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A SINGULAR CRITICAL POTENTIAL FOR THE SCHRÖDINGER OPERATOR

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Abstract. We construct a potential V on $\mathbb{R}^d$, smooth away from one pole, and a sequence of quasi-modes for the operator $-\Delta+V$, which concentrate on this pole. No smoothing effect, Strichartz estimates nor dispersive inequalities hold for the corresponding Schrödinger equation.

1. Introduction

Consider a Schrödinger operator:

$$P = -\Delta + V, \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \tag{1}$$

on $\mathbb{R}^d$, $d \geq 1$, where V is a real potential, small at infinity, and the initial value problem for the Schrödinger equation:

$$\begin{cases} i\partial_t U - PU = 0 \\ U_{\restriction t=0} = U_0 \in L^2(\mathbb{R}^d). \end{cases} \tag{2}$$

For sufficiently smooth potentials V, the equation (2) implies a local smoothing effect:

$$||\chi U||_{L^2(]0,T[,H^{1/2}(\mathbb{R}^d))} \leq C||U_0||_{L^2(\mathbb{R}^d)}, \; \chi \in C_0^\infty(\mathbb{R}^d), \tag{3}$$

where T may be finite, and in some cases infinite, and C may depend on V, χ and T. This well-known estimate (see [2], [6] and references therein) is essential in the study of (2) and related non-linear equations. In this paper, we investigate the minimal regularity to be assumed on V such that (3) is still true. In [6], A. Ruiz and L. Vega consider the equation (2) as a perturbation of the free Schrödinger equation to show an inequality which implies (3). The following potentials fall within this scope:

- $V \in L^p, \; p \geq d/2$.
- Potentials belonging to some Morey-Campanato spaces, (which are larger than $L^{d/2}$) with a smallness condition in these spaces (see [6] for details). For example, potentials with inverse-square singularities:

$$V = \sum_{j=1..N} \frac{a_j}{|x-p_j|^2} + V_0, \; V_0 \in L^p, \; p \geq d/2, \tag{4}$$

  where the $|a_j|$'s are small enough.

Let us mention that potentials in the article above may be time-dependent but we shall not develop this point here. In [5], (see also [3]), the author studies potentials of type (4) with regular V_0, assuming only the following positivity properties on the a_j's:

$$a_j + (d/2-1)^2 > 0, \quad j = 1..N,$$

and shows non-trapping type inequality on the truncated resolvent:

$$\forall \lambda \in \mathbb{C}, \; \operatorname{Re}\lambda^2 \geq C, \; \operatorname{Im}\lambda^2 \neq 0, \quad ||\chi(P-\lambda^2)^{-1}\chi||_{L^2 \to L^2} \leq \frac{C}{|\lambda|}, \quad \chi \in C_0^\infty(\mathbb{R}^d), \tag{5}$$

One may deduce from (5) estimates of type (3) (cf [2]).

The preceding works do not study potentials $V \in L^p$, $p < d/2$, and potentials of type (4) but with poles of higher order. In this paper, we show that the space $L^{d/2}$ and the inverse-square singularities are critical, by constructing an unipolar potential whose pole is of the order of $(\log r)^2/r^2$, $r = |x|$, near 0, and such that (3), (5) and some other usual estimates on solutions of Schrödinger equations do not hold.

Theorem 1. *Let $d \geq 1$, $N \geq 0$ be integers. There exist:*

- *a radial, positive potential V on $\mathbb{R}^d$, which has compact support and such that:*

$$V \in C^\infty(\mathbb{R}^d \backslash \{0\}) \tag{6}$$

$$\frac{|\log r|^2}{Cr^2} \leq V(r) \leq \frac{C|\log r|^2}{r^2},\ r \leq r_0, \tag{7}$$

