x such that the event

- *and* the set of sites z such that $d(0, z) < d(x, z)$ is infinite,
- *and* the set of sites z such that $d(0, z) > d(x, z)$ is infinite

has positive probability. The proof of this result relies on the existence of a directional asymptotic speed in the related first-passage percolation model.

The next step is to transfer the coexistence result for these sources to *any* two initial sources; this is done by a modification argument of the configuration around the sources using a finite energy property for the passage times. Roughly speaking, this result expresses the fact that non-coexistence is due to a *local* advantage obtained by one infection at the first moments of the competition. The two topological ends result is shown by a similar modification argument. These results are proved separately in Section 4 for diffuse passage times and in Section 5 for integer passage times.

The last section is finally devoted to the study of a probabilistic cellular automata describing a discrete competition model between two infection types, related to the chemical distance in super-critical Bernoulli percolation on $\mathbb{Z}^d$.

We start now with a reminder of the result of existence of directional asymptotic speeds in classical first-passage percolation, and an extension of this result to first-passage percolation on an infinite Bernoulli cluster.

2. Reminder on the directional asymptotic speed results

In classical first-passage percolation, one has the well-known directional asymptotic speed result: if $(t(e))_{e\in\mathbb{E}^d}$ are i.i.d. non-negative integrable random variables, then for every $x \in \mathbb{Z}^d$, there exists $\mu(x) \geq 0$ such that a.s :

$$\lim_{n\to\infty} \frac{t(0, nx)}{n} = \lim_{n\to\infty} \frac{\mathbb{E}\ t(0, nx)}{n} = \mu(x).$$

This result has been extended in full details in a previous work of the authors [3] to first-passage percolation on an infinite Bernoulli cluster. The aim of this section is to introduce an adapted framework and to recall, without proofs, the results needed in this paper.

Grid structure of $\mathbb{Z}^d$. In the following, $d \geq 2$. We denote by $\mathbb{Z}^d$ the graph whose set of vertices is $\mathbb{Z}^d$, and where we put a non-oriented edge between each pair $\{x, y\}$ of *neighbor* points in $\mathbb{Z}^d$, *i.e.* points whose Euclidean distance is equal to 1. This set of edges is denoted by $\mathbb{E}^d$. A *(simple) path* is a sequence $\gamma = (x_1, x_2, \dots, x_n, x_{n+1})$ of distinct points such that x_i and x_{i+1} are neighbors and e_i is the edge between x_i and x_{i+1}. The number n of edges in γ is called the *length* of γ and is denoted by $|\gamma|$.

For any set X, and $u \in \mathbb{Z}^d$, we define the *translation operator* θ_u on $X^{\mathbb{E}^d}$ by the relation

$$\forall \omega \in X \quad \forall e \in \mathbb{E}^d \quad (\theta_u \omega)_e = \omega_{u.e},$$

where $u.e$ denotes the natural action of $\mathbb{Z}^d$ on $\mathbb{E}^d$: if $e = \{a, b\}$, then $u.e = \{a + u, b + u\}$.

Assumptions and construction of first-passage percolation.

Denote by $p_c(d)$ the critical threshold for Bernoulli percolation on the edges $\mathbb{E}^d$ of $\mathbb{Z}^d$, and choose $p \in (p_c, 1]$. On $\Omega_E = \{0,1\}^{\mathbb{E}^d}$, consider the measure $\mathbb{P}_p$:

$$\text{on } \Omega_E = \{0,1\}^{\mathbb{E}^d}, \ \ \mathbb{P}_p = (p\delta_1 + (1-p)\delta_0)^{\otimes \mathbb{E}^d}.$$

A point ω in Ω_E is a *random environment* for first-passage percolation. An edge $e \in \mathbb{E}^d$ is said to be *open* in the environment ω if $\omega_e = 1$, and *closed* otherwise. A path is said to be *open* in the environment ω if all its edges are open in ω. The *clusters* of a environment ω are the connected components of the graph induced on $\mathbb{Z}^d$ by the open edges in ω. As $p > p_c(d)$, there almost surely exists one and only one infinite cluster, denoted by C_∞. On $\Omega_S = (\mathbb{R}_+)^{\mathbb{E}^d}$, consider a probability measure $\mathbb{S}_\nu$ such that:

$$\text{on } \Omega_S = (\mathbb{R}_+)^{\mathbb{E}^d}, \ \ \mathbb{S}_\nu \text{ is stationary and ergodic}$$

with respect to the previously introduced family of translations of the grid. We suppose moreover that $\mathbb{S}_\nu$ satisfies integrability and dependence conditions:

$$m = \sup_{e \in \mathbb{E}^d} \int \eta_e \, d\mathbb{S}_\nu(\eta) < +\infty. \tag{1}$$

$$\exists \alpha > 1, \ \exists A, B > 0 \text{ such that } \forall \Lambda \subseteq \mathbb{E}^d, \ \mathbb{S}_\nu \left(\sum_{e \in \Lambda} \eta_i \geq B|\Lambda| \right) \leq \frac{A}{|\Lambda|^\alpha}. \tag{2}$$

For instance, if $\mathbb{S}_\nu$ is the product measure $\nu^{\otimes \mathbb{E}^d}$, assumption (2) follows from the Marcinkiewicz-Zygmund inequality as soon as the passage time of an edge has a moment of order strictly greater than 2 – see e.g. Theorem 3.7.8 in [8].

Our probability space will then be $\Omega = \Omega_E \times \Omega_S$. A point in Ω will be denoted (ω, η), with ω corresponding to the environment, and η assigning to each edge a non-negative *passage time* which represents the time needed to cross the edge. The final probability is:

$$\text{on } \Omega = \Omega_E \times \Omega_S, \ \ \mathbb{P} = \mathbb{P}_p \otimes \mathbb{S}_\nu.$$

In the context of first-passage percolation, as we are interested in asymptotic results concerning travel time from the origin to points that tend to infinity, it is natural to condition $\mathbb{P}_p$ on the event that 0 is in the infinite cluster:

$$\overline{\mathbb{P}}_p(.) = \mathbb{P}_p(.|0 \in C_\infty) \quad \text{and} \quad \overline{\mathbb{P}} = \overline{\mathbb{P}}_p \otimes \mathbb{S}_\nu.$$

For $B \in \mathcal{B}(\Omega_E)$, with $B \subset \{0 \leftrightarrow \infty\}$ and $\mathbb{P}_p(B) > 0$, we will also define the probability measure $\overline{\mathbb{P}}_B$ by

$$\forall C \in \mathcal{B}(\Omega) \quad \overline{\mathbb{P}}_B(C) = \frac{\mathbb{P}(C \cap (B \times \Omega_S))}{\mathbb{P}_p(B)}.$$

Examples. The previous assumptions of the generalized first-passage percolation model include:

- The case of classical i.i.d. first-passage percolation: take $p = 1$, *i.e.* all the edges of $\mathbb{Z}^d$ are open, and $\mathbb{S}_\nu = \nu^{\otimes \mathbb{E}^d}$, where ν is a probability measure on $\mathbb{R}_+$.
- The case of classical i.i.d. first-passage percolation, but allowing the passage times to take the value ∞ with positive probability: take $p_c(d) < p < 1$, a probability measure ν on $\mathbb{R}_+$, and set $\mathbb{S}_\nu = \nu^{\otimes \mathbb{E}^d}$. This is equivalent to consider $p = 1$ and $\mathbb{S}_\nu = \tilde{\nu}^{\otimes \mathbb{E}^d}$, where $\tilde{\nu}$ is a probability measure on $\mathbb{R}_+ \cup \{\infty\}$ that charges ∞ with probability $1 - p$.
- The case of stationary first-passage percolation, as considered by Boivin in [1]: take $p = 1$ and $\mathbb{S}_\nu$ a stationary probability measure.

The travel time. The *chemical distance* $D(x, y)$ between x and y in $\mathbb{Z}^d$ only depends on the Bernoulli percolation structure ω and is defined as follows: $D(x, y)(\omega) = \inf_\gamma |\gamma|$, where the infimum is taken on the set of paths whose extremities are x and y and that are open in the environment ω. It is of course only defined when x and y are in the same percolation cluster, and represents then the minimal number of open edges needed to link x and y in the environment ω. Otherwise, we set by convention $D(x, y) = +\infty$.

For $(\omega, \eta) \in \Omega$, and $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we define the *travel time* from x to y:

$$d(x, y)(\omega, \eta) = \inf_\gamma d(\gamma) = \inf_\gamma \sum_{e \in \gamma} \eta_e,$$

where the infimum is taken on the set of paths whose extremities are x and y and that are open in the environment ω. Of course $d(x, y) = +\infty$ if and only if $D(x, y) = +\infty$.

A path γ from x to y which realizes the distance $d(x, y)$ is called a *finite geodesic*. An infinite path $\gamma = (x_i)_{i \geq 0}$ is called a *semi-infinite geodesic* if $(x_n, x_{n+1}, \dots, x_p)$ is a finite geodesic for every $n \leq p$.

Directional asymptotic speed results. In classical first-passage percolation, we study, for each $u \in \mathbb{Z}^d \backslash \{0\}$, the travel time $d(0, nu)$ as n goes to infinity. Here, as all points in $\mathbb{Z}^d$ are not necessarily accessible from 0, we must introduce the following definitions:

Definition 2.1. *For each $u \in \mathbb{Z}^d \backslash \{0\}$ and $B \in \mathcal{B}(\Omega_E)$, let*

$$T_u^B(\omega) = \inf\{n \geq 1; \theta_{nu}\omega \in B\},$$

define the associated random translation operator on $\Omega = \Omega_E \times \Omega_S$

$$\Theta_u^B(\omega, \eta) = \left(\theta_u^{T_u^B(\omega)}(\omega), \theta_u^{T_u^B(\omega)}(\eta)\right)$$

and the composed version $T_{n,u}^B(\omega) = \displaystyle\sum_{k=0}^{n-1} T_u^B\left((\Theta_u^B)^k \omega\right).$

Note that T_u^B only depends on the environment ω, and not on the passage times η, whereas the operator Θ_u^B acts on the whole configuration (ω, η). The next step is to study the asymptotic behavior of such quantities:

Lemma 2.2. *Θ_u^B is a $\overline{\mathbb{P}}$-preserving transformation, is ergodic for $\overline{\mathbb{P}}$ and*

$$\mathbb{E}_{\overline{\mathbb{P}}}\, T_u^B = \frac{1}{\mathbb{P}_p(B)} \quad \textit{and} \quad \frac{T_{n,u}^B}{n} \to \frac{1}{\mathbb{P}_p(B)} \quad \overline{\mathbb{P}}\ \textit{a.s.}$$

Proof. The idea is to prove that classical ergodic theorems can be applied. □

We turn now to the study of the quantity analogous to $d(0, nu)$ in the classical first-passage percolation:

Lemma 2.3. *Let $B \in \mathcal{B}(\Omega_E)$, with $B \subset \{0 \leftrightarrow \infty\}$ and $\mathbb{P}_p(B) > 0$. For $u \in \mathbb{Z}^d \backslash \{0\}$, there exists a constant $f_u^B \geq 0$ such that*

$$\frac{d\left(0, T_{n,u}^B(\omega)u\right)(\omega, \eta)}{n} \to f_u^B \quad \overline{\mathbb{P}}_B\ \textit{a.s.}$$

The convergence also holds in $L^1(\overline{\mathbb{P}}_B)$. Moreover, $f_u^B \leq \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, T_u^B u) < +\infty$.

Proof. These results are proved with full details when $B = \{0 \leftrightarrow \infty\}$ in [3]. Since the proof is essentially the same, we omit it. □

Now, for each $u \in \mathbb{Z}^d \backslash \{0\}$, we define the asymptotic speed in the direction u by

$$\mu(u) = \mathbb{P}_p(0 \leftrightarrow \infty) f_u^A$$

for the choice $A = \{0 \leftrightarrow \infty\}$. We also define $\mu(0) = 0$.

