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Superstrings on NS5 backgrounds, deformed AdS$_3$ and holography*

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Abstract: We study a non-standard decoupling limit of the D1/D5-brane system, which interpolates between the near-horizon geometry of the D1/D5 background and the near-horizon limit of the pure D5-brane geometry. The S-dual description of this background is actually an exactly solvable two-dimensional (worldsheet) conformal field theory: {null-deformed $SL(2,\mathbb{R})$}$\times SU(2)\times T^4$ or $K3$. This model is free of strong-coupling singularities. By a careful treatment of the $SL(2,\mathbb{R})$, based on the better-understood $SL(2,\mathbb{R})/U(1)$ coset, we obtain the full partition function for superstrings on $SL(2,\mathbb{R})\times SU(2)\times T^4/\mathbb{Z}_2$. This allows us to compute the partition functions for the $J^3\bar{J}^3$ and $J^2\bar{J}^2$ deformations, as well as the full line of supersymmetric null deformations, which links the $SL(2,\mathbb{R})$ conformal field theory with linear-dilaton theory. The holographic interpretation of this setup is a renormalization-group flow between the decoupled NS5-brane world-volume theory in the ultraviolet (little string theory), and the low-energy dynamics of super Yang–Mills string-like instantons in six dimensions.

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Contents

1. Introduction

So far only few exact, solvable string supersymmetric backgrounds with a neat brane interpretation are known. The most popular is certainly the near-horizon limit of the NS5-brane background [1], which is an exact worldsheet conformal field theory based on $SU(2)_k \times U(1)_Q$ (a three-sphere plus a linear dilaton), and preserves 16 supercharges thanks to the $N = 4$ superconformal algebra on the worldsheet [2]. This background includes a strong-coupling region, that can be excised by distributing the five-branes either on a circle [3] [4] or on a spherical shell [5].

Another well-known exact string vacuum is the near-horizon geometry of NS5-branes wrapped on a four-torus (or on a $K3$ manifold) and fundamental strings [6] [7] [8] [9]. In type IIB string theory, one can use S-duality to map this solution to the D1/D5-brane system. The supersymmetry of this background is enhanced from 8 to 16 supercharges in the near-horizon limit. In this case the exact conformal field theory is $SL(2,\mathbb{R}) \times SU(2) \times U(1)^4$. However, until recently, the $SL(2,\mathbb{R})$ CFT [10] [11] was poorly understood (see [12] and references therein). Substantial progress in the determination of the correct Hilbert space of this theory was made in [13], [14] and [15]. The key ingredient, first used in [16], was the observation that one must add all the representations obtained by the spectral flow of the affine algebra $\widehat{SL}(2,\mathbb{R})_L \times \widehat{SL}(2,\mathbb{R})_R$. This allows to reconcile the unitarity bound on the spin of the $SL(2,\mathbb{R})$ representations ($0 < j \leq k/2$) with the requirement that the operator product algebra be closed.

A partition function for bosonic strings on thermal AdS_3 (i.e. $H_3^+/\mathbb{Z}$) was proposed in [14], by using the older result by [17]. In [18], the partition function for the axial coset $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ – whose target-space interpretation is a Euclidian two-dimensional black hole [19] – was analyzed in the same spirit; it allowed to extract the full spectrum in agreement with previous semi-classical analysis. However, as a consequence of the non-compact nature of the group, these partition functions are plagued with a divergence, which should be handled with care in order to obtain sensible results. Finding a modular-invariant partition function for $SL(2,\mathbb{R})$ that reproduces the spectrum found in [13] is to our knowledge still an open problem. One of the aims of the present paper is to fill this gap, which is a first step towards the complete understanding of superstrings on $SL(2,\mathbb{R}) \times SU(2) \times T^4$ or $K3$ as well as deformations of this background. The structure of the partition function will be understood from a different viewpoint, by using the orbifold language, and the supersymmetrization will be discussed by considering the extended superconformal algebra on the worldsheet.

