Stochastic differential equations with non-Lipschitz coefficients: I. Pathwise uniqueness and Large deviations

Shizan FANG and Tusheng ZHANG

I.M.B, UFR Sciences et techniques, Université de Bourgogne, 9 avenue Alain Savary, B.P. 47870, 21078 Dijon, France.
Department of Mathematics, University of Manchester, Oxford road, Manchester, M13 9PL, England.

Abstract

We study a class of stochastic differential equations with non-Lipschitzian coefficients. A unique strong solution is obtained and a large deviation principle of Freidlin-Wentzell type has been established.

1 Introduction

Let $\sigma: R^d \to R^d \otimes R^m$ and $b: R^d \to R^d$ be continuous functions. It is well-known that the following Itô s.d.e:

$$dX(t) = \sigma(X(t))\, dW_t + b(X(t))\, dt, \quad X(0) = x_o \tag{1}$$

has a weak solution up to a lifetime ζ (see [SV], [IW, p.155-163]), where $t \to W_t$ is a R^m-valued standard Brownian motion. It is also known that under the assumption of linear growth of coefficients σ and b, the lifetime $\zeta = +\infty$ almost surely. If the s.d.e (1) has the pathwise uniqueness, then it admits a strong solution (see [IW, p.149], [RY, p.341]). So the study of pathwise uniqueness is of great interest. It is a classical result that under the Lipschitz conditions, the pathwise uniqueness holds and the solution of s.d.e. (1) can be constructed using Picard interation; morever the solution depends on the initial values continuousely. The main tool to these studies is the Gronwall lemma. When the coefficients σ and b do not satisfy the Lipschitz conditions, the use of Gronwall lemma meets a serious difficulty. Therefore, there are very few results of pathwise uniqueness of solutions of s.d.e. beyond the Lipschitzian (or locally Lipschitzian) conditions in the literature except in the one dimensional case (see [IW, p.168], [RY, CH IX-3]). In the case of ordinary differential equations, the Gronwall lemma was generalized in order to establish the uniqueness result (see e.g. [La]). However the method is not applicable to s.d.e.. In this work, we shall deal with a class of non-Lipschitzian s.d.e.. Namely, we shall assume that

$$(H1)\quad \begin{cases} ||\sigma(x)-\sigma(y)||^2 & \leq \quad C\,|x-y|^2\,\log\frac{1}{|x-y|}, \qquad \text{for } |x-y|<1, \\ \quad |b(x)-b(y)| & \leq \quad C\,|x-y|\,\log\frac{1}{|x-y|}, \qquad \text{for } |x-y|<1 \end{cases}$$

where $|\cdot|$ denotes the Euclidean distance in R^d and $||\sigma||^2=\sum_{i=1}^d\sum_{j=1}^m\sigma_{ij}^2$.

We will prove the pathwise uniqueness of solutions under $(H.1)$ and non explosion of the solution under the growth condition $|x|\log|x|$. The results are valid for any dimension. Our idea is to construct a family of positive increasing functions $(\Phi_\rho)_{\rho>0}$ on R_+ so that the Gronwall lemma can be applied to the composition of the functions (Φ_ρ) with appropriate processes. This family of positive functions plays a crucial role. In this work, we will also establish a Freidlin-Wentzell type large deviation principle for the solutions of the s.d.e's (see [FW]). As a by-product, it is seen that the unique solution of the s.d.e. can be obtained by Euler approximation. Our strategy for the large deviation is to prove that the Euler approximations to the s.d.e. is exponentially fast. The method of estimating moments used in the literature ([DS],[DZ], [S]) wouldn't work here because of the non-Lipschitzian feature of the coefficients. We again appeal to a family of positive functions $(\Phi_{\rho,\lambda})_{\rho>0}$. The proof of the uniform convergence of solutions of the corresponding skeleton equations over compact level sets is also tricky due to the non-Lipschitzian feature.

The organization of the paper is as follows. In section 2, we shall discuss the case of ordinary differential equations; although the results about non explosion and uniqueness are not new, but our method can also be used to study the dependence of initial values and the non confluence of the equations. In section 3, we shall consider the s.d.e. The pathwise uniqueness and the criterion of non-explosion will be established. However, the supplementary difficulties will appear when we deal with the dependence with respect to initial values and the non confluence of s.d.e., that we shall study in a forthcoming paper. The ordinary differential equation

$$\frac{dX(t)}{dt}=b(X(t)),\quad X(0)=x_o \tag{2}$$

gives rise to a dynamical system on R^d. In section 4, we shall consider its small perturbation by a white noise. Namely, we shall consider the s.d.e

$$dX^\varepsilon(t)=\varepsilon^{\frac{1}{2}}\,\sigma(X^\varepsilon(t))\,dW_t+b(X^\varepsilon(t))\,dt,\quad X^\varepsilon(0)=x_o \tag{3}$$

and state a large deviation principle for $(X^\varepsilon(t))_{t\in[0,1]}$. Section 5 and 6 are devoted to the proof of the large deviation principle. Section 5 is for the case of bounded coefficients. Section 6 is for the general case.

2 Ordinary differential equations

Let $b : R^d \to R^d$ be a continuous function. It is essentially due to Ascoli-Arzela theorem that the differential equation (2) has a solution up to a lifetime ζ. The following result weakens the linear growth condition for non explosion.

Theorem 2.1 *Let $r : R_+ \to R_+$ be a continuous function such that (i) $\lim_{s\to+\infty} r(s) = +\infty$, (ii) $\int_0^\infty \frac{ds}{sr(s)+1} = +\infty$.*

Assume that it holds

$$|b(x)| \le C\ (|x|r(|x|^2) + 1). \tag{4}$$

Then the lifetime is infinte: $\zeta = +\infty$.

Proof. Define for $\xi \ge 0$,

$$\psi(\xi) = \int_0^\xi \frac{ds}{sr(s)+1} \quad \text{and } \Phi(\xi) = e^{\psi(\xi)}.$$

We have

$$\Phi'(\xi) = \frac{\Phi(\xi)}{\xi r(\xi)+1}. \tag{5}$$

Let $\xi_t = |X_t|^2$, where $X(t)$ is a solution to (2). Then

$$\frac{d}{dt}\Phi(\xi_t) = 2\Phi'(\xi_t)\langle X_t, b(X(t))\rangle, \tag{6}$$

where $\langle\ ,\ \rangle$ denotes the inner product in R^d. By assumption (4), we have

$$\left|\frac{d}{dt}\Phi(\xi_t)\right| \le 2C\Phi'(\xi_t)|X_t|\,(|X_t|r(\xi_t)+1). \tag{7}$$

By (i), it holds that

$$\sup_{s\ge 0}\frac{s^2r(s^2)+s}{s^2r(s^2)+1} < +\infty.$$

Therefore for some constant $C_2 > 0$,

$$\left|\frac{d}{dt}\Phi(\xi_t)\right| \le 2C_2\Phi(\xi_t). \tag{8}$$

It follows that for $t < \zeta$,

$$\Phi(\xi_t) \le \Phi(|x_o|^2) + 2C_2\int_0^t \Phi(\xi_s)\,ds.$$

By Gronwall lemma, we have

$$\Phi(\xi_t) \le \Phi(|x_o|^2)\, e^{2C_2\, t}. \tag{9}$$

If $\zeta < +\infty$, letting $t \uparrow \zeta$ in (9), we get $\Phi(\xi_\zeta) \le \Phi(|x_o|^2)\, e^{2C_2\zeta}$ which is impossible because of $\xi_\zeta = +\infty$, $\Phi(+\infty) = +\infty$.

Remark. By considering the inequality $\frac{d}{dt}\Phi(\xi_t) \ge -2C_2\, \Phi(\xi_t)$, we have

$$\Phi(\xi_t) \ge \Phi(|x_o|) - 2C_2 \int_0^t \Phi(\xi_s)\, ds,$$

which yields to

$$\Phi(\xi_t) \ge \Phi(|x_o|) e^{-2C_2 t}.$$

If we denote by $X_t(x_o)$ the solution to (2) with initial value x_o, then we get $\lim_{|x_o|\to+\infty} \Phi(|X_t(x_o)|) = +\infty$, which implies that

$$\lim_{|x_o|\to+\infty} |X_t(x_o)| = +\infty. \tag{10}$$

In what follows, to be simplified, we shall assume that the solutions of (2) have non explosion.

Theorem 2.2 *Let $r :]0,1[\to R_+$ be a continuous function such that*

(i) $\lim_{s\to 0} r(s) = +\infty$;

(ii) $\int_0^a \frac{ds}{sr(s)} = +\infty$ *for any* $a > 0$.