- *an increasing sequence $(\lambda_n)_{n \geq n_0}$ of positive real numbers, diverging to $+\infty$,*
- *a sequence of radial C^∞ functions $(u_n)_{n \geq n_0}$, whose support is of the following form:*

$$\left\{ c_1 \frac{\log \lambda_n}{\lambda_n} \leq r \leq c_2 \frac{\log \lambda_n}{\lambda_n} \right\};$$

such that:

$$(-\Delta + V)u_n - \lambda_n^2 u_n = f_n \tag{8}$$

$$||u_n||_{L^1} = 1 \tag{9}$$

$$\forall j \in \mathbb{N},\ ||\frac{d^j}{dr^j} f_n||_{L^\infty} = O(\lambda_n^{j-N}),\ n \to +\infty. \tag{10}$$

Corollary 1. *Let N be any integer greater than 1, $P = -\Delta + V$, where V is the potential of the preceding theorem, and χ a function in $C_0^\infty(\mathbb{R}^d)$ which does not vanish in 0. Then:*

- *(lack of local Strichartz estimates)*

$$\forall q, q_0 \in [1, +\infty],\ q > q_0,\ \forall T > 0,\ \forall C > 0,\ \exists U_0 \in C_0^\infty,\ ||\chi U(t)||_{L^1(]0,T[, L^q(\mathbb{R}^d))} > C||U_0||_{L^{q_0}(\mathbb{R}^d)}; \tag{11}$$

- *(lack of local smoothing effect)*

$$\forall \sigma > 0\ \forall T > 0,\ \forall C > 0,\ \exists U_0 \in C_0^\infty,\ ||\chi U(t)||_{L^1(]0,T[, H^\sigma(\mathbb{R}^d))} > C||U_0||_{L^2(\mathbb{R}^d)}; \tag{12}$$

- *(lack of local dispersion)*

$$\forall q \in]1, +\infty],\ \forall T > 0,\ \forall C > 0,\ \exists U_0 \in C_0^\infty,\ ||\chi U(T)||_{L^q(\mathbb{R}^d)} > C||U_0||_{L^{q'}(\mathbb{R}^d)}; \tag{13}$$

- *(lack of Strichartz estimates with loss of derivative)*

$$\forall q \in [1, +\infty],\ \forall \sigma \in [0, 1],\ \frac{1}{2} - \frac{1}{q} > \frac{\sigma}{d}, \tag{14}$$
$$\forall T > 0,\ \forall C > 0,\ \exists U_0 \in C_0^\infty,\ ||\chi U(t)||_{L^1(]0,T[, L^q(\mathbb{R}^d))} > C||U_0||_{D(P^{\sigma/2})}.$$

In the preceding statements, we wrote $U(t)$ the solution of the equation (2) with initial value U_0 and q' the conjugate exponent of q which is defined by: $1/q + 1/q' = 1$.

Remarks.

- If $d > 2$, the hypothesis (6) and (7) implie:

$$V \in \bigcap_{p < d/2} L^p.$$

- Of course, the sequence $(f_n)_{n \geq n_0}$ invalidates (5) when $N \geq 2$.
- It will be clear in the proof of the theorem than one may construct quasi-modes of infinite order (i.e. such that(10) holds with $O(\lambda_n^{-\infty})$ instead of $O(\lambda_n^{j-N})$) by taking a singularity just a bit stronger:

$$\frac{|\log r|^{2+\varepsilon}}{Cr^2} \leq V(r) \leq \frac{C|\log r|^{2+\varepsilon}}{r^2},\ \varepsilon > 0.$$

With a still stronger singularity, one may force f_n to be exponentially decreasing in $-\lambda_n$.