The above two string backgrounds are in fact members of a family of conformal field theories interpolating between them both in space–time and in moduli space. These theories can be viewed as exact marginal deformations of the $SL(2,\mathbb{R})$ WZW model, driven by a left-right combination of null currents (i.e. currents generating null subgroups) [20]. The endpoint of this deformation gives the linear dilaton and two light-cone free coordinates [21]. This geometry corresponds actually to a near-horizon limit for the NS5-branes *only* (such a limit was also mentioned in [22]). We explain in this paper how this background can be obtained by a particular decoupling limit of the NS5/F1 background, actually the *little-string-theory decoupling limit in the presence of macroscopic fundamental strings*. In the D-brane picture, this limit involves also the decompactification of the torus. An important achievement of the present work is that this model can also be viewed as a regularization of the strong-coupling region of the NS5-brane theory, which thus provides an alternative to [5], with a better-controlled worldsheet conformal field theory.

This class of backgrounds clearly preserves a fraction of target-space supersymmetry, which is generically one quarter, enhanced to one half on the endpoints of the deformation. An interesting feature of this deformation is that it is completely fixed by the requirement of $N = 2$ superconformal invariance on the worldsheet. We will show that it reduces to an orthogonal rotation between the worldsheet bosons and fermions interpolating between the $N = 2$ superconformal algebras of $U(1)_Q \times \mathbb{R}^{1,1} \times SU(2)$ and $SL(2,\mathbb{R}) \times SU(2)$.

There has been in the last years a considerable renewal of interest for these theories because, besides their intrinsic interest as exact string backgrounds, they enter in several gauge/gravity dualities. The celebrated AdS/CFT correspondence [23] is a conjectured equivalence between the near-horizon geometry of the D3- or D1/D5-brane background (respectively $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_3 \times S^3 \times T^4$) and the extreme infra-red theory living on their world-volume, a superconformal gauge theory with maximal supersymmetry. In a similar fashion, a holographic duality between the decoupled NS5-brane world-volume theory – the so-called little string theory (LST) – and the linear-dilaton background has been conjectured in [24] [25].

The holographic interpretation of our setup is clear. The ultraviolet region of the holographic "gauge" dual corresponds to the asymptotic geometry, and is therefore the decoupled world-volume theory living on the NS5-branes. This theory is not a field theory, since there is no ultraviolet field-theoretic fixed point, and contains string-like excitations [26], hence the name "little string theory" [25]. In the type IIB case, this theory is described in the infra-red by a gauge theory in six dimensions with $U(N_5)$ gauge group and $N = (1, 1)$ supersymmetry. The standard NS5 background corresponds to a renormalization-group flow towards a free fixed point in the infra-red, whose dual picture is a strong-coupling region in the gravitational background.

In the case of the null deformation of $SL(2,\mathbb{R})$, the addition of fundamental strings in the background corresponds in the dual theory to a configuration of string-like instantons of the low-energy gauge theory. Therefore the physics near the infra-red fixed point is governed by the dynamics of the moduli space of these instantons. The effective theory in 1+1 dimensions is superconformal, hence the dilaton stops running. This CFT has been studied intensively in the last years, in particular in the context of black-hole quantum mechanics (see [27] for references).

The paper is organized as follows. In Sec. 2 we present the precise decoupling limit which leads to the background of interest, and explain why this background is an exact conformal theory. Section 3 is devoted to bosonic strings in AdS_3, with special emphasis to the derivation of a partition function for the $SL(2,\mathbb{R})$, where the spectrum is read off unambiguously at finite or infinite radius. Then we introduce the worldsheet fermions and discuss in Sec. 4 the superstrings in the background $\mathrm{AdS}_3 \times S^3 \times T^4/\mathbb{Z}_2$ with particular attention to the construction of extended worldsheet superconformal algebras. In Sec. 5 we give the partition functions for the $J^3\bar{J}^3$ and $J^2\bar{J}^2$ deformations of $SL(2,\mathbb{R})$, and study in detail its null deformation, which is the main motivation of this article. Null deformations of the supersymmetric background $\mathrm{AdS}_3 \times S^3 \times T^4$ are extensively discussed in Sec. 6, with a particular attention to the requirement of preserving $N=2$ superconformal algebra. Section 7 gives a brief outlook of the holographic interpretation of this superstring vacuum.