Corollary 2.4. *Let $B \in \mathcal{B}(\Omega_E)$, with $B \subset \{0 \leftrightarrow \infty\}$ and $\mathbb{P}_p(B) > 0$. For $u \in \mathbb{Z}^d \backslash \{0\}$, we have:*

$$\begin{aligned} \frac{d(0, (T_{n,u}^B(\omega)u)(\omega, \eta)}{n} &\to \frac{\mu(u)}{\mathbb{P}_p(B)} \quad \overline{\mathbb{P}}_B\ \textit{a.s.} \\ \frac{d(0, (T_{n,u}^B(\omega)u)(\omega, \eta)}{T_{n,u}^B(\omega)} &\to \mu(u) \quad \overline{\mathbb{P}}_B\ \textit{a.s.} \end{aligned}$$

Proof. We use the fact that $\left(\frac{d(0, T_{n,u}^B u)}{T_{n,u}^B}\right)_{n \geq 0}$, as a subsequence of $\left(\frac{d(0, T_{n,u}^A u)}{T_{n,u}^A}\right)_{n \geq 0}$, admits the same almost sure limit $\mu(x)$, and the lemma 2.2. □

In [3], it has been proved that μ enjoys the properties that are usual in classical i.i.d. first-passage percolation: μ is a semi-norm. In classical i.i.d. first-passage percolation with passage time law ν, it is well-known that μ is a norm as soon as $\nu(0) < p_c(d)$. In [3], we gave a long discussion about conditions on $\mathbb{S}_\nu$ implying that μ is a norm. Particularly, if $\mathbb{S}_\nu$ is a product measure $\nu^{\otimes \mathbb{Z}^d}$, μ is a norm as soon as $p\nu(0) < p_c(d)$.

3. Coexistence result

Consider the first-passage percolation model on $\mathbb{Z}^d$ previously introduced. For every pair x and y of distinct points in $\mathbb{Z}^d$, say that the event $\text{Coex}(x, y)$ happens if

$$\{z \in \mathbb{Z}^d; d(x, z) < d(y, z)\} \text{ and } \{z \in \mathbb{Z}^d; d(x, z) > d(y, z)\}$$

are both infinite sets.

The goal of the paper is to prove that for every pair of distinct points $x, y \in \mathbb{Z}^d$, $\mathbb{P}(\text{Coex}(x, y)) > 0$. Our proofs always require the assumption that μ is not identically null and we guess that this assumption is close to be optimal. Let us detail a particular case where $\mu = 0$ and coexistence never occurs. Suppose that $d = 2$ and $\mathbb{S}_\nu = \nu^{\otimes \mathbb{E}^d}$, with $p\nu(0) > p_c(2) = \frac{1}{2}$. In this case, μ is identically null, as previously noted. Consider two distinct points $x, y \in \mathbb{Z}^2$. Since $p\nu(0) > p_c(2)$, there almost surely exists an infinite cluster of open edges with passage time zero. It is known that in dimension 2, the supercritical infinite cluster almost surely contains a circuit that surrounds x and y and disconnects them from infinity – see Harris [6] or for instance Grimmett's book [4]. Clearly, the points in this circuit are equally d-distant from x (*resp.* y). So, if x reaches the circuit before y, it necessarily also reaches every point outside the circuit before y. Similarly, if x and y reach the circuit at the same time, all the points outside the circuit will also be reached at the same time by x and y. In both cases, coexistence does not occur.

The next theorem gives conditions that ensures that coexistence possibly occurs for some (random) x, y.

Theorem 3.1. *Let $d \geq 2$, $p > p_c(d)$, $\mathbb{S}_\nu$ a stationary ergodic probability measure on $(\mathbb{R}_+)^{\mathbb{E}^d}$ satisfying (1) and (2), and μ be the related semi-norm describing the directional asymptotic speeds.*

Let $B \in \mathcal{B}(\Omega_E)$, with $B \subset \{0 \leftrightarrow \infty\}$ and $\mathbb{P}_p(B) > 0$, and $y \in \mathbb{Z}^d$. We have:

$$\text{if } \mathbb{E}\ d(0, T^B_{1,y} y) < \frac{2\mu(y)}{\mathbb{P}_p(B)}, \quad \text{then } \overline{\mathbb{P}}_B(\mathit{Coex}(0, T^B_{1,y} y)) > 0.$$

Moreover, if $x \in \mathbb{Z}^d$ is such that $\mu(x) > 0$, then $y = rx$ satisfies to the previous condition provided that r is large enough.

Note that when $p = 1$, which corresponds to classical first-passage percolation, we can take $B = \Omega_E$, and then $T^B_{1,y} y$ is simply equal to y.

Before beginning the proof, we want to describe an elementary and clever trick used by Pemantle and Häggström in [5] that will also be useful here. Consider figure 3. The left picture deals with our problem: if we prove that when M_n goes

Figure 1. The symmetry argument

to the infinity on the right (*resp.* on the left), then M_n is infinitely often closer (*resp.* more distant) from B than from A with a probability strictly larger than 0.5, then coexistence holds with positive probability.

Now consider the right picture: for fixed n, $(d(0,J_n),d(0,I_n))$ has the same law that $(d(A,M_n),d(B,M_n))$, so the event $\{d(0,J_n)>d(0,I_n)\}$ occurs with the same probability as the event $\{d(A,M_n)>d(B,M_n)\}$. So, if we show that $\mathbb{P}(d(0,J_n)>d(0,I_n))>1/2$ for infinitely many n, the result is proved.

As Häggström and Pemantle said, the idea is that there are sites arbitrarily far away from the origin which strongly feel from which source the infection is coming. Their *modus operandi*, in the case of i.i.d. exponentials on $\mathbb{Z}^2$, was to control the infection rate "from the right to the left" and the infection rate "from the left to the right". The main idea of the proof which follows is that the advantage of the closest source can be quantified using the existence of a directional asymptotic speed in first-passage percolation. Concretely, we will use the law of $d(A,M_n)-d(B,M_n)$ (in fact, the law of $d(0,J_n)-d(0,I_n)$) instead of those of $\{d(A,M_n)>d(B,M_n)\}$ (or $\{d(0,J_n)>d(0,I_n)\}$) to carry the information.

Proof. (*Theorem 3.1*) Choose $y\in\mathbb{Z}^d\backslash\{0\}$ such that

$$\mathbb{E}\ d(0,T^B_{1,y}y)<\frac{2\mu(y)}{\mathbb{P}_p(B)}. \tag{3}$$

Let us note

$$\begin{aligned} S_0 &= \overline{\lim_{\|z\|_1\to+\infty}}\ \{d(0,z)<d(T^B_{1,y}y,z)<+\infty\},\\ S_1 &= \overline{\lim_{\|z\|_1\to+\infty}}\ \{+\infty>d(0,z)>d(T^B_{1,y}y,z)\}. \end{aligned}$$

It is obvious that $\mathrm{Coex}(0,T^B_{1,y}y)=S_0\cap S_1$. Intuitively, one expects that the difference between $d(0,z)$ and $d(T^B_{1,y}y,z)$ will be more important if $z\in\mathbb{R}y$, and we will effectively consider such z. For the convenience of the reader, we also note, for $n\in\mathbb{Z}_+$ and $x\in\mathbb{Z}^d$, $\tilde{T}_{n,x}=T^B_{n,x}x$. Define $\tilde{T}_{0,x}=0$, and for $n\geq 0$,

$$\begin{cases} X_n=d(0,\tilde{T}_{n,y})-d(\tilde{T}_{1,y},\tilde{T}_{n,y}),\\ X'_n=d(\tilde{T}_{1,y},\tilde{T}_{n,-y})-d(0,\tilde{T}_{n,-y}).\end{cases}$$

By triangular inequality, one has $|X_n|\leq d(0,\tilde{T}_{1,y})$ and $|X'_n|\leq d(0,\tilde{T}_{1,y})$.

Note that for $\omega\notin S_0$, $X_n(\omega)\leq 0$ as soon as n is large enough, whereas for $\omega\notin S_1$, $X'_n(\omega)\leq 0$ for large n. It follows that for $\omega\notin S_0\cap S_1$, $X_n(\omega)+X'_{n-1}(\omega)\leq d(0,\tilde{T}_{1,y})(\omega)$ for large n. Let us define

$$Q_n=\sum_{k=1}^n(X_k+X'_{k-1}),\quad Z_n=\frac{Q_n}{n}\quad\text{and}\quad Z=\overline{\lim_{n\to+\infty}}\ Z_n.$$

The previous remark implies easily that

$$\forall\omega\notin S_0\cap S_1\quad Z(\omega)\leq d(0,\tilde{T}_{1,y})(\omega). \tag{4}$$

By lemma 2.3, $d(0,\tilde{T}_{1,y})$ is integrable under $\overline{\mathbb{P}}_B$. Since $|Z_n|\leq d(0,\tilde{T}_{1,y})$, it follows (for instance by Fatou's lemma) that

$$\mathbb{E}_{\overline{\mathbb{P}}_B}Z=\mathbb{E}\ \overline{\lim_{n\to+\infty}}\ Z_n\geq\overline{\lim_{n\to+\infty}}\ \mathbb{E}_{\overline{\mathbb{P}}_B}Z_n.$$

Since $d(\tilde{T}_{1,y}, \tilde{T}_{n,y}) = d(0, \tilde{T}_{n-1,y}) \circ \Theta_y^B$, it follows from the invariance of $\overline{\mathbb{P}}_B$ under Θ_y^B that

$$\begin{aligned}\mathbb{E}_{\overline{\mathbb{P}}_B} X_n &= \mathbb{E}\left(d(0, \tilde{T}_{n,y}) - d(\tilde{T}_{1,y}, \tilde{T}_{n,y})\right) \\ &= \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n,y}) - \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n-1,y}).\end{aligned}$$

Then, it follows that $\mathbb{E}_{\overline{\mathbb{P}}_B}(X_1 + X_2 + \cdots + X_n) = \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n,y})$. Similarly, as $d(\tilde{T}_{1,y}, \tilde{T}_{n,-y}) = d(0, \tilde{T}_{n+1,-y}) \circ \Theta_y^B$,

$$\begin{aligned}\mathbb{E}_{\overline{\mathbb{P}}_B} X'_n &= \left(\mathbb{E}_{\overline{\mathbb{P}}_B} d(\tilde{T}_{1,y}, \tilde{T}_{n,-y}) - d(0, \tilde{T}_{n,-y})\right) \\ &= \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n+1,-y}) - \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n,-y}),\end{aligned}$$

and $\mathbb{E}_{\overline{\mathbb{P}}_B}(X'_0 + X'_1 + \cdots + X'_{n-1}) = \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n,-y}) = \mathbb{E}_{\overline{\mathbb{P}}_B} d(0, \tilde{T}_{n,y})$, using for the last equality the fact that $\overline{\mathbb{P}}_B$ is invariant under $(\Theta_y^B)^n$ and the fact that a distance is symmetric.

Then, $\mathbb{E}_{\overline{\mathbb{P}}_B} Z_n = \frac{2\mathbb{E}_{\overline{\mathbb{P}}_B} d(0,\tilde{T}_{n,y})}{n}$. Since, via Corollary 2.4, $\frac{\mathbb{E}_{\overline{\mathbb{P}}_B} d(0,\tilde{T}_{n,y})}{n}$ converges to $\frac{\mu(y)}{\mathbb{P}_p(B)}$, it follows that

$$\mathbb{E}_{\overline{\mathbb{P}}_B} Z \geq \frac{2\mu(y)}{\mathbb{P}_p(B)}. \tag{5}$$

Putting together (3), (4) and (5), we see that $\overline{\mathbb{P}}_B(S_0 \cap S_1) = 0$ – or equivalently $\overline{\mathbb{P}}_B((S_0 \cap S_1)^c) = 1$ – would yield to a contradiction. This concludes the proof of the first assertion.

