Fractional supersymmetric Quantum Mechanics as a set of replicas of ordinary supersymmetric Quantum Mechanics[1]

M. Daoud[2], M. Kibler[3]

Institut de Physique Nucléaire de Lyon,
IN2P3-CNRS et Université Claude Bernard,
43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

Abstract

A connection between fractional supersymmetric quantum mechanics and ordinary supersymmetric quantum mechanics is established in this Letter.

0. Although *ordinary* supersymmetric Quantum Mechanics (sQM) was introduced more than 20 years ago, its extension as *fractional* sQM is still the object of numerous works. The parentage between ordinary sQM and fractional sQM needs to be clarified. In particular, we may ask the question: Can fractional sQM be reduced to ordinary sQM as far as spectral analyses are concerned? It is the aim of this work to study a connection between fractional sQM of order k and ordinary sQM corresponding to $k = 2$. We consider here the case where the number of supercharges is equal to 1 (corresponding to 2 supercharges related via Hermitean conjugation).

1. Our definition of *fractional* sQM of order k, with $k \in \mathbf{N} \setminus \{0, 1\}$, is as follows. Following Refs. [1-4], a doublet of linear operators $(H, Q)_k$, with H a self-adjoint operator and Q a supersymmetry operator, acting on a separable Hilbert space and satisfying the relations

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow \; Q_-^\dagger = Q_+), \quad Q_\pm^k = 0 \tag{1a}$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \cdots + Q_+Q_-^{k-1} = Q_-^{k-2}H \tag{1b}$$

$$[H, Q_\pm] = 0 \tag{1c}$$

is said to define a k-fractional supersymmetric quantum-mechanical system (see also Refs. [5-8]). The operator H is the Hamiltonian of the system spanned by the two (dependent) supercharge operators Q_- and Q_+. In the special case $k = 2$, the system described by a doublet of type $(H, Q)_2$ is referred to as an ordinary supersymmetric

[1]Accepted for publication in Physics Letters A.

[2]Permanent address: Laboratoire de Physique de la Matière Condensée, Faculté des Sciences, Université Ibn Zohr, BP 28/S, Agadir, Morocco.

[3]Correspondence to: M. Kibler; e-mail: m.kibler@ipnl.in2p3.fr.

quantum-mechanical system ; it corresponds to a Z_2-grading with fermionic and bosonic states.

2. We now introduce a generalized Weyl-Heisenberg algebra W_k, with $k \in \mathbf{N} \setminus \{0, 1\}$, from which we can construct k-fractional supersymmetric quantum-mechanical systems. The algebra W_k is spanned by four linear operators, *viz.*, X_- (annihilation operator), X_+ (creation operator), N (number operator) and K (Z_k-grading operator). The operators X_- and X_+ are connected via Hermitean conjugation; N is a self-adjoint operator and K is a unitary operator. The four operators satisfy the relationships

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N)\, \Pi_s, \ [N, X_\pm] = \pm X_\pm, \ [K, X_\pm]_{q^{\pm 1}} = 0, \ [K, N] = 0, \ K^k = 1 \tag{2}$$

Here, the functions $f_s : N \mapsto f_s(N)$ are arbitrary functions subjected to the constraints $f_s(N)^\dagger = f_s(N)$. Furthemore, the Hermitean operators Π_s are defined by

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st}\, K^t$$

where

$$q = \exp\left(\frac{2\pi \mathrm{i}}{k}\right)$$

is a root of unity, so that they are projection operators for the cyclic group C_k. Finally, $[K, X_\pm]_{q^{\pm 1}}$ stands for the deformed commutator $K X_\pm - q^{\pm 1} X_\pm K$.

3. The operators X_-, X_+ and K can be realized in terms of k pairs $(b(s)_-, b(s)_+)$ of deformed bosons with

$$[b(s)_-, b(s)_+] = f_s(N)$$

and one pair (f_-, f_+) of k-fermions with

$$[f_-, f_+]_q = 1, \quad f_\pm^k = 0$$

The f's commute with the b's. Of course, we have $b(s)_+ = b(s)_-^\dagger$ but $f_+ \neq f_-^\dagger$ except for $k = 2$. The k-fermions introduced in [9] and recently discussed in [10] are objects interpolating between fermions and bosons (the case $k = 2$ corresponds to ordinary fermions and the case $k \to \infty$ to ordinary bosons); the k-fermions also share some features of the anyons introduced in [11,12]. For k arbitrary in $\mathbf{N} \setminus \{0, 1\}$, the realization

$$K = [f_-, f_+]$$

$$X_- = \left(f_- + \frac{f_+^{k-1}}{[k-1]_q!}\right) \sum_{s=0}^{k-1} b(s)_-\, \Pi_s$$

$$X_+ = \left(f_- + \frac{f_+^{k-1}}{[k-1]_q!}\right)^{k-1} \sum_{s=0}^{k-1} b(s)_+\, \Pi_s$$

has been discussed in Ref. [8]. Here, we have $[n]_q! = [1]_q \, [2]_q \, \cdots \, [n]_q$ (with $[0]_q! = 1$) and the symbol $[\ ]_q$ is defined by

$$[n]_q = \frac{1 - q^n}{1 - q}$$

where $n \in \mathbf{N}$.