2. A new decoupling limit for the D1/D5-brane system

In this section, we will present a decoupling limit for the D1/D5-brane system or conversely the NS5/F1 dual configuration. In this limit, we obtain a line of exact conformal theories, which turn out to be connected by a marginal deformation. Supersymmetry properties and spectra will be analyzed later.

2.1 The supergravity solution and a partial near-horizon limit

We consider the D1/D5-brane system in type IIB string theory. The D5-branes extend over the coordinates $x, x^6, \ldots, x^9$, whereas the D1-branes are smeared along the four-torus spanned by $x^6, \ldots, x^9$. The volume of this torus is asymptotically $V = (2\pi)^4 \alpha'^2 v$. With these conventions, in the sigma-model frame, the supergravity solution at hand reads (metric, dilaton and Ramond–Ramond field strength):

$$d\tilde{s}^2 = \frac{1}{\sqrt{H_1 H_5}}\left(-dt^2 + dx^2\right) + \sqrt{H_1 H_5}\left(dr^2 + r^2 d\Omega_3^2\right) + \sqrt{\frac{H_1}{H_5}}\sum_{i=6}^{9}(dx^i)^2, \tag{2.1}$$

$$\mathrm{e}^{2\tilde{\phi}} = g_{\mathrm{s}}^2 \frac{H_1}{H_5}, \tag{2.2}$$

$$F_{[3]} = -\frac{1}{g_{\mathrm{s}}} dH_1^{-1} \wedge dt \wedge dx + 2\alpha' N_5 \Omega_3 \tag{2.3}$$

(Ω_3 is the volume form of the three-sphere) with

$$H_1 = 1 + \frac{g_{\mathrm{s}}\alpha' N_1}{v r^2}\ , \quad H_5 = 1 + \frac{g_{\mathrm{s}}\alpha' N_5}{r^2}.$$

The four-torus can be replaced by a Calabi–Yau two-fold $K3$, provided that the charges of the D1 and D5 branes are of the same sign. The near-horizon ($r \to 0$) string coupling constant and the ten-dimensional gravitational coupling constant are

$$g_{10}^2 = g_{\mathrm{s}}^2 \frac{N_1}{v N_5}\ , \quad 2\kappa_{10}^2 = (2\pi)^7 \mathrm{e}^{2\langle\tilde{\phi}\rangle}\alpha'^4.$$

The standard decoupling limit of Maldacena, which leads to the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [23], is

$$\begin{aligned} &\alpha' \to 0, \\ &U \equiv r/\alpha' \text{ fixed}, \\ &v \text{ fixed}. \end{aligned}$$

In this limit, the holographic description is a two-dimensional superconformal field theory living on the boundary of AdS_3 that corresponds to the world-volume theory of the D1/D5 system compactified on a T^4 whose volume is held fixed in Planck units [9] [28].

In order to reach a decoupling limit that corresponds to the near-horizon geometry for the D5-branes only, one has to consider the limit:

$$\begin{aligned} &\alpha' \to 0, \\ &U = r/\alpha' \text{ fixed}, \\ &g_{\mathrm{s}}\alpha' \text{ fixed}, \\ &\alpha'^2 v \text{ fixed}. \end{aligned} \tag{2.4}$$

The last condition is equivalent to keeping fixed the six-dimensional string coupling constant:

$$g_6^2 = \frac{g_{\mathrm{s}}^2}{v}.$$

Since the gravitational coupling constant vanishes in this limit, the world-volume theory decouples from the bulk. The geometrical picture of the setup is the following: as $v \to \infty$, the torus decompactifies and the density of D-strings diluted in the world-volume of the D5-branes goes to zero.