The second assertion is a direct consequence of Corollary 2.4. □

One can be a bit perplexed by the fact that the position of the source which may coexist with a source at the origin is a random variable. The goal of the next result is to come back to deterministic sources. Intuitively, one can guess that the larger the distance between the two sources is, the higher the probability of coexistence will be. This is the spirit of the next result.

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, suppose moreover that μ is not identically null. Then, we have:*

- *For $x \in \mathbb{Z}^d$ with $\mu(x) \neq 0$, there is an infinite set of odd values for $n \in \mathbb{Z}_+$ such that $\mathbb{P}(Coex(0, nx)) > 0$.*
- $\mathbb{P}(\exists x, y \in \mathbb{Z}^d,\ Coex(x, y)) = 1.$

Let us say a word on the unexpected apparition of odd integers. Of course, the result would be the same with the set of integers and generally, this additional constraint does not bring much. Nevertheless, one will see later that, in the competition context, this additional property sometimes prevents the two infections from reaching a point at the very same time; it will also plays a fundamental role in the proof of Theorem 5.1.

Proof. Let $x \in \mathbb{Z}^d$ be such that $\mu(x) > 0$ and $N \in \mathbb{Z}_+$. Let $A = \{0 \leftrightarrow \infty\}$ and $B = A \cap \{T_{-x}^A \text{ is odd}\}$. We have, from the FKG inequalities,

$$\overline{\mathbb{P}}_p(B) \geq \overline{\mathbb{P}}_p(T_{-x}^A = 1) = \overline{\mathbb{P}}_p(-x \leftrightarrow \infty) \geq \mathbb{P}_p(-x \leftrightarrow \infty) > 0.$$

By lemma 2.3 and Theorem 2.4, $\frac{\mathbb{E}\ d(0,T^B_{1,rx})}{r}$ tends to $\frac{\mu(x)}{\mathbb{P}_p(B)}$, so we can find an odd integer $r \geq N$ with $\frac{\mathbb{E}\ d(0,T^B_{1,rx}rx)}{r} < \frac{2\mu(x)}{\mathbb{P}_p(B)}$. By Theorem 3.1, one has $\overline{\mathbb{P}}(S_0 \cap S_1) > 0$.

By its definition, $T^B_{1,rx}$ almost surely takes its values in the set of non-negative odd integers. Then, we can write

$$\overline{\mathbb{P}}_B(S_0 \cap S_1) = \sum_{k \text{ odd}} \overline{\mathbb{P}}_B(S_0 \cap S_1 \cap \{T^B_{1,rx} = k\}).$$

Then, there exists an odd integer $k \in \mathbb{Z}_+$, with $\overline{\mathbb{P}}_B(S_0 \cap S_1 \cap \{T^B_{1,rx} = k\}) > 0$. So, if we note $n = kr$, we have $n \geq r \geq N$, n is odd and

$$\mathbb{P}(\mathrm{Coex}(0, nx)) \geq \mathbb{P}_p(B)\overline{\mathbb{P}}_B(S_0 \cap S_1 \cap \{T^B_{1,rx} = k\}) > 0.$$

The second point is a consequence of the ergodicity assumption. □

4. Mutual unbounded growth and existence of two disjoint geodesics for diffuse passage times

The aim of this section is to prove the possibility of coexistence in general two-type first-passage percolation, and to study the existence of two semi-infinite geodesics in the corresponding spanning tree.

We first give our assumptions on the law of the passage times $\mathbb{S}_\nu$ and define the two-type competition model.

Assumptions. We consider first-passage percolation on $\mathbb{Z}^d$, with $d \geq 2$. The open edges are given by a realization of a Bernoulli percolation on the edges $\mathbb{E}^d$ of $\mathbb{Z}^d$ with parameter $p \in (p_c(d), 1]$:

$$\text{on } \Omega_E = \{0,1\}^{\mathbb{E}^d}, \ \ \mathbb{P}_p = (p\delta_1 + (1-p)\delta_0)^{\otimes \mathbb{E}^d}.$$

The passage times of the edges are given by a probability measure $\mathbb{S}_\nu$:

$$\text{on } \Omega_S = (\mathbb{R}_+)^{\mathbb{E}^d}, \ \ \mathbb{S}_\nu \text{ is stationary and ergodic.}$$

Finally, we consider the product measure $\mathbb{P} = \mathbb{P}_p \otimes \mathbb{S}_\nu$ on $\Omega_E \times \Omega_S$. We also need two distinct initial sources s_1, s_2 in $\mathbb{Z}^d$.

Lemma 4.1. *If $x \in \mathbb{Z}^d$ is such that $d(s_1, x) < d(s_2, x)$ then for every y in an optimal path realizing $d(s_1, x)$, we have $d(s_1, y) < d(s_2, y)$.*

Proof. Denote by $\gamma(s_1, x)$ an optimal path from s_1 to x, and suppose that there exists $y \in \gamma(s_1, x)$ such that $d(s_2, y) \leq d(s_1, y)$. Then, by triangular inequality,

$$d(s_2, x) \leq d(s_2, y) + d(y, x) \leq d(s_1, y) + d(y, x)$$

but as $y \in \gamma(s_1, x)$, $d(s_1, x) = d(s_1, y) + d(y, x)$ and then $d(s_2, x) \leq d(s_1, x)$, which is a contradiction. □

This allows us to define the following two-type first-passage percolation model.

Definition 4.2. *Under the previous assumptions, we set:*

$$A_1 = \{x \in \mathbb{Z}^d, d(s_1, x) < d(s_2, x)\}, \quad \textit{and} \quad A_2 = \{x \in \mathbb{Z}^d, d(s_2, x) < d(s_1, x)\}.$$

A_i is the set of sites in $\mathbb{Z}^d$ that are finally infected by type i infection. The time of infection of $x \in \mathbb{Z}^d$ is $t(x) = \inf\{d(s_i, x), 1 \leq i \leq 2\}$. We say that x is finally infected if $t(x) < \infty$.

Note that the set of finally infected points could be larger than the union of A_1 and A_2: we can not a priori *exclude that a point x could be reached simultaneously by the two infections, in which case we call it an infected point without defining an infection type.*

We say that the two infections mutually grow unboundedly if the two sets A_1 and A_2 are both infinite.

Thanks to lemma 4.1, the mutual unbounded growth of a two-type first-passage percolation starting from s_1, s_2 is equal to the event $\mathrm{Coex}(s_1, s_2)$ defined in section 3. Note also that A_1 and A_2 are connected sets, and if they are infinite, by a classical compactness argument, one can find, from each $x \in A_i$, a semi-infinite geodesic which is completely in A_i.

We add in this section assumptions that will ensure the uniqueness of optimal paths in first-passage percolation under $\mathbb{P}_p \otimes \mathbb{S}_\nu$. If Λ is a finite subset of $\mathbb{E}^d$, denote by $\mathcal{F}_{\Lambda^c}$ the σ-algebra generated by $\{(\omega_e, \eta_e), e \notin \Lambda\}$. We suppose

$$\forall \Lambda \text{ finite subset of } \mathbb{E}^d, \forall e \in \Lambda, \forall a \in \mathbb{R}_+, \ \ \mathbb{S}_\nu(\eta_e = a | \mathcal{F}_{\Lambda^c}) = 0. \tag{6}$$

Lemma 4.3. *Under the additional assumption (6), we have:*

i. *If γ and γ' are paths that differ at least from one edge, then*

$$\mathbb{P}\left(d(\gamma) = \sum_{e\in\gamma} \eta_e = d(\gamma') = \sum_{e\in\gamma'} \eta_e < \infty \right) = 0.$$

ii. *For every x and y in $\mathbb{Z}^d$, the travel time $d(x,y)$, when it is finite, is a.s. realized by a unique path.*

iii. *For every $\alpha \in \mathbb{R}$, if x, x', y, y' are distinct points in $\mathbb{Z}^d$, $\mathbb{P}(d(x,y) - d(x',y') = \alpha) = 0$.*

iv. *If $\inf_{1\le i\le 2} d(s_i, x)$ is finite, it is realized by a unique source s_i.*

Proof. These are classical and not too difficult consequences of assumption (6). □

Now, if $t(x) < \infty$, then x is reached first by a unique infection; the path of infection $\gamma(x)$ is the unique path from the corresponding source to x that realizes $t(x)$. The set of eventually infected points is in this case the union of A_1 and A_2. In other words, we can define uniquely, for each eventually infected point, its type of infection and its optimal path.

The union of $(\gamma(x))_{x\in\mathbb{Z}^d, t(x)<\infty}$ is then a random forest of two trees $T(s_1)$ and $T(s_2)$, respectively rooted at s_1 and s_2 and respectively spanning A_1 and A_2.

The next theorem ensures the irrelevance of the positions of the two initial sources in determining whether mutual unbounded growth occurs with positive probability or not. Its proof is based on a modification of the configuration around the sources, sufficiently strong to change the initial sources, and sufficiently slight to ensure that some geodesics are not modified outside a finite box.

Lemma 4.4. *Consider $\mathbb{Z}^d$, with $d \ge 2$ and $p \in (p_c(d), 1]$. Choose a stationary ergodic probability measure $\mathbb{S}_\nu$ on $\Omega_S = (\mathbb{R}_+)^{\mathbb{E}^d}$ satisfying to the non-atomic assumption (6) and to:*

$$\forall \Lambda \textit{ finite subset of } \mathbb{E}^d, \forall e \in \Lambda, \forall \varepsilon > 0, \ \ \mathbb{S}_\nu(\eta_e \le \varepsilon | \mathcal{F}_{\Lambda^c}) > 0 \ a.s. \tag{7}$$

If $p = 1$, we add the assumption that the support of the passage time is conditionally unbounded:

$$(8) \qquad \forall \Lambda \textit{ finite subset of } \mathbb{E}^d, \forall e \in \Lambda, \forall M > 0, \;\; \mathbb{S}_\nu(\eta_e \geq M | \mathcal{F}_{\Lambda^c}) > 0 \textit{ a.s.}$$

Then if s_1, s_2 and s'_1, s'_2 are two pairs of distinct points in $\mathbb{Z}^d$,

$$\mathbb{P}(\mathit{Coex}(s_1, s_2)) > 0 \Leftrightarrow \mathbb{P}(\mathit{Coex}(s'_1, s'_2)) > 0.$$

Let us comment the two assumptions (7) and (8). They have the form of finite energy properties, which are usual in modification arguments: it enables to force the occurrence of a wished event inside a finite box. But they also enable the passage time of an edge to take as small – and as large when $p = 1$ – values as we like. This is rather a technical assumption that could probably be relaxed. For instance, assumptions (7) and (8) are satisfied for $\mathbb{S}_\nu = \nu^{\otimes \mathbb{E}^d}$ with ν is equivalent to Lebesgue's measure on $\mathbb{R}_+$.

The next result says that the mutual unbounded growth in the two-type first-passage percolation model and the existence of two distinct semi-infinite geodesics in the embedded spanning tree in the corresponding first-passage percolation model are equivalent.