Assume that

$$|b(x) - b(y)| \le C\, |x - y|\; r(|x - y|^2) \quad \text{for } |x - y| < 1. \tag{11}$$

Then the differential equation (2) *has an unique solution.*

Proof. Let $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ be two solutions of the equation (2). Set $\eta_t = X(t) - Y(t)$ and $\xi_t = |\eta_t|^2$. Let $\rho > 0$, consider

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s) + \rho} \quad \text{and} \quad \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.$$

We have

$$\Phi'_\rho(\xi) = \frac{\Phi_\rho(\xi)}{\xi\; r(\xi) + \rho}. \tag{12}$$

Let

$$\tau = \inf\{t > 0,\ \xi_t \ge 1/2\,\}.$$

By assumption (11), we have

$$|\langle \eta_t, b(X(t)) - b(Y(t))\rangle| \le C\ \xi_t\, r(\xi_t), \quad t < \tau.$$

Therefore according to (12), for $t < \tau$, by chain rule,

$$\Phi_\rho(\xi_t) \le 1 + 2C \int_0^t \Phi_\rho(\xi_s)\, ds,$$

which implies that $\Phi_\rho(\xi_t) \le e^{2C\,t}$ for $t < \tau$. Letting $\rho \downarrow 0$, we get that $e^{\psi_0(\xi_t)} \le e^{2Ct}$. Now by hypothesis (ii), we obtain that $\xi_t = 0$ for all $t < \tau$. If $\tau < +\infty$, letting $t \uparrow \tau$, we get

$$\frac{1}{2} = \xi_\tau = 0,$$

which is absurd. Therefore $\xi_t = 0$ for all $t \ge 0$. In other words, $X(t) = Y(t)$ for $t \ge 0$.

Example 2.3 Define

$$f(x_1, x_2) = \sum_{k\ge 1} \frac{\sin kx_1 \cdot \sin kx_2}{k^2}.$$

Obviously the function f is continuous on R^2. We have

$$|f(X) - f(Y)| \le C\,|X - Y| \log \frac{1}{|X - Y|} \quad \text{for } |X - Y| < \frac{1}{e} \tag{13}$$

where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. In fact,

$$f(X) - f(Y) = \sum_{k=1}^{\infty} \Big\{ \frac{(\sin kx_1 - \sin ky_1)\sin kx_2}{k^2} + \frac{(\sin kx_2 - \sin ky_2)\sin ky_1}{k^2} \Big\}.$$

It follows that

$$|f(X) - f(Y)| \le 2 \sum_{k=1}^{\infty} \Big\{ \frac{|\sin(k\frac{x_1 - y_1}{2})|}{k^2} + \frac{|\sin(k\frac{x_2-y_2}{2})|}{k^2} \Big\}.$$

Lemma 2.4 *For $0 < \theta < 1/e$, we have*

$$V(\theta) := \sum_{k=1}^{\infty} \frac{|\sin k\theta|}{k^2} \le C_1\, \theta \log \frac{1}{\theta}. \tag{14}$$

Proof. Consider $\phi(s) = \frac{\sin s\theta}{s^2}$. We have

$$\phi'(s) = \frac{s^2\theta \cos s\theta - 2s \sin s\theta}{s^4}.$$

Then $|\phi'(s)| \le \frac{3\theta}{s^2}$. Let $W(\theta) = \int_1^{+\infty} \frac{|\sin s\theta|}{s^2}\, ds$. We have

$$\begin{aligned} |V(\theta) - W(\theta)| &\le \sum_{k=1}^{+\infty} \int_k^{k+1} |\phi(s) - \phi(k)|\, ds \\ &\le 3\theta \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2\theta}{2}. \end{aligned} \tag{15}$$

Now

$$W(\theta) = \theta \int_\theta^{+\infty} \frac{|\sin t|}{t^2}\, dt \le \theta \int_\theta^1 \frac{\sin t}{t}\frac{dt}{t} + \theta \int_1^{+\infty} \frac{ds}{s^2}$$

which is dominated by

$$\theta\Big(\log\frac{1}{\theta}+1\Big).$$

Therefore, according to (15)

$$V(\theta)\le\theta\Big(\log\frac{1}{\theta}+1+\frac{\pi^2}{2}\Big),$$

which is less that $2(\frac{\pi^2}{2}+1)\,\theta\log\frac{1}{\theta}$ for $0<\theta<\frac{1}{e}$.

Now applying (14), for $|x_1-y_1|+|x_2-y_2|<1/e$,

$$\begin{aligned}
&|f(X)-f(Y)|\\
\le\;& 4C_1\Big(\tfrac{|x_1-y_1|}{2}\log\tfrac{1}{|x_1-y_1|}+\tfrac{|x_2-y_2|}{2}\log\tfrac{1}{|x_2-y_2|}\Big)\\
\le\;& 4C_1\Big(\tfrac{|x_1-y_1|+|x_2-y_2|}{2}\log\tfrac{2}{|x_1-y_1|+|x_2-y_2|}\Big) \qquad (16)
\end{aligned}$$

where the last inequality was due to the concavity of the function $\xi\log\frac{1}{\xi}$ over $]0,1[$. Therefore (13) holds for some constant $C>0$.

In what follows, we shall study the dependence of initial values.

Theorem 2.5 *Assume that the conditions* (4) *and* (11) *hold. Then* $x_o\to X_t(x_o)$ *is continuous.*

Proof. Let $\varepsilon>0$. Consider a small parameter $0<\delta<\varepsilon$. Let $(x_o,y_o)\in R^d\times R^d$ such that $|x_o-y_o|<\delta$. Set $\eta_t=X_t(x_o)-X_t(y_o)$ and $|\xi_t|=|\eta_t|^2$. Define

$$\tau(x_o,y_o)=\inf\{t>0,\ \xi_t\ge\varepsilon\ \}.$$

Consider

$$\psi_\rho(\xi)=\int_0^\xi\frac{ds}{sr(s)+\rho}\quad\text{and}\quad\Phi_\rho(\xi)=e^{\psi_\rho(\xi)}.$$

As in proof of theorem 2.2, we have for $t<\tau(x_o,y_o)$,

$$\Phi_\rho(\xi_t)\le\Phi_\rho(\xi_o)e^{2C\,t}.$$

Taking $\rho=|x_o-y_o|$, we get

$$\Phi_\rho(\xi_t)\le e^\rho\,e^{2C\,t},\quad\text{for } t<\tau(x_o,y_o). \qquad (17)$$

Fix the point x_o. If $\liminf_{y_o\to x_o}\tau(x_o,y_o)=\tau<+\infty$, we can choose $y_n\to x_o$ such that $\lim_{n\to+\infty}\tau(x_o,y_n)=\tau$. Applying (17) for (x_o,y_n) and letting $t\uparrow\tau(x_o,y_n)$, we get

$$\Phi_{\rho_n}(\varepsilon)=\Phi_{\rho_n}(\xi_{\tau(x_o,y_n)})\le e^{\rho_n}\,e^{2C\,\tau(x_o,y_n)}.$$

where $\rho_n=|x_o-y_n|$. Letting $n\to+\infty$, we have

$$+\infty=\Phi_o(\varepsilon)\le e^{2C\,\tau}$$

which is absurd. Therefore

$$\lim_{y_o\to x_o}\tau(x_o,y_o)=+\infty,$$

which means that for any $t > 0$, there exists $\delta > 0$ such that for $|y_o - x_o| < \delta$, $\tau(x_o, y_o) > t$. In other words,

$$|X_t(x_o) - X_t(y_o)| \leq \varepsilon.$$

Proposition 2.6 *Assume that the conditions* (4) *and* (11) *hold. Then for $x_o \neq y_o$, we have $X_t(x_o) \neq X_t(y_o)$ for all $t \geq 0$.*

Proof. Let $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t = |\eta_t|^2$. Without loss of generality, assume that $0 < \xi_o < 1/2$. Let

$$\tau = \inf\{t > 0,\ \xi_t \geq \frac{3}{4}\}.$$

By starting from τ again , it is enough to prove that $\xi_t > 0$ for $t < \tau$. Consider

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s) + \rho} \quad \text{and} \quad \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.$$

By assumption (10), for $t < \tau$, we get

$$\left|\frac{d\Phi_\rho(\xi_t)}{dt}\right| \leq 2C\Phi_\rho(\xi_t).$$

It follows that $\Phi_\rho(\xi_t) \geq \Phi_\rho(\xi_o) - 2C\int_0^t \Phi_\rho(\xi_s)ds$ or

$$\Phi_\rho(\xi_t) \geq \Phi_\rho(\xi_o)e^{-2Ct} \quad \text{for} \quad t < \tau. \tag{18}$$

For $\rho > 0$ small enough, $\Phi_\rho(\xi_o)e^{-2Ct} > 1$. It follows that $\Phi_\rho(\xi_t) > 1$ or $\xi_t > 0$.