The string coupling remains finite in this near-horizon limit, while the asymptotic region is strongly coupled. A perturbative description, valid everywhere is obtained by S-duality. The supergravity solution (2.1), (2.2) in the S-dual frame reads:

$$ds^2 = \mathrm{e}^{-\tilde{\phi}} d\tilde{s}^2 = \frac{1}{g_{\mathrm{s}}} \left\{ \frac{1}{H_1}\left(-dt^2 + dx^2\right) + \alpha'^2 H_5 \left(dU^2 + U^2 d\Omega_3^2\right) + \sum_{i=6}^{9} (dx^i)^2 \right\}, \tag{2.5}$$

$$\mathrm{e}^{2\phi} = \frac{1}{g_{\mathrm{s}}^2} \frac{H_5}{H_1} \tag{2.6}$$

with (in the limit (2.4) under consideration)

$$H_1 = 1 + \frac{g_{\mathrm{s}} N_1}{\alpha' v U^2}\ , \quad H_5 = \frac{g_{\mathrm{s}} N_5}{\alpha' U^2}. \tag{2.7}$$

The expression (2.3) for the antisymmetric tensor remains unchanged but it stands now for a NS flux and we will trade $F_{[3]}$ for $H_{[3]}$.

We now introduce the new variables:

$$u = \frac{1}{U}\ , \quad X^{\pm} = X \pm T = \frac{x \pm t}{g_6\sqrt{N_1 N_5}},$$

and the following mass scale:

$$M^2 = \frac{g_{\mathrm{s}} N_1}{\alpha' v}. \tag{2.8}$$

In these coordinates, the solution (2.5), (2.6) and (2.3), with (2.7), reads:

$$\begin{aligned}
\frac{ds^2}{\alpha'} &= N_5\left\{\frac{du^2}{u^2}+\frac{dX^2-dT^2}{u^2+1/M^2}+d\Omega_3^2\right\}+\frac{1}{\alpha' g_{\rm s}}\sum_{i=6}^{9}(dx^i)^2,\\
{\rm e}^{2\phi} &= \frac{1}{g_{10}^2}\frac{u^2}{u^2+1/M^2},\\
\frac{H_{[3]}}{\alpha'} &= N_5\left\{\frac{2u}{\left(u^2+1/M^2\right)^2}du\wedge dT\wedge dX+2\Omega_3\right\}.
\end{aligned} \tag{2.9}$$

This is the geometry of a deformed AdS_3 times an $S^3\times T^4$. Asymptotically ($u\to 0$), it describes the near-horizon geometry of the NS5-brane background, $U(1)_Q\times\mathbb{R}^{1,1}\times SU(2)\times T^4$ *in its weakly coupled region*. In the $u\to\infty$ limit, the background becomes that of the NS5/F1 near-horizon: $SL(2,\mathbb{R})\times SU(2)\times T^4$, with a finite constant dilaton. In some sense, we are regulating the strong-coupling region of the NS5-brane background by adding an appropriate condensate of fundamental strings. As we already advertised, this regularization is an alternative to the one proposed in [5]; it avoids the spherical target-space wall of the latter, and replaces it by a smooth transition, driven by a marginal worldsheet deformation, as will become clear in Sec. 2.2.

Before going into these issues, we would like to address the question of supersymmetry. The configuration displayed in Eqs. (2.9) preserves by construction one quarter of supersymmetry. Consider indeed IIB supergravity. The unbroken supersymmetries correspond to the covariantly constant spinors for which the supersymmetry variations of the dilatino and gravitino vanish:

$$\delta\lambda = \left[\gamma^\mu\partial_\mu\phi\,\sigma^3-\frac{1}{6}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right]\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=0, \tag{2.10}$$

$$\delta\psi_\mu = \left[\partial_\mu+\frac{1}{4}\left({w_\mu}^{ab}-{H_\mu}^{ab}\sigma^3\right)\Gamma_{ab}\right]\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=0. \tag{2.11}$$

where σ^3 is the third Pauli matrix. The two supersymmetry generators η_1 and η_2 have the same chirality: $\Gamma^{11}\eta_{1,2}=\eta_{1,2}$.

Let us for example concentrate on the dilatino variation, Eq. (2.10):

$$\begin{aligned}
\delta\lambda &= \left[\Gamma^2\,{\rm e}_2^u\,\partial_u\phi\,\sigma^3-H_{uxt}\,{\rm e}_1^u\,{\rm e}_2^x\,{\rm e}_0^t\,\Gamma^1\Gamma^2\Gamma^0+H_{\theta\varphi\chi}\,{\rm e}_3^\theta\,{\rm e}_4^\varphi\,{\rm e}_5^\chi\,\Gamma^3\Gamma^4\Gamma^5\right]\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\\
&= \left[\frac{1/M^2}{u^2+1/M^2}\Gamma^2\,\sigma^3-\frac{u^2}{u^2+1/M^2}\Gamma^1\Gamma^2\Gamma^0+\Gamma^3\Gamma^4\Gamma^5\right]\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix},
\end{aligned}$$