- Classical Strichartz estimates:

$$||u||_{L^p(]0,T[,L^q(\mathbb{R}^d))} \leq C||u_0||_{L^2(\mathbb{R}^d)},\ p > 2,\ \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

are invalidated by (11), with the exception of the case $q = 2$, $p = +\infty$ which is always true by the L^2 conservation law. Likewise, (14) shows that Strichartz estimates with loss of derivative (see [2]) of the form:

$$||u||_{L^p(]0,T[,L^q(\mathbb{R}^d))} \leq C||u_0||_{D(P^{\sigma/2})},\ \frac{1}{2} - \frac{1}{q} > \frac{\sigma}{d},\ \sigma \in [0,1],$$

are false. In this inequality, the limit case $\sigma/d = 1/2 - 1/q$ is an immediate consequence of the Sobolev inclusion

$$L^q(\mathbb{R}^d) \subset H^\sigma(\mathbb{R}^d) \subset D(P^{\sigma/2}),$$

and of the conservation of the norm $D(P^{\sigma/2})$ of any solution of (2). Notice that this last inclusion implies that (14) is still true in usual Sobolev spaces, i.e when $D(P^{\sigma/2})$ is replaced by $H^\sigma(\mathbb{R}^n)$.
- The potential V and quasi-modes u_n being of compact support, as close to 0 as desired, the preceding counter-example is still valid in any regular domain, for a Laplace operator with Dirichlet or Neumann Dirichlet boundary condition.
- Examples of linear Schrödinger equations which do not admit the classical dispersive inequality or Strichartz estimates are given in [1] and [4]. In these two cases, the particular behaviour of the equation arises from the metric which defines the Laplace operator. Let us mention that the idea to use quasi-modes in relation with Strichartz inequalities is owed to M. Zworski (see [8]).

Finally, we would like to point out that for the potential V introduced in the theorem, one may define the operator P without ambiguity. The operator $-\Delta + V$ has a natural meaning on $C_0^\infty(\mathbb{R}^d\backslash\{0\})$, and is essentially self-adjoint on this space, under a positivity asumption implied by (7). We call P its unique self-adjoint extension, which is of course positive. We refer to Reed and Simon [7, Th X.30] for any precision.

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Section 2 of the paper is devoted to the proof of the corollary, and section 3 to that of the theorem.

2. Proof of the corollary.

According to Hölder's inequality, for any function v on $\mathbb{R}^d$, with a finite volume support:

$$||v||_{L^{q_0}} \leq B^{1/q_0 - 1/q}||v||_{L^q}, \quad q > q_0, \tag{15}$$

where B is the volume of the support. Let $U_n(t) = e^{-i\lambda_n^2 t}u_n$. Then:

$$i\partial_t U_n - PU_n = -e^{-i\lambda_n^2 t}f_n, \quad U_{n\restriction t=0} = u_n$$

$$U_n(t) = e^{-itP}u_n + ie^{-i\lambda_n^2 t}\int_0^t e^{i(s-t)P}f_n ds. \tag{16}$$

Assume that (11) is false. According to (16):

$$\int_0^T ||\chi U_n(t)||_{L^q(\mathbb{R}^d)}dt \leq \int_0^T ||\chi e^{-itP}u_n||_{L^q(\mathbb{R}^d)}dt + \int_0^T\int_0^t ||\chi e^{i(s-t)P}f_n||_{L^q(\mathbb{R}^d)}dsdt.$$

Since the support of u_n concentrates in 0, and χ does not vanish in 0, the left term of this inequality is, for n large enough, greater than:

$$\frac{1}{C}\int_0^T ||u_n||_{L^q(\mathbb{R}^d)}dt \geq \frac{1}{C}\left(\frac{\log\lambda_n}{\lambda_n}\right)^{d(1/q-1/q_0)}||u_n||_{L^{q_0}(\mathbb{R}^d)}.$$

Using the negation of (11), the right term is dominated by:

$$||u_n||_{L^{q_0}(\mathbb{R}^d)} + ||f_n||_{L^{q_0}(\mathbb{R}^d)}.$$

Hence we obtain, using (10):

$$\left(\frac{\lambda_n}{\log \lambda_n}\right)^{d(1/q_0-1/q)} ||u_n||_{L^{q_0}} = O\left(||u_n||_{L^{q_0}} + \lambda_n^{-N}\right),$$

which leads to the announced contradiction since $N \geq 1$ and $q > q_0$, the norm of u_n in L^{q_0} being greater than 1 by (9).