Lemma 4.5. *Under the same assumptions as in Theorem 4.4,*

$$\begin{aligned} & \exists s_1, s_2 \in \mathbb{Z}^d \textit{ such that } \mathbb{P}(\mathit{Coex}(s_1, s_2)) > 0 \\ \Leftrightarrow \quad & \mathbb{P}\left(\begin{array}{c} \textit{there exist two edge-disjoint semi-infinite geodesics} \\ \textit{in the infection tree rooted in } 0 \end{array} \right) > 0. \end{aligned}$$

Combining these results with Theorem 3.2, we obtain:

Theorem 4.6. *Consider $\mathbb{Z}^d$, with $d \geq 2$ and $p \in (p_c(d), 1]$. Choose a stationary ergodic probability measure $\mathbb{S}_\nu$ on $\Omega_S = (\mathbb{R}_+)^{\mathbb{E}^d}$ satisfying to the integrability assumptions (1), (2) and such that the related semi-norm μ describing the directional asymptotic speeds is not identically null.*

Suppose moreover that $\mathbb{S}_\nu$ satisfies to the non-atomic assumption

$$\forall \Lambda \textit{ finite subset of } \mathbb{E}^d, \forall e \in \Lambda, \forall a \in \mathbb{R}_+, \;\; \mathbb{S}_\nu(\eta_e = a | \mathcal{F}_{\Lambda^c}) = 0,$$

and to the finite energy property

$$\forall \Lambda \textit{ finite subset of } \mathbb{E}^d, \forall e \in \Lambda, \forall \varepsilon > 0, \;\; \mathbb{S}_\nu(\eta_e \leq \varepsilon | \mathcal{F}_{\Lambda^c}) > 0 \textit{ a.s.}$$

If $p = 1$, we add the assumption that the support of the passage time is conditionally unbounded:

$$\forall \Lambda \textit{ finite subset of } \mathbb{E}^d, \forall e \in \Lambda, \forall M > 0, \;\; \mathbb{S}_\nu(\eta_e \geq M | \mathcal{F}_{\Lambda^c}) > 0 \textit{ a.s.}$$

Then,

1. $\forall x \neq y \in \mathbb{Z}^d$, $\mathbb{P}(\mathit{Coex}(x, y)) > 0$.
2. $\mathbb{P}\left(\begin{array}{c} \textit{there exist two edge-disjoint semi-infinite geodesics} \\ \textit{in the infection tree rooted in } 0 \end{array} \right) > 0.$

Examples. 1. Consider first-passage percolation on $\mathbb{Z}^d$, $d \geq 2$, with a family $(t(e))_{e \in \mathbb{E}^d}$ of i.i.d. non negative random variables with a non-atomic unbounded support containing 0, for instance an exponential law as in Richardson's model. Then

- For the two-type competition model, the probability of mutual unbounded growth is positive for every pair of distinct sources in $\mathbb{Z}^d$.

- For the first-passage percolation model with one source, the probability that the embedded spanning tree of $\mathbb{Z}^d$ has two edge-disjoint infinite branches is positive.

These results were proved by Häggström and Pemantle in the pioneer work [5] for Richardson's model in dimension 2. Our results positively answer to the questions asked by Häggström and Pemantle about extensions of their coexistence result to higher dimensions and more general distributions for passage times.

2. Consider first-passage percolation on $\mathbb{Z}^d$, $d \geq 2$, with a family $(t(e))_{e\in\mathbb{E}^d}$ of i.i.d. non negative random variables whose law has no atom excepted in ∞ (*i.e.* edges can be closed with positive probability) and has 0 in its support, for instance $t \sim p.\mathcal{U}_{[0,1]} + (1-p).\delta_\infty$, with $p_c(d) < p < 1$. Then

- For the two-type competition model, the probability of mutual unbounded growth is positive for every pair of distinct sources in $\mathbb{Z}^d$.
- For the first-passage percolation model with one source, the spanning tree of the infinite open cluster has two edge-disjoint infinite branches with positive probability.

Remarks. We evoke here some possible extensions of these results.

1. In the spirit of Deijfen and Häggström's work [2], we could have considered competition models with *fertile finite sets* as initial sources rather than *points*. As the argument is a *local* modification argument around the sources, our proof can be adapted to generalize the irrelevance of the initial sources result: if S_1, S_2 and S'_1, S'_2 are two pairs of fertile finite sets in $\mathbb{Z}^d$,

$$\mathbb{P}(\mathrm{Coex}(S_1, S_2)) > 0 \Leftrightarrow \mathbb{P}(\mathrm{Coex}(S'_1, S'_2)) > 0.$$

2. Let us say a word on *multitype* first-passage percolation. The definitions concerning the two-type first-passage percolation can be generalized in the obvious manner to consider a competition model between N infections starting from N sources $s_1, s_2, ..., s_N$ and trying to invade the sites of $\mathbb{Z}^d$. In this context, the event $\mathrm{Coex}(s_1, s_2, ..., s_N)$ is defined as the event that there finally exist an infinite set of infected points of each type of infection. Theorems 4.4 and 4.5 can be proved in the same manner for N-type first-passage percolation. The only difficulty is to ensure that the considered initial sources $s_1, s_2, ..., s_N$ are susceptible to give rise to a coexistence configuration: this means initial sources $s_1, s_2, ..., s_N$ for which it is possible to find a family of N infinite paths $(\Gamma_i)_{1\leq i\leq N}$ such that for every i, Γ_i starts from s_i and such that any two of these paths have no point in common.

Unfortunately, the coexistence result Theorem 4.6 is not available for N sources, as it relies on Theorem 3.1, which is only valid for two sources, and whose proof doesn't seem to be easy to adapt to more sources.

We can now begin the proofs of these results. As the arguments of lemmas 4.4 and 4.5 are very similar, we give the proof of lemma 4.4 in full details, and give only indications to adapt the proof for the geodesics problem.

Proof. (lemma 4.4). Choose s_1, s_2 and s'_1, s'_2 two pairs of distinct points in $\mathbb{Z}^d$ and denote by Λ an hypercubic box in $\mathbb{Z}^d$ large enough to contain s_1, s_2 and s'_1, s'_2. We also define $\partial\Lambda = \{x \in \Lambda,\ \exists y \notin \Lambda,\ \|x-y\|_1 = 1\}$.

By enlarging Λ if necessary, we can assume that s_1, s_2, s'_1, s'_2 are at a distance at least 3 from $\partial\Lambda$. For an edge $e \in \mathbb{E}^d$, we say that $e \in \Lambda$ if and only if its two extremities are in Λ and at least one is not in $\partial\Lambda$. For a point (ω,η) in $\Omega=\Omega_E\times\Omega_S=\{0,1\}^{\mathbb{E}^d}\times(\mathbb{R}_+)^{\mathbb{E}^d}$,

$$(\omega_\Lambda,\eta_\Lambda)=\{(\omega_e,\eta_e),e\in\Lambda\}\quad\text{and}\quad(\omega_{\Lambda^c},\eta_{\Lambda^c})=\{(\omega_e,\eta_e),e\in\mathbb{E}^d\backslash\Lambda\}.$$

For two point x,y that are in $\Lambda^c\cup\partial\Lambda$, we define $d_{\Lambda^c}(x,y)(\omega)$ as the infimum, among all the paths γ from x to y *whose edges are not in* Λ, of $\sum_{e\in\gamma}\eta_e$, and, when this quantity is finite, $\gamma_{\Lambda^c}(x,y)$ is the only path that realizes this infimum.

Suppose that $\mathbb{P}(\mathrm{Coex}(s_1,s_2))>0$. This is equivalent to say that

$$\mathbb{P}\left(\begin{array}{c}T(s_1)\text{ contains a infinite branch }\Gamma_1\text{ starting from }s_1,\\ T(s_2)\text{ contains a infinite branch }\Gamma_2\text{ starting from }s_2.\end{array}\right)>0.$$

Remember that the box Λ has been chosen large enough to contain s_1,s_2, so there exists on Γ_1 (*resp.* Γ_2) a last point r_1 (*resp.* r_2) to be in $\partial\Lambda$. As $\partial\Lambda$ is finite, there must exist two distinct points $r_1,r_2\in\partial\Lambda$ such that:

$$\mathbb{P}\left(\begin{array}{c}T(s_1)\text{ contains a infinite branch }\Gamma_1\text{ starting from }s_1\\ \text{and whose last point in }\partial\Lambda\text{ is }r_1,\\ T(s_2)\text{ contains a infinite branch }\Gamma_2\text{ starting from }s_2\\ \text{and whose last point in }\partial\Lambda\text{ is }r_2.\end{array}\right)>0. \tag{9}$$

Now, we must introduce the following events:

$$\begin{aligned}
C_1 &= \left\{\begin{array}{c}\text{There is a simple path }(x_i^1)_{i\geq1}\text{ in }\Lambda^c\text{ such that }\|x_1^1-r_1\|_1=1\\ \text{and }\forall i\geq1,\gamma_{\Lambda^c}(r_1,x_i^1)=(r_1,x_1^1,...,x_{i-1}^1,x_i^1).\end{array}\right\},\\
C_2 &= \left\{\begin{array}{c}\text{There is a simple path }(x_j^2)_{j\geq1}\text{ in }\Lambda^c\text{ such that }\|x_1^2-r_2\|_1=1\\ \text{and }\forall j\geq1,\gamma_{\Lambda^c}(r_2,x_j^2)=(r_2,x_1^2,...,x_{j-1}^2,x_j^2).\end{array}\right\},\\
A_i^1 &= \left\{d_{\Lambda^c}(x_i^1,r_1)+d(r_1,s_1)<d_{\Lambda^c}(x_i^1,r_2)+d(r_2,s_2)\right\},\\
A_j^2 &= \left\{d_{\Lambda^c}(x_j^2,r_2)+d(r_2,s_2)<d_{\Lambda^c}(x_j^2,r_1)+d(r_1,s_1)\right\}.
\end{aligned}$$

Let us prove that (9) implies:

$$\mathbb{P}\left(C_1\cap C_2\cap\bigcap_{i\geq1}A_i^1\cap\bigcap_{j\geq1}A_j^2\right)>0. \tag{10}$$

Indeed, suppose that the event in (9) is realized. Then the portion $(x_i^1)_{i\geq1}$ of Γ_1 after r_1 is a good candidate for C_1. As Γ_1 is a branch of the infection tree $T(s_1)$, one has $\forall i\geq1,\gamma(r_1,x_i^1)=(r_1,x_1^1,...,x_{i-1}^1,x_i^1)$, which is stronger than the assertion needed for C_1. Next, as r_1 is in Γ_1, for every $i\geq1$ we know that r_1 is a point of $\gamma(s_1,x_i^1)$, and the fact that x_i^1 is in $T(s_1)$ implies that

$$d(s_1,r_1)+d(r_1,x_i^1)=d(s_1,x_i^1)<d(s_2,x_i^1)\leq d(s_2,r_2)+d(r_2,x_i^1).$$

As r_1 (*resp.* r_2) is the last point of Γ_1 (*resp.* Γ_2) to be in Λ, we have $d(r_1,x_i^1)=d_{\Lambda^c}(r_1,x_i^1)$ (*resp.* $d(r_2,x_i^1)=d_{\Lambda^c}(r_2,x_i^1)$), and thus A_i^1 is realized. Doing the same for C_2 and A_j^2, we see that the event that appears in (9) is included in $C_1\cap C_2\cap\bigcap_{i\geq1}A_i^1\cap\bigcap_{j\geq1}A_j^2$, and thus (10) is proved.

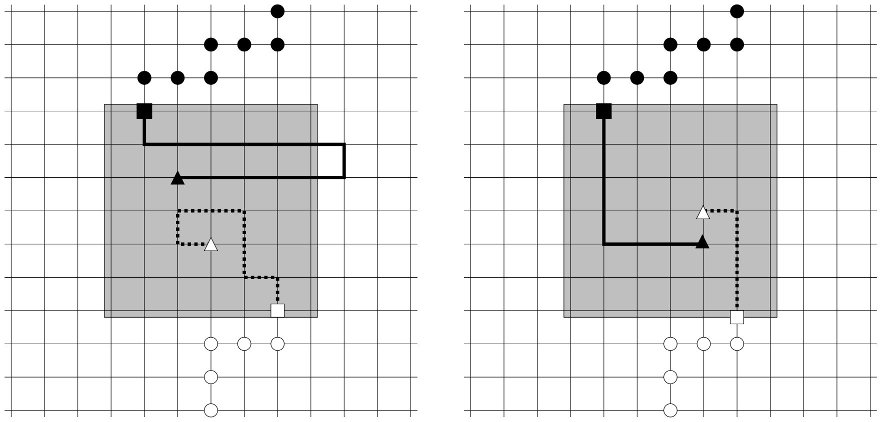

FIGURE 2. Modification of the infection trees. On the left, competition with two sources s_1 (black triangle) and s_2 (white triangle): the box B is in grey, the squares are the exiting points of the branches from Λ (black for r_1 and white for r_2) and the circles are the visible portions of the infinite branches outside Λ (black for $x^1_1, x^1_2, ..., x^1_7$ and white for $x^2_1, ..., x^2_5$). On the right, the configuration outside Λ has not changed, but we forced the branches to follow γ'_1 (in black) and γ'_2 (dashed) and thus we changed the sources into s'_1, s'_2.