4. An Hilbertean representation of W_k can be constructed in the following way. Let $\mathcal{F}$ be the Hilbert-Fock space on which the generators X_-, X_+, N and K act. Since K is a cyclic operator of order k, the space $\mathcal{F}$ can be graded as

$$\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s$$

where the subspace $\mathcal{F}_s = \{|n, s\rangle : n = 1, 2, \cdots, d\}$ is a d-dimensional space (d can be finite or infinite). The representation is given by

$$K|n, s\rangle = q^s |n, s\rangle, \quad N|n, s\rangle = n|n, s\rangle$$

$$X_-|n, s\rangle = \sqrt{F_s(n)} \begin{cases} |n-1, s-1\rangle \text{ if } s \neq 0 \\ \\ |n-1, k-1\rangle \text{ if } s = 0 \end{cases}$$

$$X_+|n, s\rangle = \sqrt{F_{s+1}(n+1)} \begin{cases} |n+1, s+1\rangle \text{ if } s \neq k-1 \\ \\ |n+1, 0\rangle \text{ if } s = k-1 \end{cases}$$

where the function F is a structure function such that

$$F_{s+1}(n+1) - F_s(n) = f_s(n) \tag{3}$$

with $F_s(0) = 0$.

5. We are now in a position to associate a k-fractional supersymmetric quantum-mechanical system to the algebra W_k characterized by a given set of functions $\{f_s : s = 0, 1, \cdots, k-1\}$. We define the supercharge Q via

$$Q \equiv Q_- = X_-(1 - \Pi_1) \Leftrightarrow Q^\dagger \equiv Q_+ = X_+(1 - \Pi_0) \tag{4}$$

There are k equivalent definitions of Q corresponding to the k circular permutations of $1, 2, \cdots, k-1$; our choice, which is such that $Q|n, 1\rangle = 0$, is adapted to the sequence $H_k, H_{k-1}, \cdots, H_1$ to be considered below. By making reapeated use of Eqs. (1), (2) and (4), we can derive the operator

$$H = (k-1) X_+ X_- - \sum_{s=3}^{k} \sum_{t=2}^{s-1} (t-1) \, f_t(N-s+t) \, \Pi_s - \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k) \, f_t(N-s+t) \, \Pi_s \tag{5}$$

which is self-adjoint and commutes with Q_- and Q_+. (Equation (5) and some other relations below include Π_k. Indeed, in view of the cyclic character of K, we have

$\Pi_k = \Pi_0$ so that the action of terms involving Π_k is quite well-defined on the space $\mathcal{F}$.) As a result, the doublet $(H, Q)_k$ associated to W_k satisfies Eq. (1) and thus defines a k-fractional supersymmetric quantum-mechanical system.

6. In order to establish a connection between *fractional* sQM (of order k) and *ordinary* sQM (of order $k = 2$), it is necessary to construct subsystems from the doublet $(H, Q)_k$ that correspond to ordinary supersymmetric quantum-mechanical systems. This may be achieved in the following way. Equation (5) can be rewritten as

$$H = \sum_{s=1}^{k} H_s \, \Pi_s \tag{6}$$

where

$$H_s \equiv H_s(N) = (k-1)F(N) - \sum_{t=2}^{k-1} (t-1)\, f_t(N-s+t) + (k-1) \sum_{t=s}^{k-1} f_t(N-s+t) \tag{7}$$

It can be shown that the operators $H_k \equiv H_0, H_{k-1}, \cdots, H_1$ turn out to be isospectral operators. By introducing

$$X(s)_- = \sum_n [H_s(n)]^{\frac{1}{2}} |n-1, s-1\rangle\langle n, s|$$

$$X(s)_+ = \sum_n [H_s(n+1)]^{\frac{1}{2}} |n+1, s\rangle\langle n, s-1|$$

it is possible to factorize H_s as

$$H_s = X(s)_+ \, X(s)_-$$

modulo the omission of the ground state $|0, s\rangle$ (which amounts to substract the corresponding eigenvalue from the spectrum of H_s). Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \, \Pi_s, \quad q(s)_+ = X(s)_+ \, \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- \, X(s)_+ \, \Pi_{s-1} + X(s)_+ \, X(s)_- \, \Pi_s \tag{8}$$