By Sobolev inequalities, (11) implies (12), taking in (11) $q_0 = 2$ and $q > 2$ close enough to 2. So (12) holds.

Let us assume that (13) is not true. Then we get, by (16):

$$\begin{aligned}||\chi u_n(T)||_{L^q(\mathbb{R}^d)} &\leq C||u_n||_{L^{q'}(\mathbb{R}^d)} + \int_0^T ||\chi e^{i(s-T)} f_n||_{L^1(\mathbb{R}^d)} ds\\ &\leq C\left(||u_n||_{L^{q'}(\mathbb{R}^d)} + ||f_n||_{L^2(\mathbb{R}^d)}\right).\end{aligned}$$

To obtain the second inequality we have bounded, up to a multiplicative constant, the $L^{q'}$ norm on the support of χ by the L^2 norm. The contradiction is again a simple consequence of (15) and the fact that $q' < q$.

We shall now prove (14). Otherwise, we would have by (16):

$$\int_0^T ||u_n(t)||_{L^q(\mathbb{R}^d)} dt \leq C||u_n||_{D(P^{\sigma/2})} + \int_0^T \int_0^t ||e^{i(s-t)P} f_n||_{D(P^{\sigma/2})} ds\, dt. \tag{17}$$

Of course, $e^{i(s-t)P}$ maps isometrically the domain of $P^{\sigma/2}$ onto itself. We have:

$$||f_n||^2_{D(P^{1/2})} = \int |\nabla f_n(x)|^2 dx + \int (1+V(x))|f_n(x)|^2 dx.$$

On the support of f_n, as on that of u_n, $|x| \geq \frac{\log \lambda_n}{C\lambda_n}$. Using the superior bound of V in (7), we get, on the support of f_n:

$$|V(r)| \leq C\frac{\lambda_n^2}{(\log \lambda_n)^2} (\log\log \lambda_n - \log \lambda_n)^2 \leq C\lambda_n^2.$$

It follows using (10) and $N \geq 2$ that:

$$||f_n||_{D(P^{1/2})} \leq C\left(||\nabla f_n||_{L^2} + \lambda_n ||f_n||_{L^2}\right) \leq C\lambda_n^{-1}. \tag{18}$$

Furthermore, the equation $Pu_n - \lambda_n^2 u_n = f_n$ implies, using once again the bounds (9) and (10) on the norms of u_n and f_n:

$$\begin{aligned}||u_n||^2_{D(P^{1/2})} &\leq ||f_n||_{L^2}||u_n||_{L^2} + \lambda_n^2||u_n||^2_{L^2}\\ &\leq C\lambda_n^2||u_n||^2_{L^2}.\end{aligned}$$

By interpolation on the norms of the left hand side, and since $0 \leq \sigma \leq 1$, we get:

$$||u_n||_{D(P^{\sigma/2})} \leq C\lambda_n^{\sigma}||u_n||_{L^2}. \tag{19}$$

According to (18), (19) and the inequality (17):

$$||u_n||_{L^q} \leq C\lambda_n^{\sigma}||u_n||_{L^2} + o(1).$$

Hence, with (15):

$$\left(\frac{\lambda_n}{\log \lambda_n}\right)^{d/2-d/q} ||u_n||_{L^2} \leq C\lambda_n^{\sigma}||u_n||_{L^2} + o(1),$$

which is absurd since $\frac{d}{2} - \frac{d}{q} > \sigma$.