where Latin indices $a,b,\ldots$ refer to the tangent-space orthonormal bases, with $\{0,1,2\}$ and $\{3,4,5\}$ corresponding to the AdS_3 and S^3 submanifolds. The two $SO(9,1)$ spinors are decomposed into $SO(1,1)\times SO(4)\times SO(4)_T$:

$$\mathbf{16}\to(+,\mathbf{2},\mathbf{2})+(+,\mathbf{2}',\mathbf{2}')+(-,\mathbf{2}',\mathbf{2})+(-,\mathbf{2},\mathbf{2}').$$

The first $SO(4)$ is the isometry group of the transverse space (coordinates $x^{2,\ldots,5}$) and $SO(4)_T$ corresponds to the four-torus.

In the infinite-deformation limit, $M^2 \to 0$, this equation projects out the $SO(4)$ spinors of one chirality:

$$\left[\sigma^3 + \Gamma^2\Gamma^3\Gamma^4\Gamma^5\right] \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0.$$

The surviving supersymmetry generators, are, for η_1, $(+, \mathbf{2}, \mathbf{2})$ and $(-, \mathbf{2}, \mathbf{2'})$, and for η_2, $(+, \mathbf{2'}, \mathbf{2'})$ and $(-, \mathbf{2'}, \mathbf{2})$.

In the opposite limit of undeformed AdS_3, $M^2 \to \infty$, we have instead:

$$\left[1 - \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\right] \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0,$$

which projects out the $SO(5,1)$ spinor (coordinates $x^{0,1,6,\ldots,9}$ of the five-brane world-volume) of left chirality, i.e. keeps the representations $(+, \mathbf{2'}, \mathbf{2'})$ and $(-, \mathbf{2}, \mathbf{2'})$ for both supersymmetry generators. For any finite value of the deformation, both projections must be imposed:

$$\left[\frac{1/M^2}{u^2 + 1/M^2}\left(\sigma^3 - \Gamma^2\Gamma^3\Gamma^4\Gamma^5\right) - \frac{u^2}{u^2 + 1/M^2}\left(\Gamma^0\Gamma^1 + \Gamma^2\Gamma^3\Gamma^4\Gamma^5\right)\right] \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0,$$

which breaks an additional half supersymmetry. The remaining supersymmetries are $(-, \mathbf{2}, \mathbf{2'})$ for η_1 and $(+, \mathbf{2'}, \mathbf{2'})$ for η_2. The gravitino equation gives no further restrictions as it should (it reduces to the Killing-spinor equation on S^3 and deformed AdS_3), and we are eventually left with one quarter of supersymmetry. Supersymmetry enhancement occurs only for the limiting backgrounds – $\mathrm{AdS}_3 \times S^3$ or three-sphere plus linear dilaton, which preserve one half of the original supersymmetry.

2.2 Exact conformal-field-theory description: a null deformation of $SL(2,\mathbb{R})$

We will now show that the deformed-AdS_3 factor in the background (2.9) is the target space of an exactly conformal sigma-model.

The action for a WZW model is in general

$$S = \frac{k}{16\pi}\int_{\partial\mathcal{B}} \mathrm{Tr}\left(g^{-1}dg \wedge * g^{-1}dg\right) + \frac{ik}{24\pi}\int_{\mathcal{B}} \mathrm{Tr}\left(g^{-1}dg\right)^3. \tag{2.12}$$

In the case of $SL(2,\mathbb{R})$, one can use the Gauss decomposition for the group elements:

$$g = g_- g_0 g_+ = \begin{pmatrix} 1 & 0 \\ x^- & 1 \end{pmatrix} \begin{pmatrix} 1/u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & x^+ \\ 0 & 1 \end{pmatrix}, \tag{2.13}$$

which provides the Poincaré coordinate system (see Appendix A). With this choice, the sigma-model action reads:

$$S = \frac{k}{2\pi}\int d^2z \left(\frac{\partial u \bar\partial u}{u^2} + \frac{\partial x^+ \bar\partial x^-}{u^2}\right). \tag{2.14}$$