Now, as C_1 and C_2 are in $\mathcal{F}_{\Lambda^c}$, conditioning on $\mathcal{F}_{\Lambda^c}$ gives:

$$\begin{aligned}&\mathbb{P}\left(C_1 \cap C_2 \cap \bigcap_{i\geq 1} A^1_i \cap \bigcap_{j\geq 1} A^2_j\right)\\ = &\int d\mathbb{P}(\omega_{\Lambda^c}, \eta_{\Lambda^c}).\mathbf{1}_{C_1}(\omega_{\Lambda^c}, \eta_{\Lambda^c}).\mathbf{1}_{C_2}(\omega_{\Lambda^c}, \eta_{\Lambda^c}).\mathbb{P}\left((\omega,\eta) \in \bigcap_{i\geq 1} A^1_i \cap \bigcap_{j\geq 1} A^2_j \middle| \mathcal{F}_{\Lambda^c}\right),\end{aligned}$$

where we can also write:

$$\begin{aligned}(11)\qquad &\mathbb{P}\left((\omega,\eta) \in \bigcap_{i\geq 1} A^1_i \cap \bigcap_{j\geq 1} A^2_j \middle| \mathcal{F}_{\Lambda^c}\right)\\ = \ &\mathbb{P}\left(\begin{array}{l}\forall i \geq 1,\ d_{\Lambda^c}(x^1_i, r_1) - d_{\Lambda^c}(x^1_i, r_2) < d(r_2, s_2) - d(r_1, s_1)\\ \forall j \geq 1,\ d(r_2, s_2) - d(r_1, s_1) < d_{\Lambda^c}(x^2_j, r_1) - d_{\Lambda^c}(x^2_j, r_2)\end{array} \middle| \mathcal{F}_{\Lambda^c}\right)\ a.s.\end{aligned}$$

Define

$$\begin{aligned}m_1 = m_1(\omega_{\Lambda^c}, \eta_{\Lambda^c}) &= \sup_{i\geq 1}(d_{\Lambda^c}(x^1_i, r_1) - d_{\Lambda^c}(x^1_i, r_2))\\ m_2 = m_2(\omega_{\Lambda^c}, \eta_{\Lambda^c}) &= \inf_{j\geq 1}(d_{\Lambda^c}(x^2_j, r_1) - d_{\Lambda^c}(x^2_j, r_2)).\end{aligned}$$

Now, we have

$$\mathbb{P}\left(C_1 \cap C_2 \cap \bigcap_{i\geq 1} A_i^1 \cap \bigcap_{j\geq 1} A_j^2 \cap \{m_1 < m_2\}\right) > 0. \tag{12}$$

Indeed, thanks to (11) and to lemma 4.3.iii,

$$\mathbb{P}\left(\bigcap_{i\geq 1} A_i^1 \cap \bigcap_{j\geq 1} A_j^2 \cap \{m_1 \geq m_2\} \middle| \mathcal{F}_{\Lambda^c}\right) = \mathbb{P}\left(d(r_2,s_2) - d(r_1,s_1) = m_1 \middle| \mathcal{F}_{\Lambda^c}\right) = 0 \ a.s.$$

and then the probabilities in (12) and in (10) are equal. It is also easy to see, if $m_1(\omega_{\Lambda^c},\eta_{\Lambda^c}) < m_2(\omega_{\Lambda^c},\eta_{\Lambda^c})$, that we can find $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ such that

$$b_1 m_1 < b_2 m_2, \quad a_2 m_2 < a_1 m_1, \quad b_1 - a_1 = 1, \quad \text{and } b_2 - a_2 = 1.$$

Define also

$$\begin{aligned} M = M(\omega_{\Lambda^c},\eta_{\Lambda^c}) &= \max\{a_1 m_1, b_2 m_2\} \\ &+ \max_{\substack{z\in\partial\Lambda \\ d_{\Lambda^c}(r_1,z)<\infty}} \{d_{\Lambda^c}(r_1,z)\} + \max_{\substack{z\in\partial\Lambda \\ d_{\Lambda^c}(r_2,z)<\infty}} \{d_{\Lambda^c}(r_2,z)\}. \end{aligned}$$

Now, we build a set $G = G(\omega_{\Lambda^c},\eta_{\Lambda^c})$ of *good* configurations $(\omega_\Lambda,\eta_\Lambda)$ inside Λ, depending on the configuration outside Λ. First, since Λ has been chosen large enough, it is possible to draw with the edges in Λ, a path γ_1' that links s_1' to r_1 and a path γ_2' that links s_2' to r_2 such that γ_1' and γ_2' have no vertex and no edge in common. Denote by $|\gamma_1'|$ (*resp.* $|\gamma_2'|$) the number of edges in γ_1' (*resp.* γ_2'). We define now G as the set of $(\omega_\Lambda,\eta_\Lambda)$ that satisfy the following conditions:

i. $\forall e \in \gamma_1'$, $\omega_e = 1$ and $a_2 m_2/|\gamma_1'| < \eta_e < a_1 m_1/|\gamma_1'|$,
ii. $\forall e \in \gamma_2'$, $\omega_e = 1$ and $b_1 m_1/|\gamma_2'| < \eta_e < b_2 m_2/|\gamma_2'|$,
iii.
- if $p < 1$, then $\forall e \in \Lambda\backslash(\gamma_1' \cup \gamma_2')$, $\omega_e = 0$.
- if $p = 1$, then $\forall e \in \Lambda\backslash(\gamma_1' \cup \gamma_2')$, $\eta_e > M$.

Under the finite energy assumptions (7) and (8), on the event $\{m_1 < m_2\}$, we have $\mathbb{P}(G(\omega_{\Lambda^c},\eta_{\Lambda^c})|\mathcal{F}_{\Lambda^c}) > 0$ a.s., so (12) implies

$$\int \mathbf{1}_{C_1}.\mathbf{1}_{C_2}.\mathbf{1}_{\{m_1<m_2\}}.\mathbb{P}\left(G|\mathcal{F}_{\Lambda^c}\right) d\mathbb{P} \quad = \quad \mathbb{P}(C_1 \cap C_2 \cap \{m_1 < m_2\} \cap G) > 0. \tag{13}$$

Let us prove now that on the event $C_1 \cap C_2 \cap \{m_1 < m_2\} \cap G$, each of the two infection trees $T(s_1')$ and $T(s_2')$ contains one infinite branch, or in other words

$$C_1 \cap C_2 \cap \{m_1 < m_2\} \cap G \subset \mathrm{Coex}(s_1', s_2').$$

Suppose then that $(\omega,\eta) \in C_1 \cap C_2 \cap \{m_1 < m_2\} \cap G$. We have, in the configuration (ω,η):

- $a_2 m_2 < d(\gamma_1') < a_1 m_1$ thanks to condition i. in the definition of G.
- $b_1 m_1 < d(\gamma_2') < b_2 m_2$ thanks to condition ii. in the definition of G.
- Thus, by difference and by the choice of a_1, b_1, a_2, b_2, we have $m_1 = m_1(b_1 - a_1) < d(\gamma_2') - d(\gamma_1') < m_2(b_2 - a_2) = m_2$.

Moreover, as soon as a path γ from s_1' to r_1 differs from γ_1' by at least one edge, it must use an edge e in $\Lambda\backslash(\gamma_1' \cup \gamma_2')$, and this edge is either closed or such that $\eta_e > M$ thanks to condition iii. in the definition of G. Thus $d(\gamma) > M \geq a_1 m_1 > d(\gamma_1')$. Then, γ_1' is the optimal path from s_1' to r_1. On the other hand, every path γ

from s'_2 to r_1 has to use an edge e in $\Lambda\backslash(\gamma'_1 \cup \gamma'_2)$, and then by the same argument $d(\gamma) > M \geq a_1 m_1 > d(\gamma'_1)$, and then $d(s'_2, r_1) > d(s'_1, r_1)$. Consequently,

$$r_1 \in T(s'_1) \quad \text{and} \quad \gamma'_1 \subset T(s'_1) \quad \text{and} \quad d(s'_1, r_1) = d(\gamma'_1).$$

In the same manner,

$$r_2 \in T(s'_2) \quad \text{and} \quad \gamma'_2 \subset T(s'_2) \quad \text{and} \quad d(s'_2, r_2) = d(\gamma'_2).$$

Let us prove now that the path $(x^1_i)_{i\geq 1}$, given by C_1, is an infinite branch of $T(s'_1)$. Let $i \geq 1$, and let γ be a path from s'_1 to x^1_i that doesn't exit from Λ in r_1 but in $z \neq r_1$. If γ is the minimal path from s'_1 to x^1_i, we must have

$$\begin{aligned} & d(s'_1, x^1_i) = d(\gamma) = d(s'_1, z) + d_{\Lambda^c}(z, x^1_i) < d(s'_1, r_1) + d_{\Lambda^c}(r_1, x^1_i) \\ \text{and then} \quad & d(s'_1, z) < d(r_1, s'_1) + d_{\Lambda^c}(r_1, x^1_i) - d_{\Lambda^c}(z, x^1_i) \leq d(r_1, s'_1) + d_{\Lambda^c}(r_1, z). \end{aligned}$$

The last inequality is just the triangular inequality for d_{Λ^c}. But the path γ must then contain at least one edge e in $\Lambda\backslash(\gamma'_1 \cup \gamma'_2)$, and so such that $\eta_e > M$ or $\omega_e = 0$, and then by definition of M, we must have

$$d(s'_1, z) > M \geq a_1 m_1 + d_{\Lambda^c}(r_1, z) > d(r_1, s'_1) + d_{\Lambda^c}(r_1, z),$$

which contradicts the previous inequality. Thus, the portion of the branch between s'_1 and x^1_i that is in Λ exits Λ in r_1: it is exactly γ'_1. It remains to see that the part of the branch from s'_1 to x_i that is not in Λ is exactly $(r_1, x_1, \ldots, x_{i-1}, x_i)$. But this condition is always satisfied in the event C_1. Thus the path $(x^1_i)_{i\geq 1}$, given by C_1, is an infinite branch of $T(s'_1)$, and in the same manner, the path $(x^2_j)_{j\geq 1}$, given by C_2, is an infinite branch of $T(s'_2)$. We have thus proved the desired inclusion $C_1 \cap C_2 \cap \{m_1 < m_2\} \cap G \subset \mathrm{Coex}(s'_1, s'_2)$.

Now, (13) ensures that $\mathbb{P}(\mathrm{Coex}(s'_1, s'_2)) > 0$, which ends the proof. □

Proof. (*Theorem 4.5*). To prove the direct implication, the proof is the same with $s'_1 = s'_2 = 0$. The only difference is to take the two paths γ'_1, γ'_2 rooted both in 0, with no other point in common and with no edge in common.

The converse implication can also be proved by an analogous modification argument. □

5. Mutual unbounded growth and existence of two distinct geodesics for integer passage times

In the previous section, the law of the passage time of an edge was supposed to admit no atom, and thus the minimal paths were unique. However, as seen in definition 4.2, it is still possible to define a two-type competition model. Choose two distinct sources $s_1, s_2 \in \mathbb{Z}^d$. We define exactly as previously the sets:

$$A_1 = \{x \in \mathbb{Z}^d, d(s_1, x) < d(s_2, x)\}, \text{ and } A_2 = \{x \in \mathbb{Z}^d, d(s_2, x) < d(s_1, x)\},$$

and say that coexistence occurs, event denoted by $\mathrm{Coex}(s_1, s_2)$, if these two sets are infinite.