3. Démonstration du théorème.

Denote by r the euclidian norm of x and by $'$ the radial derivative $\frac{d}{dr}$. We would like to find radial functions V, f_n, u_n such that:

$$f_n(r) = -u_n''(r) - \frac{d-1}{r}u_n'(r) + V(r)u_n(r) - \lambda_n^2 u_n(r), \tag{20}$$

with f_n small and λ_n diverging to infinity. We shall first change functions to get rid of the first-order derivative in this equation. Set:

$$u_n = r^{-\frac{d-1}{2}} v_n, \quad f_n = r^{-\frac{d-1}{2}} g_n, \quad W = V + \frac{d^2-4d+3}{4r^2}. \tag{21}$$

Thus (20) becomes:

$$g_n(r) = -v_n'' + Wv_n - \lambda_n^2 v_n. \tag{20'}$$

Let:

$$y(s) = e^{-\sqrt{s^2+1}},\ b(s) = -\frac{1}{(s^2+1)^{3/2}} + \frac{s^2}{s^2+1},$$

which are C^∞ solutions of the equation on $\mathbb{R}$:

$$-y''(s) + b(s)y(s) = 0 \tag{22}$$

$$y(s) > 0, \forall j \in \mathbb{N}, \quad |y^{(j)}(s)| \leq C_j e^{-|s|} \tag{23}$$

$$|b(s)| \leq 1. \tag{24}$$

We shall write:

$$y_{\omega,a}(r) = y(\omega(r-a)), \quad b_{\omega,a}(r) = \omega^2 b(\omega(r-a)),$$

where a and ω are two real parameters. We have:

$$-\, y_{\omega,a}'' + b_{\omega,a} y_{\omega,a} = 0; \tag{22'}$$

Let $q(\lambda)$ be a strictly increasing positive function defined in a neighbourhood of $+\infty$ such that:

$$\lim_{\lambda\to+\infty} q(\lambda) = +\infty \tag{25}$$

$$\lim_{\lambda\to+\infty} \frac{q(\lambda)}{\lambda} = 0. \tag{26}$$

Let $(\lambda_n)_{n\geq n_0}$ be the sequence defined by the equations:

$$10^n = q(\lambda_n),$$

so that λ_n diverges faster to $+\infty$ than 10^n.

Choose a cutoff function χ: $\chi \in C_0^\infty(]-1,1[)$, $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Set:

$$\psi_n(r) = \chi(10^n r) - \chi(10^{n+1} r),$$
$$\chi_n(r) = \chi\left(10^{n+1}\left(r - \frac{3}{10^{n+1}}\right)\right),$$

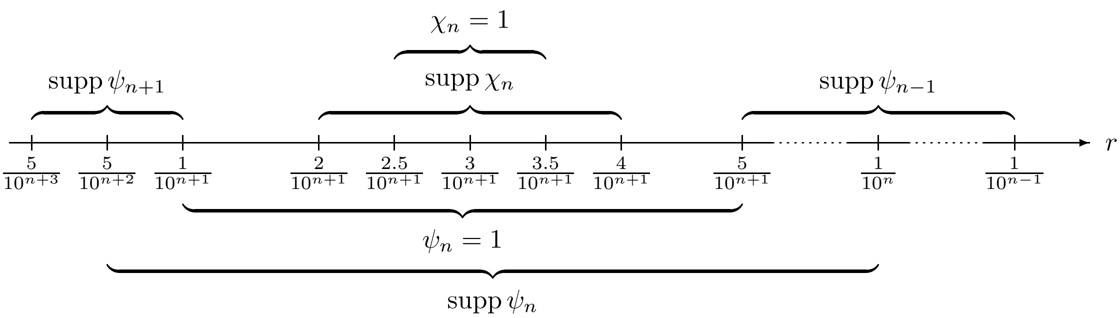

FIGURE 1. Cutoff functions near $3/10^{n+1}$

The ψ_n's form a partition of unity near 0. The support of each χ_n is included in that of ψ_n, away from those of the ψ_j's, $j \neq n$:

$$\operatorname{supp} \psi_n \subset \left[\frac{5}{10^{n+2}}, \frac{1}{10^n}\right]$$

$$\frac{1}{10^{n+1}} \leq r \leq \frac{5}{10^{n+1}} \Rightarrow \psi_n(r) = 1$$

$$\sum_{n \geq n_0} \psi_n(r) = \chi(10^{n_0} r) \tag{27}$$

$$\operatorname{supp} \chi_n \subset \left[\frac{2}{10^{n+1}}, \frac{4}{10^{n+1}}\right] \subset \{\psi_n = 1\} \tag{28}$$

$$\frac{25}{10^{n+2}} \leq r \leq \frac{35}{10^{n+2}} \Rightarrow \chi_n(r) = 1. \tag{29}$$