Now using (11) and proposition 2.6, and by the standard arguments, we obtain

Theorem 2.7 *Assume that the conditions* (4) *and* (11) *hold. Then for any $t > 0$, $x_o \to X_t(x_o)$ defines a flow of homeomorphisms of R^d.*

3 Stochastic differential equations

Let $\sigma : R^d \to R^d \otimes R^m$ be continuous function. Let $X(t)$ be a solution of the following Itô stochastic differential equation:

$$dX(t) = \sigma(X(t))\,dW_t + b(X(t))\,dt, \quad X(0) = x_o \tag{19}$$

with the lifetime $\zeta(w)$.

Theorem 3.1 *Let $r : [1, +\infty[\to R_+$ be a function of $\mathcal{C}^1$, satisfying (i) $\lim_{s\to+\infty} r(s) = +\infty$, (ii) $\int_1^\infty \frac{ds}{sr(s)+1} = +\infty$ and*

(iii) $\lim_{s\to+\infty} \frac{sr'(s)}{r(s)} = 0.$

Assume that for $|x| \geq 1$,

$$\begin{cases} ||\sigma(x)||^2 & \leq & C\Big(|x|^2\, r(|x|^2) + 1\Big), \\ |b(x)| & \leq & C\Big(|x|\, r(|x|^2) + 1\Big). \end{cases} \tag{20}$$

Then the s.d.e (19) *has no explosion:* $P(\zeta = +\infty) = 1$.

Proof. For $0 < s \leq 1$, define $r(s) = r(\frac{1}{s})$. Then there exists a constant $C > 0$ such that the condition (20) holds for any x. Consider

$$\psi(\xi) = \int_0^\xi \frac{ds}{sr(s) + 1} \quad \text{and} \quad \Phi(\xi) = e^{\psi(\xi)}, \quad \xi \geq 0.$$

We have

$$\Phi'(\xi)\,(\xi\, r(\xi) + 1) = \Phi(\xi), \tag{21}$$

$$\Phi''(\xi) = \begin{cases} \frac{\Phi(\xi)\big(1 - r(\frac{1}{\xi}) + \frac{1}{\xi} r'(\frac{1}{\xi})\big)}{(\xi r(\frac{1}{\xi}) + 1)^2} & \text{if} \quad 0 < \xi < 1, \\ \frac{\Phi(\xi)\,\big(1 - r(\xi) - \xi r'(\xi)\big)}{(\xi r(\xi) + 1)^2} & \text{if} \quad \xi > 1. \end{cases} \tag{22}$$

By conditions (i) and (iii), there exists $M > 0$ such that $\Phi''(\xi) \leq 0$ for $\xi \leq \frac{1}{M}$ or $\xi \geq M$. Remark that the function Φ is not $\mathcal{C}^2$ at the point $\xi = 1$. Fix a small $\delta > 0$, take $\tilde{\Phi} \in \mathcal{C}^2(R_+)$ such that

$$\tilde{\Phi} \geq \Phi, \quad \tilde{\Phi}(\xi) = \Phi(\xi) \quad \text{for } \xi \notin [1 - \delta, 1 + \delta]. \tag{23}$$

Denote

$$K_1 = \sup_{\xi \in [1-\delta, 1+\delta]} \Big(|\tilde{\Phi}'(\xi)| + |\tilde{\Phi}''(\xi)|\Big), \quad K_2 = \sup_{\xi \in [1-\delta, 1+\delta]} \Big(\xi\, r(\xi)\Big).$$

Then

$$|\tilde{\Phi}'(\xi)| \leq \frac{K_1(K_2 + 1)}{\Phi(1 - \delta)} \cdot \frac{\Phi(\xi)}{\xi\, r(\xi) + 1}, \quad \xi \in [1 - \delta, 1 + \delta], \tag{24}$$

and

$$|\tilde{\Phi}''(\xi)| \leq \frac{K_1(K_2 + 1)^2}{\Phi(1 - \delta)} \cdot \frac{\Phi(\xi)}{(\xi\, r(\xi) + 1)^2}, \quad \xi \in [1 - \delta, 1 + \delta]. \tag{25}$$

Let $\xi_t(w) = |X_t(w)|^2$. We have

$$\begin{aligned} d\xi_t \;&=\; 2\langle X_t, \sigma(X(t))\, dW_t\rangle + 2\langle X_t, b(X(t))\rangle\, dt \\ &\qquad + ||\sigma(X(t))||^2\, dt, \end{aligned} \tag{26}$$

and the stochastic contraction $d\xi_t \cdot d\xi_t$ is given by

$$d\xi_t \cdot d\xi_t = 4|\sigma^*(X_t) X_t|^2\, dt \tag{27}$$

where σ^* denotes the transpose matrix of σ.

Define

$$\tau_R = \inf\{t > 0,\ \xi_t \geq R\}, \quad R > 0.$$

Then $\tau_R \uparrow \zeta$ as $R \uparrow +\infty$. Let

$$I_w = \{t > 0,\ \xi_t(w) \in [\frac{1}{M}, M]\}.$$

Taking M big enough such that $[1-\delta, 1+\delta] \subset [\frac{1}{M}, M]$. By (22),

$$\tilde{\Phi}''(\xi_t) = \Phi''(\xi_t) \leq 0 \quad \text{for } t \notin I_w. \tag{28}$$

Combining (22) and (25), there exists a constant C_1 such that

$$\left|\tilde{\Phi}''(\xi_t)\right| \leq \frac{C_1 \Phi(\xi_t)}{(\xi_t\, r(\xi_t) + 1)^2}, \quad t \in I_w. \tag{29}$$

By (21) and (24), for some constant $C_2 > 0$, we have

$$\left|\tilde{\Phi}'(\xi_t)\right| \leq \frac{C_2\, \Phi(\xi_t)}{\xi_t\, r(\xi_t) + 1}, \quad t > 0. \tag{30}$$

Now by Itô formula and according to (26), (27), we have

$$\begin{aligned} \tilde{\Phi}(\xi_{t\wedge\tau_R}) &= \Phi(\xi_o) + 2\int_0^{t\wedge\tau_R} \tilde{\Phi}'(\xi_s)\langle X_s, \sigma(X(s))dW_s\rangle \\ &+ 2\int_0^{t\wedge\tau_R} \tilde{\Phi}'(\xi_s)\, \langle X_s, b(X(s))\rangle\, ds \\ &+ \int_0^{t\wedge\tau_R} \tilde{\Phi}'(\xi_s)||\sigma(X(s))||^2\, ds \\ &+ 2\int_0^{t\wedge\tau_R} \tilde{\Phi}''(\xi_s)\, |\sigma^*(X(s))X_s|^2\, ds. \end{aligned} \tag{31}$$

By (28) and (29),

$$\int_0^{t\wedge\tau_R} \tilde{\Phi}''(\xi_s)\, |\sigma^*(X(s))X_s|^2\, ds \leq \int_0^{t\wedge\tau_R} \mathbf{1}_{I_w}(s) \frac{C_1\, \Phi(\xi_s)}{(\xi_s\, r(\xi_s) + 1)^2}\, |\sigma^*(X(s))X_s|^2\, ds. \tag{32}$$

By (20),

$$\frac{|\sigma^*(X(s))X_s|^2}{(\xi_s\, r(\xi_s) + 1)^2} \leq C_3\ \frac{\xi_s\, (\xi_s\, r(\xi_s) + 1)}{(\xi_s\, r(\xi_s) + 1)^2} \tag{33}$$

which is dominated by a constant C_3. According to (32), we get

$$\int_0^{t\wedge\tau_R} \tilde{\Phi}''(\xi_s)\, |\sigma^*(X(s))X_s|^2\, ds \leq C_3 \int_0^{t\wedge\tau_R} \Phi(\xi_s)\, ds. \tag{34}$$

In the same way, for some constant $C_4 > 0$, we have

$$\frac{|\langle X_s, b(X(s))\rangle| + ||\sigma(X(s))||^2}{\xi_s\, r(\xi_s) + 1} \leq C_4, \quad s > 0. \tag{35}$$

Now using (31) and according to (30), (35) and (34), we get

$$E\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \le E\Big(\tilde{\Phi}(\xi_{t\wedge\tau_R})\Big) \le \Phi(\xi_o) + C_5 \int_0^t E(\Phi(\xi_{s\wedge\tau_R}))\, ds,$$

which implies that

$$E\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \le \Phi(\xi_o) e^{C_5\, t}.$$

Letting $R \to +\infty$, by Fatou lemma, we get

$$E\Big(\Phi(\xi_{t\wedge\zeta})\Big) \le \Phi(\xi_o) e^{C_5\, t}. \tag{36}$$

Now if $P(\zeta < +\infty) > 0$, then for some $T > 0$, $P(\zeta \le T) > 0$. Taking $t = T$ in (36), we get

$$E\Big(\mathbf{1}_{(\zeta\le T)}\Phi(\xi_\zeta)\Big) \le \Phi(\xi_o) e^{C_5\, t}$$

which is impossible, because of $\Phi(\xi_\zeta) = +\infty$.