As in the preceding section, the geodesics inside A_1 and A_2 will be useful to obtain configurations that allow to prove the irrelevance of the initial sources in determining whether coexistence happens with positive probability or not.

Assumptions. We consider first-passage percolation on $\mathbb{Z}^d$, with $d \geq 2$. The open edges are given by a realization of a Bernoulli percolation on the edges $\mathbb{E}^d$ of $\mathbb{Z}^d$ with parameter $p \in (p_c(d), 1]$:

$$\text{on } \Omega_E = \{0,1\}^{\mathbb{E}^d}, \;\; \mathbb{P}_p = (p\delta_1 + (1-p)\delta_0)^{\otimes \mathbb{E}^d}.$$

The passage times of the edges are given by a probability measure $\mathbb{S}_\nu$:

$$\text{on } \Omega_S = (\mathbb{R}_+)^{\mathbb{E}^d}, \;\; \mathbb{S}_\nu \text{ is stationary and ergodic.}$$

Finally, we consider the product measure $\mathbb{P} = \mathbb{P}_p \otimes \mathbb{S}_\nu$ on $\Omega_E \times \Omega_S$. We say that $\mathbb{S}_\nu$ satisfies condition (H_α) if

$$(H_\alpha) \quad \exists A, B > 0 \text{ such that } \quad \forall \Lambda \subseteq \mathbb{E}^d,\ \mathbb{S}_\nu \left(\eta \in \Omega_S; \sum_{e \in \Lambda} \eta_i \geq B|\Lambda| \right) \leq \frac{A}{|\Lambda|^\alpha}.$$

Note that condition (2) was exactly: there exists $\alpha > 1$ such that (H_α) holds. In order to have the coexistence result (Theorem 3.2), we suppose moreover that $\mathbb{S}_\nu$ satisfies conditions (1) and (H_α) for some $\alpha > 1$.

Theorem 5.1. *Let us suppose that $\mathbb{S}_\nu$ satisfies the previous general assumptions, and assume moreover that:*

1. The related semi-norm μ describing the directional asymptotic speeds is not identically null.

2. $\mathbb{S}_\nu$ is "discrete": there exists a subset S of $\mathbb{Z}_+$ such that $\mathbb{S}_\nu(S^{\mathbb{E}^d}) = 1$.

3. $\mathbb{S}_\nu$ satisfies the following finite energy property: for each finite subset Λ of $\mathbb{E}^d$ and each $e_\Lambda \in S^\Lambda$, we have

$$\mathbb{S}_\nu(\omega_\Lambda = e_\Lambda | \mathcal{F}_{\Lambda^c}) > 0 \quad \mathbb{S}_\nu \ a.s.$$

4. If $p = 1$, we add the assumption: S is unbounded.

5. Some stronger integrability is assumed: suppose that one of the three following conditions is fulfilled

- *(H_α) holds for some $\alpha > d^2 + 2d - 1$.*
- *$p = 1$ and the passage times of bonds have a moment of order $\alpha > d$.*
- *$p = 1$, $\mathbb{S}_\nu$ is a product measure and the passage times of bonds have a second moment.*

Then, for each pair s_1, s_2 of distinct sources in $\mathbb{Z}^d$, $\mathbb{P}(Coex(s_1, s_2)) > 0$. Moreover,

$$\mathbb{P} \left(\begin{array}{c} \textit{there exists two disjoint semi-infinite geodesics} \\ \textit{starting from } 0 \textit{ for the random distance } d \end{array} \right) > 0.$$

The last integrability condition is the only one that is specific to the discrete case: in the diffuse case of the previous section, we could give to a given edge an arbitrary small value thanks to (7), and there was no need to control the length of an optimal path. Here, as passage times are integers, we need a stronger integrability assumption that helps to control these paths. The second moment assumption is classical in i.i.d. first-passage percolation to ensure the shape theorem –see the reference article [7] lemma 3.5; the (H_α) assumption with $\alpha > d$ is the one used by Boivin in [1] for the shape theorem in stationary first-passage percolation. Finally, the (H_α) assumption with $\alpha > d^2 + 2d - 1$ is the one we use in [3] lemma 3.7 to obtain the shape theorem when the edges can be closed. Note that, if $\mathbb{S}_\nu$ is the product measure $\nu^{\otimes \mathbb{E}^d}$, assumption (H_α) follows from the Marcinkiewicz-Zygmund inequality as soon as the passage time of an edge has a moment of order strictly greater than 2α.

In any case, we obtain the following estimate:

Lemma 5.2. *There exists $K_1 > 0$ such that for every $a \in \mathbb{Z}^d$, we can construct a random integer $M(a) < +\infty$ such that*

$$a \leftrightarrow \infty \text{ and } y \leftrightarrow \infty \text{ and } \|y\|_1 \geq M(a) \Longrightarrow d(a, y) \leq K_1 \|y\|_1.$$

Examples.

- Take $p < 1$ and $\mathbb{S}_\nu = \nu^{\otimes \mathbb{E}^d}$, where the support of ν is a finite subset of $\mathbb{Z}_+^*$. As a special case, $\nu = \delta_1$ gives the classical chemical distance on a Bernoulli percolation cluster:

  Corollary 5.3 (Geodesics on a Bernoulli cluster)**.** *For each $p > p_c$, consider Bernoulli percolation with parameter p. Then, there almost surely exists a point of the infinite cluster from which we can draw two disjoint semi-infinite geodesics.*

  Proof. The considered event is translation-invariant, so its probability is null or full. By Theorem 5.1 with $\mathbb{S}_\nu = \delta_1^{\otimes \mathbb{E}^d}$, it can not be null. □

- Consider a Poisson point process on $\mathbb{R}^d$ with an intensity proportional to Lebesgue's measure. Let $\alpha \in \mathbb{Z}_+^*$ and define the passage time η_e by $\eta_e = 1 + \alpha n_e$, where n_e is the number of obstacles around e, *i.e.* the number of points of the Poisson process which are closer from e than from any other edge.

We can now begin the proof of Theorem 5.1.

Proof. (Coexistence result). The goal is to prove that for each pair s_1, s_2 of distinct sources in $\mathbb{Z}^d$, $\mathbb{P}(\text{Coex}(s_1, s_2)) > 0$.

By translation invariance, we can suppose $s_1 \in \mathbb{Z}^d \backslash \{0\}$ and $s_2 = 0$. Since μ is not identically null, we can find $x \in \mathbb{Z}^d$ such that $\|s_1\|_1$ and $\|x\|_1$ have the same parity and such that $\mu(x) \neq 0$. Thanks to Theorem 3.2, we can consider an odd integer n_0 such that $\mathbb{P}(\text{Coex}(0, n_0 x)) > 0$. Note $s_1' = n_0 x$. We are going to prove that $\mathbb{P}(\text{Coex}(0, s_1)) > 0$.

Take $K_1 > 0$ and $M(0)$ and $M(s_1')$ as defined in lemma 5.2. Since

$$\lim_{n \to +\infty} \mathbb{P}(\{M(0) \leq n\} \cap \{M(s_1') \leq n\} \cap \text{Coex}(0, s_1')) = \mathbb{P}(\text{Coex}(0, s_1')) > 0,$$

we can find an integer R_1 such that

$$\mathbb{P}\left(\{M(0) \leq R_1\} \cap \{M(s_1') \leq R_1\} \cap \text{Coex}(0, s_1')\right) > 0.$$

Let $\Lambda = \{x \in \mathbb{Z}^d; \|x\|_1 \leq R\}$, for a large integer R whose exact value will be fixed later. The idea is then to show that every configuration (ω, η) in the event $A = \{M(0) \leq R_1\} \cap \{M(s_1') \leq R_1\} \cap \text{Coex}(0, s_1')$ can be modified inside the ball Λ to get a configuration (ω', η') where $\text{Coex}(0, s_1)$ holds. A classical finite energy argument concludes the proof: at first, note that $\mathbb{P} = \mathbb{P}_p \otimes \mathbb{S}_\nu$ also enjoys the finite energy property. Now if B is a subset of Ω such that there exists a map $f : A \to B$

with $f(x)_{\Lambda^c} = x_{\Lambda^c}$ for each $x \in A$, then $P(B) > 0$, because

$$\begin{aligned}\mathbb{P}(B) &= \int_\Omega \mathbb{P}(B|\mathcal{F}_{\Lambda^c})(x)\ d\mathbb{P}(x)\\ &\geq \int_A \mathbb{P}(B|\mathcal{F}_{\Lambda^c})(x)\ d\mathbb{P}(x)\\ &\geq \int_A \mathbb{P}(\{f(x)\}|\mathcal{F}_{\Lambda^c})(x)\ d\mathbb{P}(x)\\ &> 0.\end{aligned}$$

Let us explain now the modification inside Λ. In the following, we will assume without loss of generality that the greatest common divisor of the elements of S is 1. By the lemma of Bezout, we can find a finite family of integers a_k and s_k, with $s_k \in S$, such that $\sum_k a_k s_k = 1$. Note

$$\begin{array}{ll} S_+ = \{k \in S; a_k > 0\}, & S_- = \{k \in S; a_k < 0\},\\ C_1 = \sum_{k\in S_+} a_k, & C_2 = \sum_{k\in S_-}(-a_k),\\ b_1 = \text{ smallest odd element of } S, & b_2 = \text{ smallest even element of } S,\\ C = \max(C_1, C_2), & B = \max(b_1, b_2). \end{array}$$

By convention, if S only contains odd integers, we set $b_2 = b_1$ and $B = 0$.

The next lemma is a geometrical result, and we omit its proof because it is rather tedious and not particularly illuminating:

Lemma 5.4. *Let us consider two fixed points $a_0, a_1 \in \mathbb{Z}^d$ (not necessarily distinct) and two non-negative numbers D and K. Let us note $\Lambda_n = \{x \in \mathbb{Z}^d; \|x\|_1 \leq n\}$.*

There exists $\kappa = \kappa(a_0, a_1, D, K) < +\infty$ such that the following holds as soon as $n \geq \kappa$:

For each distinct $r_0, r_1 \in \mathbb{Z}^d$ with $\|r_0\|_1 = \|r_1\|_1 = n$ and each integer l which has the same parity than $\|a_0 - a_1\|_1$ and satisfies to $|l| \leq Kn + D$, one can construct inside Λ_n two simple paths γ_0 from a_0 to r_0 and γ_1 from a_1 to r_1 with no common point (but maybe a_0 if $a_0 = a_1$) and such that:

$$|\gamma_0| \geq Kn + D,\ \ |\gamma_1| \geq Kn + D,\ \ |\gamma_0| - |\gamma_1| = l.$$

We can now define the radius

$$R = \max(\kappa(0, s_1, B, K_1 C), \|s_1'\|_1 + 2, R_1)$$

and define $\Lambda = \Lambda_R$. Consider a semi-infinite geodesic starting from 0 (*resp.* s_1') and define by r_0 (*resp.* r_1) the last point of this semi-infinite geodesic which belongs to Λ. Denote

$$L = d(r_0, 0) - d(r_1, s_1').$$

For simplicity, we will suppose, without loss of generality, that L is non-negative. Remember that s_1' has the same parity than s_1. Let us define

$$\begin{cases} b_2' = b_2 \text{ and } b_1' = b_1 & \text{if } \|s_1\|_1 \text{ does not have the same parity than } L(C_1 - C_2),\\ b_1' = b_2' = 0 & \text{otherwise.}\end{cases}$$

$$\text{and } l = L(C_1 - C_2) + b_1' - b_2'.$$

Note that $b_1 - b_2$ is odd, unless S only contains odd integers. But in that case, $C_1 - C_2$ is odd and L has the same parity than $\|r_0\|_1 + \|r_1\|_1 + \|s_1'\|_1$, that is the same parity than $\|s_1\|_1$.