We shall denote by y_n and b_n the following functions:

$$y_n(r) = y_{\frac{\lambda_n}{2}, \frac{3}{10^{n+1}}}(r) = y\left(\frac{\lambda_n}{2}\left(r - \frac{3}{10^{n+1}}\right)\right) \tag{30}$$

$$b_n(r) = b_{\frac{\lambda_n}{2}, \frac{3}{10^{n+1}}}(r) = \frac{\lambda_n^2}{4} b\left(\frac{\lambda_n}{2}\left(r - \frac{3}{10^{n+1}}\right)\right). \tag{31}$$

Each of the function $-b_n$ may be seen as a potential well which concentrates (by equation (22')) the energy of y_n near $3/10^{n+1}$. We shall construct W so that the equation $g_n(r) = 0$ is exactly $-v_n'' + b_n v_n = 0$ on a small interval (including the support of v_n) around the point $3/10^{n+1}$. The size of this interval will be of the same order as 10^{-n}, smaller than the equation parameter $\omega = \lambda_n/2$. We shall choose v_n as a cutoff of y_n. The exponential decay of y, combined with the difference of order between these two scales will induced small enough error terms for (9) and (10) to hold.

Set (see the figure):

$$W(r) = \sum_{n \geq n_0} \psi_n(r) \left(b_n(r) + \lambda_n^2\right) \tag{32}$$

$$v_n(r) = \alpha_n y_n(r) \chi_n(r), \tag{33}$$

where the α_n's are strictly positive constants yet to be determined. The functions g_n, u_n, f_n and the potential V are thus defined by (20') and (21).