Theorem 3.2 *Let $r :]0,1[\to R_+$ be $\mathcal{C}^1$-function satisfying the conditions*

(i) $\lim_{\xi\to 0} r(\xi) = +\infty$,

(ii) $\lim_{\xi\to 0} \frac{\xi\, r'(\xi)}{r(\xi)} = 0$,

(iii) $\int_0^a \frac{ds}{sr(s)} = +\infty$, *for any* $a > 0$.

Assume that for $|x-y| < 1$,

$$\begin{cases} ||\sigma(x)-\sigma(y)||^2 & \le & C\,|x-y|^2\, r(|x-y|^2), \\ |b(x)-b(y)| & \le & C\,|x-y|\, r(|x-y|^2). \end{cases} \tag{37}$$

Then the s.d.e. (19) *has the pathwise uniqueness.*

Proof. Let $\eta_t(w) = X(t) - Y(t)$ and $\xi_t(w) = |\eta_t(w)|^2$. Let $\rho > 0$. Define the function $\psi_\rho : [0,1] \to R$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s)+\rho}. \tag{38}$$

It is clear that for any $0 < \xi < 1$,

$$\psi_\rho(\xi) \uparrow \psi_0(\xi) = \int_0^\xi \frac{ds}{sr(s)} = +\infty, \quad \text{as } \rho \downarrow 0.$$

Define

$$\Phi_\rho(\xi) = e^{\psi_\rho(\xi)}. \tag{39}$$

Then we have

$$\Phi'_\rho(\xi)\,(\xi\, r(\xi) + \rho) = \Phi_\rho(\xi), \tag{40}$$

and

$$\Phi''_\rho(\xi) = \frac{\Phi_\rho(\xi)(1 - \xi\, r'(\xi) - r(\xi))}{(\xi\, r(\xi)+\rho)^2}. \tag{41}$$

By assumption $(i),(ii)$ on r, there exists $\delta > 0$ such that $\Phi''_\rho(\xi) \leq 0$ for $0 < \xi < \delta$.
Let

$$\tau = \inf\{\, t > 0, \quad \xi_t \geq \delta \,\}.$$

By Itô formula, we have

$$\begin{aligned}
\Phi_\rho(\xi_{t\wedge\tau}) &= 1 + 2\int_0^{t\wedge\tau} \Phi'_\rho(\xi_s)\langle \eta_s, (\sigma(X(s)) - \sigma(Y(s)))dW_s\rangle \\
&+ 2\int_0^{t\wedge\tau} \Phi'_\rho(\xi_s)\,\langle \eta_s, b(X(s)) - b(Y(s))\rangle\, ds \\
&+ \int_0^{t\wedge\tau} \Phi'_\rho(\xi_s)||\sigma(X(s)) - \sigma(Y(s))||^2\, ds \\
&+ 2\int_0^{t\wedge\tau} \Phi''_\rho(\xi_s)\,|(\sigma^*(X(s)) - \sigma^*(Y(s)))\eta_s|^2\, ds.
\end{aligned} \tag{42}$$

Applying the hypothesis (37), we get

$$E\Big(\Phi_\rho(\xi_{t\wedge\tau})\Big) \leq 1 + 2C\,E\Big(\int_0^{t\wedge\tau} \Phi'_\rho(\xi_s)\,\xi_s\, r(\xi_s)\ ds\Big)$$

which is smaller by (41) than

$$1 + 2C\int_0^t E\Big(\Phi_\rho(\xi_{s\wedge\tau})\Big)\, ds.$$

Now by Gronwall lemma, we get $E\Big(\Phi_\rho(\xi_{t\wedge\tau})\Big) \leq e^{2ct}$ or

$$E\Big(e^{\psi_\rho(\xi_{t\wedge\tau})}\Big) \leq e^{2Ct}. \tag{43}$$

Letting $\rho \downarrow 0$ in (43),

$$E\Big(e^{\psi_0(\xi_{t\wedge\tau})}\Big) \leq e^{2Ct}$$

which implies that for any t given,

$$\xi_{t\wedge\tau} = 0 \quad \text{almost surely.} \tag{44}$$

If $P(\tau < +\infty) > 0$, then for some $T > 0$ big enough $P(\tau \leq T) > 0$. By (44), almost surely for all $t \in Q \cap [0, T]$, $\xi_{t\wedge\tau} = 0$. It follows that on $\{\tau \leq T\}$,

$$\xi_\tau = 0$$

which is absurd with the definition of τ. Therefore $\tau = +\infty$ almost surely and for any t given, $\xi_t = 0$ almost surely. Now by continuity of samples, the two solutions are indistinguishable.

Remark 3.3 In the case of $d = m = 1$, finer results about pathwise uniqueness have been established. Namely σ was allowed to be Hölder of exponent $\geq 1/2$ (see [RY, Ch. IX-3], [IW, p.168]).

Theorem 3.4 *Assume that the coefficients σ and b satisfy the assumption* (20) *and* (37)*. Let $X(t, x_o)$ be the solution of the s.d.e.* (19) *with initial value x_o. Then for any $\varepsilon > 0$, we have*

$$\lim_{y_o\to x_o} P(\sup_{0\leq s\leq t} |X(s, y_o) - X(s, x_o)| > \varepsilon) = 0. \tag{45}$$

Proof. Fix x_o. Let δ be the parameter given in proof of theorem 3.2, consider $|y_o - x_o| < \varepsilon < \delta$. Let $\xi_t(w) = |X(t, y_o) - X(t, x_o)|^2$. Define

$$\tau_w(x_o, y_o) = \inf\{\, t > 0,\ \xi_t > \varepsilon^2 \,\}.$$

The same arguments as above yields to

$$E\Big(\Phi_\rho(\xi_{t\wedge\tau(x_o,y_o)})\Big) \le \Phi_\rho(\xi_o)\, e^{2Ct}.$$

Taking $\rho = |x_o - y_o|$, we have $E\Big(\Phi_\rho(\xi_{t\wedge\tau(x_o,y_o)})\Big) \le e^\rho\, e^{2Ct}$. Hence

$$P(\tau(x_o, y_o) < t)\Phi_\rho(\varepsilon) \le E\Big(\Phi_\rho(\xi_{t\wedge\tau(x_o,y_o)})\Big) \le e^\rho\, e^{2Ct}.$$

It follows

$$P(\sup_{0\le s\le t} |X(s, y_o) - X(s, x_o)| > \varepsilon) = P(\tau(x_o, y_o) < t) \le e^{-\psi_\rho(\varepsilon)}\, e^\rho\, e^{2Ct} \to 0$$

as $\rho = |y_o - x_o| \to 0$.

Corollary 3.5 *The diffusion process* $(X(t, x))$ *given by the solution of the s.d.e. is Feller, i.e., the associated semigroup* $(T_t, t \ge 0)$ *maps* $C_b(R^d)$ *into* $C_b(R^d)$.

Proof. It is a direct consequence of Theorem 3.4 and the definition

$$T_t f(x) = E[f(X(t, x))], \qquad f \in C_b(R^d)$$

Remark 3.6 We have a difficulty here to apply the Kolmogoroff modification theorem to obtain a version $\tilde{X}(t, x_o)$ such that $x_o \to \tilde{X}(t, x_o)$ is continuous. However the situation for the case of S^1 is well handled (see [F], [M]).