As usual, the affine symmetry $\widehat{SL}(2,\mathbb{R})_L \times \widehat{SL}(2,\mathbb{R})_R$ is generated by weight-one currents. Since the group is non-compact, there are null directions, easily identified in the

Poincaré coordinates. The corresponding $\widehat{U}(1)_L \times \widehat{U}(1)_R$ symmetries are linearly realized and generated by the following *null currents*[1]*:*

$$J = \frac{\partial x^+}{u^2} \ , \quad \bar{J} = \frac{\bar{\partial} x^-}{u^2}. \tag{2.15}$$

The $(1,1)$ operator $J(z)\bar{J}(\bar{z})$, is truly marginal and can be used to generate a line of CFT's. Along this line, i.e. for finite values of the deformation parameter $1/M^2$, the geometry back-reaction must be taken into account. Integrating the corrections (much like in [29] and [30] for compact cases) one obtains the following null-deformed $SL(2,\mathbb{R})$-sigma-model action [20]:

$$S = \frac{k}{2\pi} \int d^2 z \left(\frac{\partial u \bar{\partial} u}{u^2} + \frac{\partial x^+ \bar{\partial} x^-}{u^2 + 1/M^2} \right). \tag{2.16}$$

The affine symmetry $\widehat{SL}(2,\mathbb{R})_L \times \widehat{SL}(2,\mathbb{R})_R$ is broken down to $\widehat{U}(1)_L \times \widehat{U}(1)_R$ and only the two null currents survive, which now read:

$$J = \frac{\partial x^+}{u^2 + 1/M^2} \ , \quad \bar{J} = \frac{\bar{\partial} x^-}{u^2 + 1/M^2}. \tag{2.17}$$

The deformed target-space geometry and antisymmetric tensor are read off directly from Eq. (2.16), and turn out to coincide with the deformed-AdS_3 factor in (2.9).

There is an alternative way to reach the same conclusion. As shown in [29], marginal deformations of a WZW model correspond to $O(d,d,\mathbb{R})$ transformations acting on the Abelian isometries. In order to implement the latter in the case at hand, we rewrite the $SL(2,\mathbb{R})$-WZW action as:

$$S = \frac{k}{2\pi} \int d^2 z \left(\frac{\partial u \bar{\partial} u}{u^2} + \partial \begin{pmatrix} x^+ x^- \end{pmatrix} \cdot E \cdot \bar{\partial} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \right) \tag{2.18}$$

with

$$E = \begin{pmatrix} 0 & 1/u^2 \\ 0 & 0 \end{pmatrix}.$$

Acting on the corresponding background with the following $O(2,2,\mathbb{R})$ element[2] (see e.g. [31] for a review):

$$g = \begin{pmatrix} \mathbb{I} & 0 \\ -\Theta/M^2 & \mathbb{I} \end{pmatrix} \ , \quad \text{with} \ \ \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.19}$$

we recover (2.18) with

$$E' = g(E) = \begin{pmatrix} 0 & \frac{1}{u^2 + 1/M^2} \\ 0 & 0 \end{pmatrix},$$

which is precisely the null-deformed $SL(2,\mathbb{R})$-WZW action, Eq. (2.16).

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[1]Strictly speaking, these are not Cartan generators. See Appendix A.

[2]The simplest setup for illustrating this transformation is flat background – compactified bosons: two light-cone coordinates with a constant B-field. The original Lagrangian, $4\pi\mathcal{L} = \partial x^+ \bar{\partial} x^-$, transforms into $4\pi\tilde{\mathcal{L}} = \frac{M^2}{1+M^2} \partial x^+ \bar{\partial} x^-$; this amounts to a shift of radii.

Notice finally that the deformed $SL(2,\mathbb{R})$-WZW model under consideration can also be obtained as a coset by gauging a $U(1)$ subgroup of $SL(2,\mathbb{R})\times U(1)$ [32]. We will come back later to this CFT to compute the one-loop partition function, which demands a careful treatment of global properties of the fields. The question of supersymmetry along these deformations needs also to be recast in the present, exact CFT framework, and will be discussed in Sec. 6.