Thus, l and $\|s_1\|_1$ always have the same parity. Note that

$$\begin{aligned} |l| &\leq |L||C_1 - C_2| + B \\ &\leq \max(d(0, r_0), d(s'_1, r_1)).\max(C_1, C_2) + B \\ &\leq K_1.\max(\|r_0\|_1, \|r_1\|_1).C + B = B + CK_1R. \end{aligned}$$

So, by lemma 5.4, and by the choice we made for R, one can construct inside Λ two simple paths with no common point γ_0 from 0 to r_0, and γ_1 from s_1 to r_1 such that

$$|\gamma_0| \geq CK_1R + B, \ \ |\gamma_1| \geq CK_1R + B, \ \ |\gamma_0| - |\gamma_1| = l.$$

Let us note $k = |\gamma_0| - (LC_1 + b'_1)$. As proved in the upper bound for $|l|$, $LC_1 \leq CK_1R$. We thus have

$$k \geq CK_1R + B - (LC_1 + b'_1) \geq B - b_1 \geq 0.$$

Obviously, $|\gamma_0| = LC_1 + b'_1 + k$ and $|\gamma_1| = LC_2 + b'_2 + k$. Define also the following quantity M that will played the role of an "infinite" passage time for open edges:

$$\begin{aligned} M &= \max\left\{L\sum_{i\in S_+} a_i s_i + b'_1 b_2 + k b_1, L\sum_{i\in S_-}(-a_i)s_i + (b'_2 + k)b_1\right\} \\ &\quad + \max\left\{d(x,y)(0_\Lambda\omega_{\Lambda^c},\eta), \|x\|_1 = \|y\|_1 = R\right\}. \end{aligned}$$

Note that M is in $\mathcal{F}_{\Lambda^c}$, the σ-algebra generated by $\{(\omega_e, \eta_e), e \in \Lambda^c\}$. Now define, for every $(\omega_e, \eta_e) \in A$, the configuration $(\omega', \eta') \in \Omega$: set $(\omega'_e, \eta'_e) = (\omega_e, \eta_e)$ for $e \in \mathbb{E}^d\backslash\Lambda$ and define $(\omega'_\Lambda, \eta'_\Lambda)$ inside Λ as follows:

i. If $p < 1$, $\forall e \in \Lambda\backslash(\gamma_0 \cup \gamma_1), \omega'_e = 0$ and $\eta'_e = b_1$, but this value does not play a special role;
 if $p = 1$, $\forall e \in \Lambda\backslash(\gamma_0 \cup \gamma_1), \eta'_e > M$, and $\omega'_e = 1$.
ii. $\forall e \in \gamma_0 \cup \gamma_1, \omega'_e = 1$.
iii. Assign a passage time to edges in γ_0 as follows (remember that $|\gamma_0| = LC_1 + b'_1 + k$): first, for each $i \in S_+$, give to a_iL edges the value $\eta'_e = s_i$ and next complete giving to b'_1 other edges the value $\eta'_e = b_2$ and to k other edges the value $\eta'_e = b_1$.
iv. Assign a passage time to edges in γ_1 as follows (remember that $|\gamma_1| = LC_2 + b'_2 + k$): first, for each $i \in S_-$, give to $-a_iL$ edges the value $\eta'_e = s_i$ and next complete giving to the remaining $b'_2 + k$ edges the value $\eta'_e = b_1$.

Now we immediately obtain:

$$\begin{aligned} \sum_{e\in\gamma_0} \eta'_e &= L\sum_{i\in S_+} a_i s_i + b'_1 b_2 + k b_1, \\ \sum_{e\in\gamma_1} \eta'_e &= L\sum_{i\in S_-}(-a_i)s_i + (b'_2 + k)b_1, \\ \sum_{e\in\gamma_0} \eta'_e - \sum_{e\in\gamma_1} \eta'_e &= L\big(\sum_{i\in S} a_i s_i\big) + b'_1 b_2 - b'_2 b_1 = L = d(0, r_0) - d(s'_1, r_1). \end{aligned}$$

For $(\omega, \eta) \in A \subset \mathrm{Coex}(0, s'_1)$, there exist two infinite geodesics starting from 0 and s'_1. Let us denote by Γ_0 (*resp.* Γ_1) the part beginning at r_0 (*resp.* r_1) in the geodesic starting from 0 (*resp.* s'_1) in the configuration (ω, η).

We are going to prove that $\gamma_0 \cup \Gamma_0$ (*resp.* $\gamma_1 \cup \Gamma_1$) is an infinite geodesic starting from 0 (*resp.* s_1) in the configuration (ω', η'). Let x be a point of Γ_0. Let us prove that an optimal path from 0 to x in the configuration (ω', η') is included in $\gamma_0 \cup \Gamma_0$.

Let γ be an optimal path from 0 to x in the configuration (ω', η'), and denote by z the point from which the path γ exits from Λ. We have

$$d(0,z) = \sum_{e \in \gamma} \eta'(e) \leq \sum_{e \in \gamma_0} \eta'(e) + d(r_0, z).$$

But since $\sum_{e \in \gamma_0} \eta'(e) = L \sum_{i \in S_+} a_i s_i + b_1' b_2 + k b_1$ and $d(r_0,z)(\omega', \eta') \leq d(r_0,z)(0_\Lambda \omega_{\Lambda^c}, \eta)$, it follows that $d(0,z) \leq M$. By definition of M, it ensures that γ does not use any bond in Λ, except those used in $\gamma_0 \cup \gamma_1$, and particularly, it implies that $z = r_0$, and thus an optimal path from 0 to x is included in $\gamma_0 \cup \Gamma_0$.

Similarly, let γ be an optimal path from s_1 to x in the configuration (ω', η'), and denote by z the point from which the path γ exits from Λ. We have

$$d(0,z) = \sum_{e \in \gamma} \eta'(e) \leq \sum_{e \in \gamma_1} \eta'(e) + d(r_1, z).$$

But since $\sum_{e \in \gamma_1} \eta'(e) = L \sum_{i \in S_-} (-a_i) s_i + (b_2' + k) b_1$ and $d(r_1,z)(\omega', \eta') \leq d(r_1,z)(0_\Lambda \omega_{\Lambda^c}, \eta)$, it follows that $d(s_1,z) \leq M$. By definition of M, it ensures that γ do not use any bond in Λ, except those used in $\gamma_0 \cup \gamma_1$, and particularly, it implies that $z = r_1$, and thus an optimal path from s_1 to x uses γ_1 to exit from Λ.

Let us now prove that if $x \in \Gamma_0$, $d(0,x)(\omega', \eta') < d(s_1,x)(\omega', \eta')$.

$$\begin{aligned}
d(s_1,x)(\omega', \eta') &= d(s_1, r_1)(\omega', \eta') + d(r_1, x)(0_\Lambda \omega_{\Lambda^c}, \eta'), \\
&= d(s_1, r_1)(\omega', \eta') + d(r_1, x)(0_\Lambda \omega_{\Lambda^c}, \eta), \\
d(0,x)(\omega', \eta') &= d(0, r_0)(\omega', \eta') + d(r_0, x)(0_\Lambda \omega_{\Lambda^c}, \eta'), \\
&= d(0, r_0)(\omega', \eta') + d(r_0, x)(0_\Lambda \omega_{\Lambda^c}, \eta).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& (d(s_1,x) - d(0,x))(\omega', \eta') \\
= \; & (d(s_1,r_1) - d(0,r_0))(\omega', \eta') + d(r_1,x)(0_\Lambda \omega_{\Lambda^c}, \eta) - d(r_0,x)(0_\Lambda \omega_{\Lambda^c}, \eta), \\
= \; & (d(s_1',r_1) - d(0,r_0))(\omega, \eta) + d(r_1,x)(0_\Lambda \omega_{\Lambda^c}, \eta) - d(r_0,x)(0_\Lambda \omega_{\Lambda^c}, \eta), \\
\geq \; & d(s_1',x)(\omega, \eta) - d(0,x)(\omega, \eta) > 0,
\end{aligned}$$

because $x \in \Gamma_0$, which is a part of the infinite geodesic issued from 0 in the configuration (ω, η). Thus $\gamma_0 \cup \Gamma_0$ is an infinite geodesic issued from 0 in the configuration (ω', η').

In the same manner, working symmetrically with Γ_1, we prove that $\gamma_1 \cup \Gamma_1$ is an infinite geodesic issued from s_1 in the configuration (ω', η'), and finally $(\omega', \eta') \in \mathrm{Coex}(0, s_1)$. $\square$

Proof. (Geodesics result). The goal here is to prove that

$$\mathbb{P}\left(\begin{array}{c} \text{there exists two distinct semi-infinite geodesics} \\ \text{starting from 0 for the random distance } d \end{array} \right) > 0.$$

The proof is exactly the same as the previous one. The only difference is to use the single source 0 rather than two distinct sources $0, s_1'$. The geometrical structure of the modification is once again given by lemma 5.4, and the adjustment of the values is made as before. $\square$

As seen previously, a trouble with integer passage times is that some points can be reached at the very same moment by the two distinct infections. This case can be ruled out under some extra assumptions, and this is the goal of the next result. But first, for two distinct sources $x, y \in \mathbb{Z}^d$, we say that the event Sep-Coex(x, y) happens if

$$\begin{cases} & \{z \in \mathbb{Z}^d, d(x,z) < d(y,z)\} \text{ is infinite} \\ \text{and} & \{z \in \mathbb{Z}^d, d(x,z) > d(y,z)\} \text{ is infinite} \\ \text{and} & \forall z \in \mathbb{Z}^d,\ d(x,z) \neq d(y,z) \text{ unless } d(x,z) = d(y,z) = +\infty. \end{cases}$$

We have the following result:

Lemma 5.5. *Denote by $\mathcal{O}$ the set of non-negative odd integers, and as previously, let $d \geq 2$, $p > p_c(d)$, $\mathbb{S}_\nu$ a stationary ergodic probability measure on $\mathcal{O}^{\mathbb{E}^d}$ satisfying (1) and (2). Then, for $x \in \mathbb{Z}^d$ with $\|x\|_1$ odd,*

$$\mathbb{P}(Coex(0,x) \backslash Sep\text{-}Coex(0,x)) = 0.$$

Proof. By the assumption we made on μ, the length of a path from x to y has the same parity than $\|x-y\|_1$. So, the identity $d(x,z) = d(y,z)$ can only happens if $\|x-y\|_1$ is even. □

Now, for a given point x with $\|x\|_1$ odd, the fact that $\mathbb{P}(\text{Sep-Coex}(0,x)) > 0$ can be obtained as a consequence of Theorem 3.2 or Theorem 5.1. Note also that when the assumptions of lemma 5.5 are fulfilled, μ is always a norm: since the passage time of a bond is an odd integer, it is at least equal to 1. Then, it is easy to see that for each $x \in \mathbb{Z}^d$, we have $\mu(x) \geq \|x\|_1$.

6. An example of a discrete time competing process

The last section is devoted to the study of a natural example of a non-trivial dynamical system which can be studied with the help of Theorem 5.1 and lemma 5.5.

Consider two species, say blue and yellow, which attempt to conquer the space $\mathbb{Z}^d$. At each instant, each fertile cell tries to contaminate each of its non-occupied neighbors. It succeeds with probability p. In case of success, the non-occupied cell takes the color of the infector. If a yellow cell and blue cell simultaneously succeed in contaminating a given cell, this one takes the green color. If a green cell and another cell simultaneously succeed in contaminating a given cell, this one takes the green color. At the next step, the individuals that have just been generated are fertile, but the previous generation is no more fertile. We make the following assumptions:

- the success of each attempt of contamination at a given time does not depend on the past,
- the successes of simultaneous attempts to contamination are independent.

The first assumption allow a modelization by an homogeneous Markov chain. Markov chains satisfying to the second condition are sometimes called Probabilistic Cellular Automata (PCA).

Let us define

$$S = \{0, \text{blue}, \text{yellow}, \text{green}, \text{blue}^*, \text{yellow}^*, \text{green}^*\},$$

where 0 is the state of an empty cell, blue, yellow, green the states of fertile cells, and blue*, yellow*, green* the states of unfertile cells.