4 Statement of large deviation principle

Let $\sigma : \ R^d \to R^d \otimes R^m$ and $b : \ R^d \to R^d$ be continuous functions. Consider the following Itô s.d.e:

$$dX^\varepsilon(t) = \varepsilon^{\frac{1}{2}} \sigma(X^\varepsilon(t))\, dW_t + b(X^\varepsilon(t))\, dt, \quad X_o^\varepsilon(w) = x_o \tag{46}$$

where $t \to W_t$ is a R^m-valued standard Brownian motion. In order to be more explicit, we shall work under the following assumptions,

$$(H1) \quad \begin{cases} ||\sigma(x) - \sigma(y)||^2 & \le \quad C\,|x-y|^2 \log \frac{1}{|x-y|}, \qquad \text{for } |x-y| < 1, \\ |b(x) - b(y)| & \le \quad C\,|x-y| \log \frac{1}{|x-y|}, \qquad \text{for } |x-y| < 1 \end{cases}$$

$$(H2) \qquad \begin{cases} ||\sigma(x)||^2 & \leq & C\,(|x|^2 \log|x| + 1), \\ |b(x)| & \leq & C\,(|x| \log|x| + 1) \end{cases}$$

where $|\cdot|$ denotes the Euclidean distance in R^d and $||\sigma||^2 = \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2$. The rest of this paper is to establish a large deviation principle for solutions of above s. d. e.

Let $C_x([0,1], R^m)$ be the space of continuous functions from $[0,1]$ into R^m with initial value x. If $g \in C_0([0,1], R^m)$ is absolutely continuous, set $e(g) = \int_0^1 |\dot{g}(t)|^2 dt$. Let $F(g)$ be the solution to the differential equation

$$\begin{aligned} F(g)(t) &= x_0 + \int_0^t b(F(g)(s))ds \\ &\quad + \int_0^t \sigma(F(g)(s))\dot{g}(s)ds, \quad 0 < t < \infty \end{aligned} \tag{47}$$

the uniqueness and non explosion being obtained as in section 2 for the differential equation (2). See also lemma 5.3 below.

Theorem 4.1 *Let μ_ε be the law of $X^\varepsilon(\cdot)$ on $C_{x_0}([0,1], R^d)$. Assume $(H1)$ and $(H2)$. Then $\{\mu_\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with the following good rate function*

$$I(f) = \inf\left\{\frac{1}{2}e(g)^2; F(g) = f\right\}, \qquad f \in C_{x_0}([0,1], R^d) \tag{48}$$

i.e.,

(i) for any closed subset $C \subset C_{x_0}([0,1], R^d)$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(C) \leq -\inf_{f \in C} I(f) \tag{49}$$

(ii) for any open subset $G \subset C_{x_0}([0,1], R^d)$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geq -\inf_{f \in G} I(f) \tag{50}$$

The proof of the theorem will be given in Section 5 and 6.

5 Large deviations when σ, b are bounded

The theory of large deviations for diffusion processes under Lipschitzian coefficients is well established (see [A], [S]). Some new developments in infinite dimensional situations are discussed in [FZ1,2], [Z1,2]. The main task of this work is to handle the non Lipschitzian feature. For $n \geq 1$, let $X_n^\varepsilon(\cdot)$ to be the solution to

$$\begin{aligned} X_n^\varepsilon(t) &= x_0 + \int_0^t b(X_n^\varepsilon(\frac{[ns]}{n}))ds \\ &\quad + \varepsilon^{\frac{1}{2}} \int_0^t \sigma\left(X_n^\varepsilon\left(\frac{[ns]}{n}\right)\right) dW_s \end{aligned} \tag{51}$$

We need the following lemma from Stroock [S,P.81].

Lemma 5.1. *Let $\alpha(\cdot)$ and $\beta(\cdot)$ be $(\mathcal{F}_t)_{t\geq 0}$ -progressively measurable with values in $R^d \otimes R^m$ and R^d respectively. Assume $||\alpha(\cdot)|| \leq A$ and $|\beta(\cdot)| \leq B$, and set $\xi(t) = \int_0^t \alpha(s)dW_s + \int_0^t \beta(s)ds$. Then for $T > 0$ and $R > 0$ satisfying $d^{\frac{1}{2}}BT < R$:*

$$P(\sup_{0\leq t\leq T} |\xi(t)| \geq R) \leq 2d \exp\big(-(R - d^{\frac{1}{2}}BT)^2/2A^2dT\big) \tag{52}$$

Proposition 5.2. *In addition to (H.1), we also assume that b, σ are bounded. For any $\delta > 0$, it holds that*

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq 1} |X^\varepsilon(t) - X_n^\varepsilon(t)| > \delta) = -\infty \tag{53}$$

Proof. We may and will assume $\delta < e^{-1} < 1$. Let $Y_n^\varepsilon(t) := X^\varepsilon(t) - X_n^\varepsilon(t)$ and $\xi_n^\varepsilon(t) = |Y_n^\varepsilon(t)|^2$. We have

$$\begin{aligned} Y_n^\varepsilon(t) &= \int_0^t \Big[b(X^\varepsilon(s)) - b(X_n^\varepsilon(\frac{[ns]}{n})) \Big] ds \\ &\quad + \varepsilon^{\frac{1}{2}} \int_0^t \Big[\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\frac{[ns]}{n})) \Big] dW_s \end{aligned} \tag{54}$$

and

$$\begin{aligned} d\xi_n^\varepsilon(t) &= 2\varepsilon^{\frac{1}{2}} \langle Y_n^\varepsilon(t), \Big(\sigma(X^\varepsilon(t)) - \sigma(X_n^\varepsilon(\frac{[nt]}{n})) \Big) dW_t \rangle \\ &\quad + 2\langle Y_n^\varepsilon(t), \Big(b(X^\varepsilon(t)) - b(X_n^\varepsilon(\frac{[nt]}{n})) \Big) \rangle dt \\ &\quad + \varepsilon ||\sigma(X^\varepsilon(t)) - \sigma(X_n^\varepsilon(\frac{[nt]}{n}))||^2 dt \end{aligned} \tag{55}$$

The stochastic contraction $d\xi_n^\varepsilon \cdot d\xi_n^\varepsilon$ is given by

$$d\xi_n^\varepsilon \cdot d\xi_n^\varepsilon = 4\varepsilon| \Big(\sigma(X^\varepsilon(t)) - \sigma(X_n^\varepsilon(\frac{[nt]}{n})) \Big)^* Y_n^\varepsilon(t)|^2 \, dt \tag{56}$$

where σ^* denotes the transpose of σ. Let $\rho > 0$. Define the function $\psi_\rho : [0,1] \to R$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s \log \frac{1}{s} + \rho}. \tag{57}$$

It is clear that for any $0 < \xi < 1$,

$$\psi_\rho(\xi) \uparrow \psi_0(\xi) = \int_0^\xi \frac{ds}{s \log \frac{1}{s}} = +\infty, \quad \text{as } \rho \downarrow 0.$$

Define for $\lambda > 0$

$$\Phi_{\rho,\lambda}(\xi) = e^{\lambda \psi_\rho(\xi)}. \tag{58}$$

Then we have

$$\Phi'_{\rho,\lambda}(\xi)\, (\xi \log \frac{1}{\xi} + \rho) = \lambda \Phi_{\rho,\lambda}(\xi), \tag{59}$$

and

$$\Phi''_{\rho,\lambda}(\xi) = \lambda^2\Phi_{\rho,\lambda}(\xi)\frac{1}{(\xi\log\frac{1}{\xi}+\rho)^2} + \lambda\Phi_{\rho,\lambda}(\xi)\frac{(1-\log\frac{1}{\xi})}{(\xi\log\frac{1}{\xi}+\rho)^2}$$

$$\leq \lambda^2\Phi_{\rho,\lambda}(\xi)\frac{1}{(\xi\log\frac{1}{\xi}+\rho)^2} \quad \text{if } \xi \leq e^{-1}. \tag{60}$$

Now, choose a positive constant $\delta_1 < e^{-1}$ satisfying $\delta_1 \log\frac{1}{\delta_1} < \rho$. Define $\tau_n^\varepsilon = \inf\{t \geq 0; |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{[nt]}{n})| \geq \delta_1\}$, and set $\xi_{n,\delta_1}^\varepsilon(t) = \xi_n^\varepsilon(t\wedge\tau_n^\varepsilon), t \geq 0$. Putting $T_n^\varepsilon = \inf\{t \geq 0, |\xi_{n,\delta_1}^\varepsilon(t)| \geq \delta^2\}$, we have,

$$\begin{aligned} P(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)| > \delta) &= P(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)| > \delta, \tau_n^\varepsilon \leq 1) \\ &\quad + P(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)| > \delta, \tau_n^\varepsilon > 1) \\ &\leq P(\tau_n^\varepsilon \leq 1) + P(T_n^\varepsilon \leq 1) \end{aligned} \tag{61}$$