3. Bosonic strings on AdS_3

This section is devoted to the bosonic part of the AdS_3 (i.e. $SL(2,\mathbb{R})$) component of the previous backgrounds. Despite many efforts and achievements (a short summary is given Appendix A), our understanding is not completely satisfactory. We show here how to reach information about $SL(2,\mathbb{R})$ starting from the better-understood $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ axial gauging. This enables us to provide a partition function for the $SL(2,\mathbb{R})$, which carries full information about the spectrum. Under this form, the $SL(2,\mathbb{R})$ WZW model resembles a $\mathbb{Z}_N$ freely acting orbifold of $T^2\times S^1\times S^1$ over $U(1)$, at large N. Spectra and partition functions of $SL(2,\mathbb{R})$ deformations will be addressed in Sec. 5.

3.1 $SL(2,\mathbb{R})$ from $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$

There is a tight link between the spectrum of string theory on AdS_3 and the spectrum of the axial-gauged coset $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ – non-compact parafermions. The states in the coset are those of the $SL(2,\mathbb{R})$ CFT with the restriction $J^3_n|\mathrm{state}\rangle=\bar{J}^3_n|\mathrm{state}\rangle=0$ for $n>0$, and the conditions on the zero modes $J^3_0+\bar{J}^3_0=-wk$ and $J^3_0-\bar{J}^3_0=n$. It is therefore possible to reconstruct the $SL(2,\mathbb{R})$ starting from its axial gauging, much like in the case of compact parafermions, where the $SU(2)/U(1)$ gauging enables for reconstructing the $SU(2)$ WZW model [33]. In the non-compact case, however, the coset was shown to be a unitary conformal field theory [34], whereas this holds for the $SL(2,\mathbb{R})$ only if Virasoro conditions are imposed [11] [35] [36] [37]. The physical states can be chosen, up to a spurious state, to be annihilated by the positive modes of the time-like current $J^3_{n>0}$, $\bar{J}^3_{n>0}$. This is the same as for the coset, except for the zero modes.

Our aim is here to show how a partition function for the $SL(2,\mathbb{R})$ can be reached starting from the partition function of the $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ proposed in [18]. We start with the WZW action (2.12) for $g\in SL(2,\mathbb{R})$ parameterized with Euler angles (see Eq. (A.6)). We will gauge the $U(1)$ axial subgroup $g\to hgh$ with $h=\mathrm{e}^{i\lambda\sigma_2/2}$. The action for the gauged model is

$$S(g,A)=S(g)+\frac{k}{2\pi}\int d^2z\,\mathrm{Tr}\left(A\bar{\partial}gg^{-1}+\bar{A}g^{-1}\partial g-Ag\bar{A}g^{-1}-A\bar{A}\right).$$

The gauge field is Hodge-decomposed as:

$$A=\partial(\tilde{\rho}+\rho)+\frac{i}{\tau_2}(u_1\bar{\tau}-u_2)\ ,\quad \bar{A}=\bar{\partial}(\tilde{\rho}-\rho)-\frac{i}{\tau_2}(u_1\tau-u_2).$$

After field redefinitions, the gauged-fixed action is given by an $SL(2,\mathbb{R})$ times a compact boson, with global constraints and a (b,c) ghost system. This theory is unitary and the corresponding target space is Euclidean.

The partition function has been computed by using path-integral techniques in [17] and [18]. We would like to summarize the method and remind the final result. Following [18] the model is transformed, for technical convenience, into a $U(1)$-gauging of the – non-unitary – $H_3^+ = SL(2,\mathbb{C})/SU(2)$ CFT:

$$\begin{aligned} S = &\frac{k}{2\pi}\int d^2z\left(\partial\phi\bar\partial\phi + \left(\partial\bar v + \bar v\partial\phi\right)\left(\bar\partial v + v\bar\partial\phi\right)\right) \\ &+ \frac{k}{2\pi}\int d^2z\partial\rho\bar\partial\rho + \frac{1}{\pi}\int d^2z\left(b\bar\partial c + \tilde b\partial\tilde c\right). \end{aligned}$$