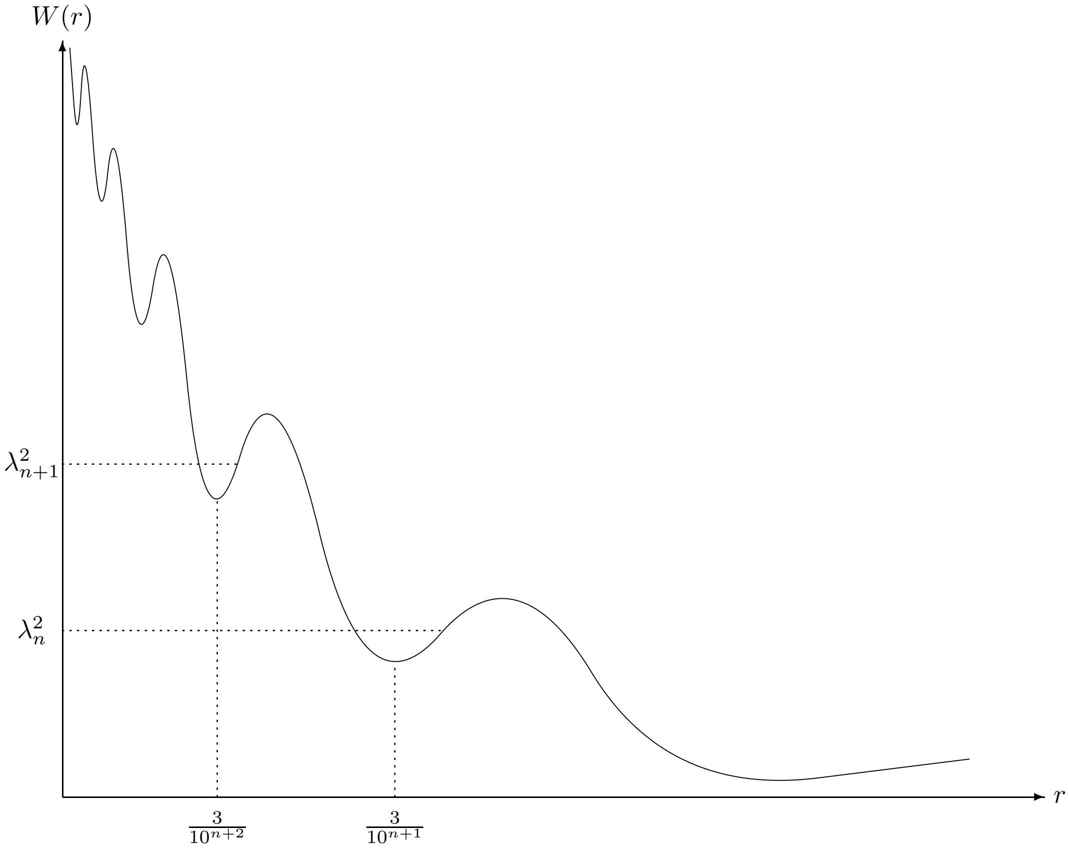

FIGURE 2. The potential W

Lemma 1. *The following inequalities hold:*

$$\forall j \in \mathbb{N},\ \exists C > 0,\ ||\frac{d^j f_n}{dr^j}||_{L^\infty(\mathbb{R}^d)} \leq C\alpha_n \lambda_n^{1+j}\left(q(\lambda_n)\right)^{\frac{d+1}{2}} e^{-\frac{\lambda_n}{20q(\lambda_n)}} \tag{34}$$

$$||u_n||_{L^1(\mathbb{R}^d)} \geq \frac{\alpha_n}{C}\lambda_n^{-1}\left(q(\lambda_n)\right)^{-\frac{d-1}{2}}. \tag{35}$$

Proof. According to (28), on the support of χ_n, W is equal to $b_n + \lambda_n^2$. Hence:

$$g_n(r) = -v_n''(r) + b_n(r)v_n(r). \tag{36}$$

Thus, using (22'):

$$f_n(r) = r^{-\frac{d-1}{2}} g_n(r) = r^{-\frac{d-1}{2}} \alpha_n \left(y_n(r)\chi_n''(r) + 2y_n'(r)\chi_n'(r)\right).$$

The derivative of f_n of order j is then of the form:

$$\frac{d^j f_n}{dr^j} = \sum_{j_1+j_2+j_3=j+1} \beta_{j_1,j_2,j_3} \frac{d^{j_1}}{dr^{j_1}} y_n \frac{d^{j_2}}{dr^{j_2}} \chi_n' \frac{d^{j_3}}{dr^{j_3}} r^{-\frac{d-1}{2}}. \tag{37}$$

On the support of χ_n':

$$\left|r - \frac{3}{10^{n+1}}\right| > \frac{1}{10^{n+1}} = \frac{1}{10q(\lambda_n)}.$$

So according to the bounds (23) of y and its derivatives:

$$|y_n^{(j_1)}(r)| \leq C\lambda_n^{(j_1)} e^{-\frac{\lambda_n}{2}\left|r-\frac{3}{10^{n+1}}\right|}$$
$$\leq \lambda_n^{j_1} e^{-\frac{\lambda_n}{20q(\lambda_n)}}$$

Furthermore,

$$|\chi_n^{(j_2+1)}| \leq C\left(q(\lambda_n)\right)^{j_2+1} \quad \leq C\lambda_n^{j_2} q(\lambda_n)$$
$$\left|\left(\frac{d}{dr}\right)^{j_3}\left(r^{-\frac{d-1}{2}}\right)\right| \leq C\left(q(\lambda_n)\right)^{\frac{d-1}{2}+j_3} \leq C\lambda_n^{j_3}\left(q(\lambda_n)\right)^{\frac{d-1}{2}},$$

for on the support of χ_n, $r \geq \frac{1}{10^{n+1}}$. These three inequalities, together with (37), implies (34).