Observe,

$$P(\tau_n^\varepsilon \leq 1) \leq \sum_{k=1}^{n} P\Big(\sup_{\frac{k-1}{n}\leq t\leq\frac{k}{n}} |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{k-1}{n})| \geq \delta_1\Big). \tag{62}$$

Using Lemma 5.1 and the boundness of σ, b, there exists a constant $c_{\delta_1} > 0$ such that

$$P(\tau_n^\varepsilon \leq 1) \leq n\exp(-nc_{\delta_1}/\varepsilon). \tag{63}$$

Hence,

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\tau_{n,\rho}^\varepsilon \leq 1) = -\infty \tag{64}$$

For notational simplicity, write T for T_n^ε and τ for τ_n^ε. By Ito's formula,

$$\begin{aligned} &\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(t\wedge T)) \\ = &\ 1 + 2\varepsilon^{\frac{1}{2}}\int_0^{t\wedge T\wedge\tau}\Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))\Big\langle Y_n^\varepsilon(s), \Big(\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\tfrac{[ns]}{n}))\Big)\, dW_s\Big\rangle \\ + &\ 2\int_0^{t\wedge T\wedge\tau}\Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))\Big\langle Y_n^\varepsilon(s), b(X^\varepsilon(s)) - b(X_n^\varepsilon(\tfrac{[ns]}{n}))\Big\rangle ds \\ + &\ \varepsilon\int_0^{t\wedge T\wedge\tau}\Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))||\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\tfrac{[ns]}{n}))||^2 ds \\ + &\ 2\varepsilon\int_0^{t\wedge T\wedge\tau}\Phi''_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))\,|\Big(\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\tfrac{[ns]}{n}))\Big)^* Y_n^\varepsilon(s)|^2\, ds. \end{aligned} \tag{65}$$

Note that for $s \leq T\wedge\tau$, $|Y_n^\varepsilon(s)| \leq \delta_1 \leq e^{-1} < 1$ and $|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})| \leq \delta_1 < e^{-1}$. Therefore, for $s \leq T\wedge\tau$,

$$|\Big\langle Y_n^\varepsilon(s), b(X^\varepsilon(s)) - b(X_n^\varepsilon(\frac{[ns]}{n}))\Big\rangle|$$

$$= |\Big\langle Y_n^\varepsilon(s), b(X^\varepsilon(s)) - b(X_n^\varepsilon(s))\Big\rangle| + |\Big\langle Y_n^\varepsilon(s), b(X_n^\varepsilon(s)) - b(X_n^\varepsilon(\frac{[ns]}{n}))\Big\rangle|$$

$$\leq C|Y_n^\varepsilon(s)|^2\log(\frac{1}{|Y_n^\varepsilon(s)|})+C|X_n^\varepsilon(s)-X_n^\varepsilon(\frac{[ns]}{n})|\log(\frac{1}{|X_n^\varepsilon(s)-X_n^\varepsilon(\frac{[ns]}{n})|})$$

$$\leq \frac{1}{2}C\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\delta_1\log(\frac{1}{\delta_1})\leq C(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho), \qquad (66)$$

where we have used the fact that the function $x\log(\frac{1}{x})$ is increasing on $[0,e^{-1}]$. Furthermore,for $s\leq T\wedge\tau$,

$$||\sigma(X^\varepsilon(s))-\sigma(X_n^\varepsilon(\frac{[ns]}{n}))||^2$$

$$\leq 2||\sigma(X^\varepsilon(s))-\sigma(X_n^\varepsilon(s)))||^2+2||\sigma(X_n^\varepsilon(s))-\sigma(X_n^\varepsilon(\frac{[ns]}{n}))||^2$$

$$\leq 2C\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+2C|X_n^\varepsilon(s)-X_n^\varepsilon(\frac{[ns]}{n})|^2\log(\frac{1}{|X_n^\varepsilon(s)-X_n^\varepsilon(\frac{[ns]}{n})|})$$

$$\leq C(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho) \qquad (67)$$

Similarly,

$$|\left(\sigma(X^\varepsilon(s))-\sigma(X_n^\varepsilon(\frac{[ns]}{n}))\right)^* Y_n^\varepsilon(s)|^2$$

$$\leq |Y_n^\varepsilon(s)|^2||\left(\sigma(X^\varepsilon(s))-\sigma(X_n^\varepsilon(\frac{[ns]}{n}))\right)^*||^2$$

$$\leq C\xi_n^\varepsilon(s)(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho)\leq C(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho)^2 \qquad (68)$$

Taking these inequalities into account, it follows from (65) that

$$\begin{aligned} & E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(t\wedge T))] \\ \leq \quad & CE\Big[\int_0^{t\wedge T}\Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))(\xi_{n,\delta_1}^\varepsilon(s)\log(\tfrac{1}{\xi_{n,\delta_1}^\varepsilon(s)})+\rho)ds\Big] \\ + \quad & C\varepsilon E\Big[\int_0^{t\wedge T}\Phi''_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))(\xi_{n,\delta_1}^\varepsilon(s)\log(\tfrac{1}{\xi_{n,\delta_1}^\varepsilon(s)})+\rho)^2ds\Big] \end{aligned} \qquad (69)$$

which is smaller by (59) and (60) than

$$1+C\varepsilon(\lambda^2+\lambda)\int_0^t E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s\wedge T))]ds$$

By Gronwall's inequality, we obtain that

$$E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(1\wedge T))]\leq e^{C(\varepsilon\lambda^2+\lambda)} \qquad (70)$$

On the other hand,

$$E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(1\wedge T))]\geq E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(T)),T\leq 1]=e^{\lambda\psi_\rho(\delta^2)}P(T\leq 1) \qquad (71)$$

Combining (70) with (71), we have

$$P(T\leq 1)\leq e^{-\lambda\psi_\rho(\delta^2)}e^{C(\varepsilon\lambda^2+\lambda)}$$

Taking $\lambda = \frac{1}{\varepsilon}$ it follows that

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon log P(T\leq 1)\leq -\psi_\rho(\delta^2)+2C$$

This together with (64) implies that

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon log P(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)|>\delta)\leq -\psi_\rho(\delta^2)+2C \tag{72}$$

Sending ρ to 0 completes the proof.

For $n\geq 1$, define a map $F_n(\cdot): C_0([0,1],R^m)\to C_{x_0}([0,1],R^d)$ by

$$\begin{aligned} F_n(\omega)(0) &= x_0 \\ F_n(\omega)(t) &= F_n(\omega)\Big(\frac{k}{n}\Big)+b\Big(F_n(\omega)\Big(\frac{k}{n}\Big)\Big)\Big(t-\frac{k}{n}\Big) \\ &\quad +\sigma\Big(F_n(\omega)\Big(\frac{k}{n}\Big)\Big)\Big(\omega(t)-\omega\Big(\frac{k}{n}\Big)\Big) \end{aligned} \tag{73}$$

for $\frac{k}{n}\leq t\leq \frac{k+1}{n}$. It is easy to see that F_n is a continuous map from $C_0([0,1],R^m)$ to $C_{x_0}([0,1],R^d)$.

Lemma 5.3. $\lim_{n\to\infty}\sup_{\{g;e(g)\leq\alpha\}}\sup_{0\leq t\leq 1}|F_n(g)(t)-F(g)(t)|=0$.

Proof . Note that for g with $e(g)\leq\alpha$,

$$\begin{aligned} F_n(g)(t) &= x_0+\int_0^t b(F_n(g)(\frac{[ns]}{n}))ds \\ &+ \int_0^t \sigma\Big(F_n(g)\Big(\frac{[ns]}{n}\Big)\Big)\dot{g}(s)ds \end{aligned} \tag{74}$$

Thus,

$$\begin{aligned} &F_n(g)(t)-F(g)(t) \\ &=\int_0^t [b(F_n(g)\Big(\frac{[ns]}{n}\Big))-b(F(g)(s))]ds \\ &+\int_0^t [\sigma(F_n(g)\Big(\frac{[ns]}{n}\Big))-\sigma(F(g)(s))]\dot{g}(s)ds \end{aligned} \tag{75}$$