Since we will study the evolution of a system which starts with only two cells, we will only deal with configurations in which a finite numbers of cells are nonempty. So, we will deal with a classical Markov chain on the denumerable set

$$C = \{\xi \in S^{\mathbb{Z}^d}; \quad \exists \Lambda \text{ finite}, \quad \xi_k = 0 \text{ for } k \in \mathbb{Z}^d \backslash \Lambda\}.$$

We now define for color $\in$ Act $= \{\text{blue}, \text{yellow}, \text{green}\}$:

$$n(\text{color}, x)(\xi) = |\{y \in \mathbb{Z}^d; \|x - y\|_1 = 1 \text{ and } \xi_y = \text{color}\}|,$$

and

$$s(\text{color}, x) = 1 - (1-p)^{n(\text{color},x)},$$

which represents the probability that at least one neighbor of x succeeds in infecting x with the given color.

The considered dynamics form an homogeneous PCA with space state S and whose local evolution rules are given by

$$p_x(s,t) = \begin{cases} (1-p)^{n(\text{yellow},x)+n(\text{blue},x)+n(\text{green},x)} & \text{if } s = t = 0 \\ s(\text{blue}, x)(1-p)^{n(\text{yellow},x)+n(\text{green},x)} & \text{if } s = 0 \text{ and } t = \text{blue} \\ s(\text{yellow}, x)(1-p)^{n(\text{blue},x)+n(\text{green},x)} & \text{if } s = 0 \text{ and } t = \text{yellow} \\ s(\text{green}, x) + (1 - s(\text{green}, x))s(\text{blue}, x)s(\text{yellow}, x) & \text{if } s = 0 \text{ and } t = \text{green} \\ 1 & \text{if } s \in \{\text{blue}, \text{yellow}, \text{green}\} \text{ and } t = s^* \\ 1 & \text{if } s \in \{\text{blue}^*, \text{yellow}^*, \text{green}^*\} \text{ and } t = s \\ 0 & \text{otherwise.} \end{cases}$$

In term of Markov chains, it means that the transition matrix is defined by

$$\forall (\xi, \omega) \in C \times C \quad p(\xi, \omega) = \prod_{k \in \mathbb{Z}^d} p_k(\xi_k, \omega_k).$$

The product is convergent because only a finite numbers of terms differs from 1.

With the help of the tools that we have developed above, we will prove the following theorem:

Theorem 6.1. *Let $p > p_c$. For $s_{yellow}, s_{blue} \in \mathbb{Z}^d$ with $s_{blue} \neq s_{yellow}$, let us denote by $\mathbb{P}_{p,s_{yellow},s_{blue}}$ the law of a PCA $(X_n)_{n \geq 0}$ following the dynamics described above, and starting a configuration with exactly two non-empty cells: a blue cell at site s_{blue}, a yellow cell at site s_{yellow}, the others cells being empty. Then,*

$$\mathbb{P}_{p,s_{yellow},s_{blue}}(\forall n \in \mathbb{Z}_+ \; \exists (x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d, \; X_n(x) = \textit{blue and } X_n(y) = \textit{yellow}) > 0.$$

If moreover $\|s_{yellow} - s_{blue}\|_1$ is odd, green cells never appear.

The following lemma gives the link between this PCA and our competing model.

Lemma 6.2. *Consider a probability space where lives a family $(\omega_e)_{e \in \mathbb{E}^d}$ of independent Bernoulli variables with parameter p, which defines a random chemical distance D.*

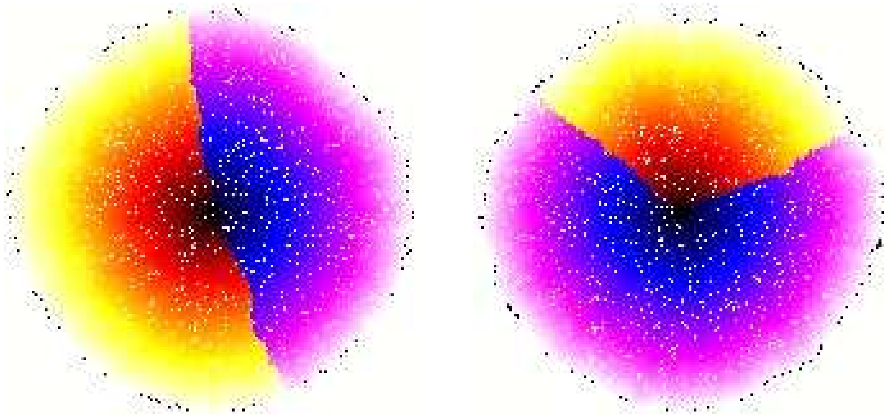

FIGURE 3. Two samples of simulation of the competing process when $p = 0.6$. The process is stopped when the border of a given box is attained by one of the two species. The color in the picture is determined by the time of coloring and the type of the cell.

Let $s_{yellow}, s_{blue} \in \mathbb{Z}^d$ with $s_{blue} \neq s_{yellow}$. Define

$$X_n(x) \quad = \quad \begin{cases} \textit{blue} & \textit{if } n = D(s_{blue}, x) < D(s_{yellow}, x) \\ \textit{yellow} & \textit{if } n = D(s_{yellow}, x) < D(s_{blue}, x) \\ \textit{green} & \textit{if } n = D(s_{yellow}, x) = D(s_{blue}, x) \\ \textit{blue}^* & \textit{if } D(s_{blue}, x) < \min(D(s_{yellow}, x), n) \\ \textit{yellow}^* & \textit{if } D(s_{yellow}, x) < \min(D(s_{blue}, x), n) \\ \textit{green}^* & \textit{if } D(s_{yellow}, x) = D(s_{blue}, x) < n \\ 0 & \textit{otherwise.} \end{cases}$$

Then, $(X_n)_{n\geq 0}$ is an homogeneous PCA with space state

$$S = \{0, \textit{blue}, \textit{yellow}, \textit{green}, \textit{blue}^*, \textit{yellow}^*, \textit{green}^*\}$$

associated to the probabilities $p_x(s,t)$ defined above.

Proof. Let us consider the map

$$\begin{array}{rcl} f : S^{\mathbb{Z}^d} \times \Omega_E & \to & S^{\mathbb{Z}^d} \\ (\xi, \omega) & \mapsto & (f_x(\xi_x, \omega))_{x\in\mathbb{Z}^d} \end{array}$$

where $f_x : S \times \Omega_E \to S$ is defined by

$$\begin{cases} f_x(s,\omega) & = & s^* \text{ for each } s \in \{\text{blue}, \text{yellow}, \text{green}\} \\ f_x(s,\omega) & = & s \text{ for each } s \in \{\text{blue}^*, \text{yellow}^*, \text{green}^*\} \\ f_x(0,t) & = & \begin{cases} \text{blue} & \text{if } \mathrm{Act} \cap \{\xi_y ; \|x-y\|_1 = 1 \text{ and } \omega_{\{x,y\}} = 1\} = \{\text{blue}\} \\ \text{yellow} & \text{if } \mathrm{Act} \cap \{\xi_y ; \|x-y\|_1 = 1 \text{ and } \omega_{\{x,y\}} = 1\} = \{\text{yellow}\} \\ \text{green} & \text{if } \mathrm{Act} \cap \{\xi_y ; \|x-y\|_1 = 1 \text{ and } \omega_{\{x,y\}} = 1\} = \{\text{blue}, \text{yellow}\} \\ \text{green} & \text{if } \mathrm{Act} \cap \{\xi_y ; \|x-y\|_1 = 1 \text{ and } \omega_{\{x,y\}} = 1\} \supset \{\text{green}\} \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

By considering Dijkstra's algorithm in the particular case where the travel times are constant, it is not difficult to see that $(X_n)_{n\ge 0}$ satisfy to the recurrence formula $X_{n+1} = f(X_n, \omega)$. To recognize $(X_n)_{n\ge 0}$ as a convenient PCA, we will build a coupling of ω with a i.i.d. sequence $(\omega^n)_{n\ge 1}$ to obtain the canonical Markov Chain representation $X_{n+1} = f(X_n, \omega^n)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\zeta^0, \omega^0, \omega^1, \omega^2, \dots$ independent $\{0,1\}^{\mathbb{E}^d}$ valued variables with $\mathrm{Ber}(p)^{\otimes \mathbb{E}^d}$ as common law.

We define $A_0 = \{s_{\text{blue}}, s_{\text{yellow}}\}$ and recursively

$$\begin{cases} B_{n+1} = & \{y \in \mathbb{Z}^d\backslash A_n \quad \exists x \in \partial A_n : \|x-y\|_1 = 1 \text{ and } \omega^n_{\{x,y\}} = 1\} \\ A_{n+1} = & A_n \cup B_{n+1}. \end{cases}$$

Note that the random set B_{n+1} is measurable with respect to the σ-algebra generated by $(\omega^0, \omega^1, \dots, \omega^n)$. We define ζ^n recursively by

$$\zeta_e^{n+1} = \quad \begin{cases} \omega_e^{n+1} & \text{if } e = \{x,y\} \text{ with } (x,y) \in \partial A_n \times \mathbb{Z}^d\backslash A_n \\ \zeta_e^n & \text{otherwise.} \end{cases}$$

By natural induction, we prove that the law of ζ^n under P is $\mathrm{Ber}(p)^{\otimes \mathbb{E}^d}$. By construction, each bond e writes $e = \{x,y\}$ with $(x,y) \in \partial A_n \times \mathbb{Z}^d\backslash A_n$ for at most one value of n. It follows that the sequence ζ^n converges in the product topology. Let us denote by ω^∞ its limit. Since the law of ζ^n under P is $\mathrm{Ber}(p)^{\otimes \mathbb{E}^d}$, it follows that the law of ω^∞ under P is also $\mathrm{Ber}(p)^{\otimes \mathbb{E}^d}$.

Now, it is not difficult to see that sequence $(X_n)_{n\ge 0}$ defined from ω^∞ as previously, satisfies to the recurrence formula $X_{n+1} = f(X_n, \omega^\infty)$, but also to $X_{n+1} = f(X_n, \omega^n)$.

It is now proved that $(X_n)_{n\ge 0}$ is an homogeneous Markov chain. The recognition of the transition matrix follows from an elementary calculus. □

We can now prove the theorem announced above.

Proof. Clearly, lemma 6.2 connects the considered PCA with the random distance studied in Theorem 5.1. Here, the passage times of open bonds are identically equal to 1, which is obviously an odd number. By lemma 5.5, this prevents from the appearance of green cells when $\|s_{\text{yellow}} - s_{\text{blue}}\|_1$ is odd. □

Remarks. If $\|s_{\text{yellow}} - s_{\text{blue}}\|_1 \neq 0$ is even and if the two species infinitely grow, there are necessarily green cells at the boundary between blue cells and yellow cells.

A natural question is the following: is it possible to have an infinite set of green cells surrounding the blue cells and the yellow cells ? The answer is yes, as soon as $\|s_{\text{yellow}} - s_{\text{blue}}\|_1 \neq 0$ is even: consider figure 6.

The picture describes a particular case when $d = 2$, but the reasoning can obviously be generalized.

In this case, the yellow flow and the blue flow immediately converge to engender a green flow. They also do not develop themselves elsewhere. If the point labelled 0 belongs to the infinite cluster, then the result is proved.

It is now to see that, conditionally to the states of the bonds imposed by this picture, the probability that 0 belongs to the infinite cluster is strictly positive, which follows from a classical modification argument.

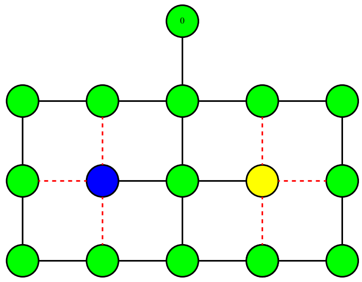

FIGURE 4. Green surrounding the two sources

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