We shall now prove (35). By the definition of y_n:

$$\frac{\lambda_n}{2}\left|r-\frac{3}{10^n}\right| \leq \frac{1}{2} \Rightarrow y_n(r) \geq m = \sup_{|s|\leq\frac{1}{2}} |y(s)|.$$

Furthermore, if r is as above, and n big enough, then $\chi_n(r) = 1$ and so:

$$u_n(r) = r^{-\frac{d-1}{2}} v_n(r) = r^{-\frac{d-1}{2}} \alpha_n y_n(r)$$

Hence:

$$||u_n||_{L^1} \geq m\alpha_n \int_{\lambda_n\left|r-\frac{3}{10^n}\right|\leq 1} r^{-\frac{d-1}{2}} r^{d-1} dr$$
$$\geq \frac{\alpha_n}{C}\left(10^{-n}\right)^{\frac{d-1}{2}} \lambda_n^{-1}.$$

□

Choose $M > 1$ and set:

$$q(\lambda) = \frac{\lambda}{M\log\lambda}. \tag{38}$$

The positive function q is strictly increasing for big λ's and satisfies (25) and (26). In addition, lemma 1 implies, bounding from above $q(\lambda_n)$ by λ_n:

$$||\frac{d^j f_n}{dr^j}||_{L^\infty} \leq C\alpha_n \lambda_n^{\frac{d+3}{2}+j-\frac{M}{20}}$$
$$||u_n||_{L^1} \geq \frac{\alpha_n}{C}\lambda_n^{\frac{d+1}{2}},$$

with new constants C, which may depend on M. So the conditions (9) and (10) of the theorem are satisfied if the constants M and α_n are well chosen. The support of u_n is that of χ_n, which is of the desired form:

$$\left\{c_1\frac{\log\lambda_n}{\lambda_n} \leq r \leq c_2\frac{\log\lambda_n}{\lambda_n}\right\},$$

if the support of χ is taken to be a segment.

The assertion (7) on the potential remains to be checked. We have the following approximation of the inverse function of q:

Lemma 2. *Let q be defined by (38). Then:*

$$q(\lambda)\log(q(\lambda)) \sim \frac{\lambda}{M},\ \lambda \to +\infty.$$

Proof.

$$\begin{aligned} q(\lambda)\log q(\lambda) &= q(\lambda)\log\left(\frac{\lambda}{M\log\lambda}\right) \\ &= q(\lambda)\left(\log\lambda - \log\log\lambda - \log M\right) \\ &\sim q(\lambda)\log\lambda, \end{aligned}$$

when λ goes to infinity. □

On the support of ψ_n, $q(\lambda_n) = 10^n$ and so:

$$\frac{1}{20r} \le q(\lambda_n) \le \frac{1}{r}$$

$$\frac{1}{20r}|\log r - \log 20| \le q(\lambda_n)\log(q(\lambda_n)) \le \frac{1}{r}|\log r|.$$

(We used that $s \mapsto s\log s$ is an increasing function for $s > e^{-1}$). Using lemma 2 and taking r close enough to 0 we get:

$$C^{-1}\frac{1}{r}|\log r| \le \lambda_n \le C\frac{1}{r}|\log r|.$$

Thus, by the definition (32) of W, $|b_n|$ being bounded from above by $\lambda_n^2/4$:

$$C^{-1}\frac{|\log r|^2}{r^2}\sum_{n\ge n_0}\psi_n(r) \le W(r) \le C\frac{|\log r|^2}{r^2}\sum_{n\ge n_0}\psi_n(r),$$

which implies, with (27), the inequality (7) on the potential W. Consequently V satisfies the same inequality.

Remark. To get quasi-modes of infinite order, we would have taken $q(\lambda) = \frac{\lambda}{(\log\lambda)^{1+\varepsilon}}$ and suitably modified lemma 2.

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