Since b,σ are bounded, we have for $t\leq 1$

$$\begin{aligned} |F_n(g)(t)-F_n(g)\Big(\frac{[nt]}{n}\Big)| &\leq \int_{\frac{[nt]}{n}}^t |b(F_n(g)(\frac{[ns]}{n}))|ds \\ &+\int_{\frac{[nt]}{n}}^t ||\sigma(F_n(g)\Big(\frac{[ns]}{n}\Big))|||\dot{g}|(s)ds \\ &\leq C_\alpha\Big(\frac{1}{n}\Big)^{\frac{1}{2}}, \end{aligned} \tag{76}$$

where C_α is a constant depending only on α and the uniform norms of b and σ. Let $Y_n^g(t)=F_n(g)(t)-F(g)(t)$ and $Z_n^g(t)=|Y_n^g(t)|^2$. For any $0<\delta<e^{-1}$, define $\tau_n(g)=\inf\{t\geq 0,|Y_n^g(t)|>\delta\}$. Given $\rho>0$, define

$$\Phi_\rho(\xi)=e^{\psi_\rho(\xi)},$$

where

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s\log\frac{1}{s} + \rho}.$$

Then

$$\Phi'_\rho(\xi)(\xi\log\frac{1}{\xi} + \rho) = \Phi_\rho(\xi) \tag{77}$$

By the chain rule,

$$\begin{aligned} &\Phi_\rho(Z_n^g(t\wedge\tau_n(g))) \\ &= 1 + 2\int_0^{t\wedge\tau_n(g)} \Phi'_\rho(Z_n^g(s))\Big\langle Y_n^g(s), b(F_n(g)(\frac{[ns]}{n})) - b(F(g)(s))\Big\rangle ds \\ &+2\int_0^{t\wedge\tau_n(g)} \Phi'_\rho(Z_n^g(s))\Big\langle Y_n^g(s), (\sigma(F_n(g)(\frac{[ns]}{n})) - \sigma(F(g)(s)))\dot{g}(s)\Big\rangle ds \end{aligned} \tag{78}$$

Using $(H.1)$ and (76), for $s \le \tau_n(g)$,

$$\begin{aligned} &|\Big\langle Y_n^g(s), b(F_n(g)\Big(\frac{[ns]}{n}\Big)) - b(F(g)(s))\Big\rangle| \\ &\le |\Big\langle Y_n^g(s), b(F_n(g)\Big(\frac{[ns]}{n}\Big)) - b(F_n(g)(s))\Big\rangle| \\ &+|\Big\langle Y_n^g(s), b(F_n(g)(s)) - b(F_n(g)(s))\Big\rangle| \\ &\le C|Y_n^g(s)||F_n(g)\Big(\frac{[ns]}{n}\Big) - (F_n(g)(s)|\log(\frac{1}{|F_n(g)(\frac{[ns]}{n}) - (F_n(g)(s)|}) \\ &+\frac{1}{2}CZ_n^g(s)\log(\frac{1}{Z_n^g(s)}) \\ &\le CC_\alpha(\frac{1}{n})^{\frac{1}{2}}\log(n^{\frac{1}{2}}) + CZ_n^g(s)\log(\frac{1}{Z_n^g(s)}) \\ &\le C(Z_n^g(s)\log(\frac{1}{Z_n^g(s)}) + \rho) \end{aligned} \tag{79}$$

for $n \ge N_\alpha^1$, where N_α^1 depends on α and ρ. Similarly, for $s \le \tau_n(g)$ and $n \ge N_\alpha^1$,

$$\begin{aligned} &|Y_n^g(s)|||\sigma(F_n(g)\Big(\frac{[ns]}{n}\Big)) - \sigma(F(g)(s))|| \\ &\le C(Z_n^g(s)\log(\frac{1}{Z_n^g(s)}) + \rho) \end{aligned} \tag{80}$$

It follows from (78) that for $n \ge N_\alpha^1$ and all $g \in \{g; e(g) \le \alpha\}$,

$$\begin{aligned} &\Phi_\rho(Z_n^g(t\wedge\tau_n(g))) \\ &\le 1 + C\int_0^{t\wedge\tau_n(g)} \Phi'_\rho(Z_n^g(s))(\frac{1}{Z_n^g(s)}) + \rho)(1 + |\dot{g}(s)|)ds \\ &\le C\int_0^t \Phi_\rho(Z_n^g(s\wedge\tau_n(g)))(1 + |\dot{g}(s)|)ds \end{aligned} \tag{81}$$

By Gronwall's lemma,

$$\Phi_\rho(Z_n^g(1 \wedge \tau_n(g))) \le Ce^{1+\int_0^1 |\dot{g}(s)|ds}$$

Since $\Phi_\rho(\xi)$ is incresing in ξ, it follows that for $n \ge N_\alpha^1$,

$$\Phi_\rho(\sup_{g\in\{g;e(g)\le\alpha\}} Z_n^g(1 \wedge \tau_n(g))) \le Ce^{1+\alpha} \tag{82}$$

Consequently, for any $\rho > 0$,

$$\limsup_{n\to\infty} \Phi_\rho(\sup_{g\in\{g;e(g)\le\alpha\}} Z_n^g(1 \wedge \tau_n(g))) \le Ce^{1+\alpha} \tag{83}$$

To complete the proof it suffices to show that for any $\delta > 0$ there exists an integer N such that if $n \ge N$, then $\tau_n(g) > 1$ for all $g \in \{g; e(g) \le \alpha\}$. This is now a consequence of (83). In fact, otherwise, there exists $\delta > 0$, a subsequence $\{n_k, k \ge 1\}$ of positive integers and $g_{n_k} \in \{g; e(g) \le \alpha\}$ such that $\tau_{n_k}(g_{n_k}) > 1$. This implies that

$$\Phi_\rho(\sup_{g\in\{g;e(g)\le\alpha\}} Z_{n_k}^g(1 \wedge \tau_{n_k}(g))) \ge \Phi_\rho(Z_{n_k}^{g_{n_k}}(1 \wedge \tau_{n_k}(g_{n_k}))) \ge \Phi_\rho(\delta^2)$$

Combing this with (83), we get

$$\Phi_\rho(\delta^2) \le Ce^{1+\alpha} \tag{84}$$

for all ρ. This leads to a contradiction since the left side of (84) tends to infinity as ρ goes to 0. The proof is complete.

Proof of Theorem 4.1 when b, σ are bounded Notice that $X_n^\varepsilon(s) = F_n(\varepsilon^{\frac{1}{2}}W)(s)$, where W is the Brownian motion. The theorem follows from Proposition 5.2 , Lemma 5.3 and Theorem 4.2.23 in [DZ].

6 Large deviations: general case

In this section we will remove the boundeness assumptions on b and σ. We begin with

Proposition 6.1 *Assume (H.2). Then*

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\le t\le 1} |X^\varepsilon(t) - x_0| > R) = -\infty \tag{85}$$

where $X^\varepsilon(\cdot)$ is the solution to equation (51).

Proof. Let δ_0 be a fixed small positive constant, say $\delta_0 < \frac{1}{2}$. Let $f \in C^1(R_+)$ be a strictly positive C^1 function on R_+ that satisfies

$$f(s) = \begin{cases} -s \log s & \text{if } 0 \le s \le 1 - \delta_0 \\ s \log s & \text{if } s \ge 1 + \delta_0 \end{cases}$$

From now on, we will use C to denote a generic constant which may change from line to line. It is easy to see that there exists a positive contant C such that

$$s|\log s| + 1 \le C(f(s)+1), \quad f'(s) \ge -C \quad \text{for } s \ge 0 \tag{86}$$

Define

$$\psi(\xi) = \int_0^\xi \frac{ds}{f(s)+1} \quad \xi \ge 0.$$

and put

$$\Phi_\lambda(\xi) = e^{\lambda\psi(\xi)}, \quad \lambda \ge 0$$

It follows from (86) that

$$\Phi_\lambda'(\xi) = \lambda\Phi_\lambda(\xi)\frac{1}{f(\xi)+1} \le C\lambda\Phi_\lambda(\xi)\frac{1}{(\xi|\log\xi|+1)}, \tag{87}$$

and

$$\begin{aligned}\Phi_\lambda''(\xi) &= \lambda^2\Phi_\lambda(\xi)\frac{1}{(f(\xi)+1)^2} - \lambda\Phi_\lambda(\xi)\frac{f'(\xi)}{(f(\xi)+1)^2} \\ &\le C(\lambda^2+\lambda)\Phi_\lambda(\xi)\frac{1}{(\xi|\log\xi|+1)^2}\end{aligned} \tag{88}$$