It is then a simple matter of calculation to prove that $h(s)$ is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = q(s)_- q(s)_+ + q(s)_+ q(s)_-, \quad [h(s), q(s)_\pm] = 0$$

Consequently, the doublet $(h(s), q(s))_2$, with $q(s) \equiv q(s)_-$, satisfies Eq. (1) with $k = 2$ and thus defines an ordinary sypersymmetric quantum-mechanical system (corresponding to $k = 2$).

7. The Hamiltonian $h(s)$ is amenable to a form more appropriate for discussing the link between ordinary sQM and fractional sQM. Indeed, we can show that

$$X(s)_- \, X(s)_+ = H_s(N+1) \tag{9}$$

Then, by combining Eqs. (2), (3), (7) and (9), Eq. (8) leads to the important relation

$$h(s) = H_{s-1} \, \Pi_{s-1} + H_s \, \Pi_s \tag{10}$$

to be compared with the expansion of H in terms of supersymmetric partners H_s (see Eq. (6)).

8. To close this Letter, let us sum up the obtained results and offer some conclusions.

Starting from a Z_k-graded algebra W_k, characacterized by a set $\{f_s : s = 0, 1, \cdots, k-1\}$, it was shown how to associate a k-fractional supersymmetric quantum-mechanical system $(H, Q)_k$ characterized by an Hamiltonian H and a supercharge Q.

The extended Weyl-Heisenberg algebra W_k covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given set $\{f_s : s = 0, 1, \cdots, k-1\}$ yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

$$\forall s \in \{0, 1, \cdots, k-1\} \ : \ f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [13]) and the case

$$\forall s \in \{0, 1, \cdots, k-1\} \ : \ f_s(N) = aN + b \text{ where } (a, b) \in \mathbf{R}^2$$

to systems with translational shape-invariant potentials (in the sense of Ref. [14]). For instance, the case $(a = 0, b > 0)$ corresponds to the harmonic oscillator potential, the case $(a < 0, b > 0)$ to the Morse potential and the case $(a > 0, b > 0)$ to the Pöshl-Teller potential. For these various potentials, the part of W_k spanned by X_-, X_+ and N can be identified with the ordinary Weyl-Heisenberg algebra for $(a = 0, b \neq 0)$, with the su(1,1) Lie algebra for $(a > 0, b > 0)$ and with the su(2) Lie algebra for $(a < 0, b > 0)$. These matters shall be the subject of a forthcoming paper.

The Hamiltonian H for the system $(H, Q)_k$ was developed as a superposition of k isospectral supersymmetric partners $H_0, H_1, \cdots, H_{k-1}$.

The system $(H, Q)_k$ itself, corresponding to k-fractional sQM, was expressed in terms of $k-1$ sub-systems $(h(s), q(s))_2$, corresponding to ordinary sQM. The Hamiltonian $h(s)$ is given as a sum involving the supersymmetric partners H_{s-1} and H_s (see Eq. (10)). Since the supercharges $q(s)_\pm$ commute with the Hamiltonian $h(s)$, it follows that

$$H_{s-1} X(s)_- = X(s)_- H_s, \quad H_s X(s)_+ = X(s)_+ H_{s-1} \tag{11}$$

As a consequence, the operator $X(s)_+$ (respect. $X(s)_-$) makes it possible to pass from the spectrum of H_{s-1} (respect. H_s) to the one of H_s (respect. H_{s-1}). This result is quite familiar for ordinary sQM (corresponding to $s = 2$). Note that Eq. (11) is reminiscent of the intertwining method based on the Darboux transformation and on the factorization method which are useful for studying superintegrability of quantum systems.

For $k = 2$, the operator $h(1)$ is nothing but the total Hamiltonian H corresponding to ordinary sQM. For arbitrary k, the other operators $h(s)$ are simple replicas (except for the ground state of $h(s)$) of $h(1)$. It is in this sense that k-fractional sQM can be considered as a set of $k-1$ replicas of ordinary sQM typically described by $(h(s), q(s)_\pm)_2$. Along this vein, it is to be emphasized that

$$H = q(2)_-\, q(2)_+ + \sum_{s=2}^{k} q(s)_+\, q(s)_-$$

which can be identified to $h(2)$ for $k = 2$.

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