Let $\eta^\varepsilon(t) = X^\varepsilon(t) - x_o$ and $\xi^\varepsilon(t) = |\eta^\varepsilon(t)|^2$. Define

$$\tau_R = \inf\{t > 0, |\eta^\varepsilon(t)| \ge R\}, \quad R > 0.$$

Now by Itô formula, we have

$$\begin{aligned}\Phi_\lambda(\xi^\varepsilon(t\wedge\tau_R)) &= 1 + 2\varepsilon^{\frac{1}{2}}\int_0^{t\wedge\tau_R} \Phi_\lambda'(\xi^\varepsilon(s))\langle\eta^\varepsilon(s), \sigma(X^\varepsilon(s))dW_s\rangle \\ &+ 2\int_0^{t\wedge\tau_R} \Phi_\lambda'(\xi^\varepsilon(s))\,\langle\eta^\varepsilon(s), b(X^\varepsilon(s))\rangle\, ds \\ &+ \varepsilon\int_0^{t\wedge\tau_R} \Phi_\lambda'(\xi^\varepsilon(s))||\sigma(X^\varepsilon(s))||^2\, ds \\ &+ 2\varepsilon\int_0^{t\wedge\tau_R} \Phi_\lambda''(\xi^\varepsilon(s))\,|\sigma^*(X^\varepsilon(s))\eta^\varepsilon(s)|^2\, ds.\end{aligned} \tag{89}$$

By $(H2)$, there exists $C_1 > 0$ such that

$$\begin{cases} ||\sigma(x)||^2 & \le & C_1\,(|x-x_o|^2\,\log|x-x_o|+1), \\ |b(x)| & \le & C_1\,(|x-x_o|\,\log|x-x_o|+1). \end{cases}$$

It follows that

$$\frac{|\sigma^*(X^\varepsilon(s))\eta^\varepsilon(s)|^2}{(\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1)^2} \le C_1\,\frac{\xi^\varepsilon(s)\,(\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1)}{(\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1)^2}$$

which is dominated by a constant C. According to (88), we get

$$\int_0^{t\wedge\tau_R} \Phi_\lambda''(\xi^\varepsilon(s))\,|\sigma^*(X^\varepsilon(s))\eta^\varepsilon(s)|^2\, ds \le C(\lambda^2+\lambda)\int_0^{t\wedge\tau_R} \Phi_\lambda(\xi^\varepsilon(s))\, ds. \tag{90}$$

In the same way, for some constant $C > 0$, we have

$$\frac{|\langle \eta^\varepsilon(s), b(X^\varepsilon(s))\rangle| + ||\sigma(X^\varepsilon(s))||^2}{\xi^\varepsilon(s)\,|\log \xi^\varepsilon(s)| + 1} \leq C, \quad s > 0. \tag{91}$$

Combining above inequalities together, we get

$$E(\Phi_\lambda(\xi^\varepsilon(t\wedge\tau_R))) \leq 1 + C(\varepsilon\lambda^2+\lambda)\int_0^t E(\Phi_\lambda(\xi^\varepsilon(t\wedge\tau_R)))\,ds,$$

which implies that

$$E(\Phi_\lambda(\xi^\varepsilon(1\wedge\tau_R))) \leq e^{C(\varepsilon\lambda^2+\lambda)}.$$

Let $\lambda = \frac{1}{\varepsilon}$. It follows that

$$P(\sup_{0\leq t\leq 1}|X^\varepsilon(t)-x_0| > R)\Phi_{\frac{1}{\varepsilon}}(R) \leq E(\Phi_\lambda(\xi^\varepsilon(1\wedge\tau_R)))$$

$$\leq e^{2C\frac{1}{\varepsilon}}$$

This gives that

$$\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\leq t\leq 1}|X^\varepsilon(t)-x_0| > R) \leq C - \psi(R) \tag{92}$$

Note that $\lim_{R\to\infty}\psi(R) = +\infty$. Letting R tend to $+\infty$ in (92) proves the proposition.

For $R > 0$, define $m_R = \sup\{|b(x)|, ||\sigma(x)||; |x| \leq R\}$ and $b_i^R = (-m_R - 1)\vee b_i\wedge(m_R+1)$, $\sigma_{i,j}^R = (-m_R-1)\vee\sigma_{i,j}\wedge(m_R+1)$, $1 \leq i \leq d$, $0 \leq j \leq m$. Put $b_R = (b_1^R, b_2^R, ..., b_d^R)$ and $\sigma_R = (\sigma_{i,j}^R)_{1\leq i\leq d, 1\leq j\leq m}$. Then for $|x| \leq R$,

$$b_R(x) = b(x), \quad \sigma_R(x) = \sigma(x).$$

and b_R, σ_R satisfy (H.1) and (H.2) with the same constants.
Let $X_R^\varepsilon(\cdot)$ be the solution to

$$\begin{aligned} X_R^\varepsilon(t) &= x_0 + \int_0^t b_R(X_R^\varepsilon(s))ds \\ &+ \varepsilon^{\frac{1}{2}}\int_0^t \sigma_R(X_R^\varepsilon(s))dW_s,\ t > 0. \end{aligned} \tag{93}$$

For g with $e(g) < \infty$, let $F_R(g)$ be the solution to

$$\begin{aligned} F_R(g)(t) &= x_0 + \int_0^t b_R(F_R(g)(s))ds \\ &+ \int_0^t \sigma_R(F_R(g)(s))\dot{g}(s)ds \end{aligned} \tag{94}$$

Define

$$I_R(f) = \inf\left\{\frac{1}{2}e(g)^2; F_R(g) = f\right\}, \qquad f \in C_{x_0}([0,1]\to R^d) \tag{95}$$

If $\sup_{0\le t\le 1}|F(g)(t)|\le R$, then $F(g)=F_R(g)$. Therefore,

$$I(f)=I_R(f), \quad \text{for } f \text{ with } \sup_{0\le t\le 1}|f(t)|\le R \tag{96}$$

Lemma 6.2 *$I(\cdot)$ is a good rate function on $C_{x_0}([0,1],R^d)$,i.e., for any $\alpha\ge 0$, the level set $\{f;I(f)\le\alpha\}$ is compact.*

Proof. Arguing as in Lemma 5.3, it is easy to see that for $\alpha>0$, $\sup_{\{g;e(g)\le\alpha\}}||F(g)||_\infty\le R$ for some constant R. Thus $F_R(\cdot)=F(\cdot)$ on $\{g;e(g)\le\alpha\}$. On the other hand, It is easy to see that $F_R(\cdot)$ is continuous on the level set $\{g;e(g)\le\alpha\}$, so is $F(\cdot)$. This is sufficient to conclude that $I(\cdot)$ is a good rate functional since $e(\cdot)$ is.

Proof of Theorem 4.1 in the Unbounded Case.

Let μ_ε^R denote the law of $X_R^\varepsilon(\cdot)$ on $C_{x_0}([0,1],R^d)$. According to previous section, $\{\mu_\varepsilon^R,\varepsilon>0\}$ satisfies a large deviation principle with good rate function $I_R(\cdot)$. Note that μ_ε^R and μ_ε coincide on the ball $\{f;||f||_\infty\le R\}$. For $R>0$ and a closed subset $C\subset C_{x_0}([0,1],R^d)$, set $C_R=C\cap\{f;||f||_\infty\le R\}$. Then,

$$\begin{aligned}\mu_\varepsilon(C) &\le \mu_\varepsilon(C_R)+P(\sup_{0\le t\le 1}|X^\varepsilon(t)|>R)\\ &= \mu_\varepsilon^R(C_R)+P(\sup_{0\le t\le 1}|X^\varepsilon(t)|>R)\end{aligned} \tag{97}$$

By the large deviation principle for $\{\mu_\varepsilon^R,\varepsilon>0\}$,

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon(C)\\ &\le (-\inf_{f\in C_R}I_R(f))\vee(\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|X^\varepsilon(t)|>R))\\ &= (-\inf_{f\in C_R}I(f))\vee(\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|X^\varepsilon(t)|>R))\end{aligned} \tag{98}$$

Applying Proposition 6.1 and Letting $R\to\infty$, we obtain

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon(C)\le(-\inf_{f\in C}I(f)) \tag{99}$$

which is the upper bound.
Let G be an open subset of $C_{x_0}([0,1]\to R^d)$. Fix any $\phi_0\in G$. Choosing $\delta>0$ such that $B(\phi_0,\delta)=\{f;||f-\phi_0||_\infty\le\delta\}\subset G$. Let $R=||\phi_0||_\infty+\delta$. Since

$$B(\phi_0,\delta)\subset\{f;||f||_\infty\le R\}$$

We have

$$\begin{aligned}-I(\phi_0)=-I_R(\phi_0) &\le \limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon^R\Big(B(\phi_0,\delta)\Big)\\ &= \limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon\Big(B(\phi_0,\delta)\Big)\\ &\le (\limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon(G))\end{aligned} \tag{100}$$

Since ϕ_0 is arbitrary, it follows that

$$-\inf_{f\in G} I(f) \leq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \tag{101}$$

which is the lower bound.

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