The first part of the action is indeed the $H_3^+ = SL(2,\mathbb{C})/SU(2)$ sigma-model. The various fields acquire non-trivial holonomies from the gauge field, and can be decomposed as:

$$\begin{aligned} \phi &= \hat\phi + \frac{1}{4\tau_2}\left[(u_1\bar\tau - u_2)z + (u_1\tau - u_2)\bar z\right], \\ \rho &= \hat\rho + \frac{1}{4\tau_2}\left[(u_1\bar\tau - u_2)z + (u_1\tau - u_2)\bar z\right], \\ v &= \hat v \exp -\frac{1}{4\tau_2}\left[(u_1\bar\tau - u_2)z + (u_1\tau - u_2)\bar z\right]. \end{aligned}$$

The fields v and $\bar v$ give the following contribution to the partition function:

$$\begin{aligned} \det\left|\partial + \frac{1}{2\tau_2}(u_1\bar\tau - u_2) + \partial\hat\phi\right|^{-2} &= \det\left|\partial + \frac{1}{2\tau_2}(u_1\bar\tau - u_2)\right|^{-2}\exp\frac{2}{\pi}\int d^2z\partial\hat\phi\bar\partial\hat\phi \\ &= 4\eta\bar\eta\frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(u_1\tau - u_2))^2}}{\left|\vartheta_1(u_1\tau - u_2|\tau)\right|^2}\exp\frac{2}{\pi}\int d^2z\partial\hat\phi\bar\partial\hat\phi, \end{aligned}$$

where $\vartheta_1(\nu|\tau)$ is an elliptic theta function (see Appendix D). The periodicity properties of this determinant allows for breaking u_1 and u_2 into an integer and a compact real: $u_1 = s_1 + w$, $u_2 = s_2 + m$, $s_i \in [0,1)$. Taking finally into account the contributions of the free bosons ϕ and ρ and that of the ghosts, leads to the result [18]:

$$Z_{SL(2,\mathbb{R})/U(1)_{\mathrm{A}}} = 4\sqrt{k(k-2)}\,\eta\bar\eta\int d^2s\frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{\left|\vartheta_1(s_1\tau - s_2|\tau)\right|^2}\sum_{m,w=-\infty}^{+\infty}\mathrm{e}^{-\frac{k\pi}{\tau_2}\left|(s_1+w)\tau - (s_2+m)\right|^2}. \tag{3.1}$$

We can recast the latter in terms of the free-boson conformal blocks (B.1):

$$Z_{SL(2,\mathbb{R})/U(1)_{\mathrm{A}}} = 4\sqrt{(k-2)\tau_2}\,\eta\bar\eta\int d^2s\frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{\left|\vartheta_1(s_1\tau - s_2|\tau)\right|^2}\sum_{m,w\in\mathbb{Z}}\zeta\begin{bmatrix} w+s_1 \\ m+s_2\end{bmatrix}(k), \tag{3.2}$$

which meets our intuition that there are one compact and one non-compact bosons in the cigar geometry.

A few remarks are in order here. The integration over s_1, s_2 should be thought of as a constraint on the Hilbert space, which defines the non-compact parafermionic $\mathbb{Z}$ charge. The allowed parafermionic charges are $m = n/2$, $\bar m = -n/2$ for the unflowed sector, and $\tilde m = (n - wk)/2$ and $\tilde{\bar m} = -(n + wk)/2$ for the w-flowed sector. Another important issue is the logarithmic divergence originating from $s_1 = s_2 = 0$, and due to the non-compact nature of the group. Baring this divergence, the partition function (3.1) is modular-invariant, as can be easily checked by using the modular properties of Jacobi functions. Its content in terms of non-compact-parafermion discrete and continuous series can be further investigated (see [18] for details). Moreover, in the large-k (flat-space) limit, $Z_{SL(2,\mathbb{R})/U(1)_{\mathrm{A}}} \sim k\left(\pi\sqrt{\tau_2}\eta\bar{\eta}\right)^{-2}$ up to an infinite-volume factor[3]: we recover two free, uncompactified bosons.

We will now show that it is possible to recover a partition function for the $SL(2,\mathbb{R})$ WZW model, starting from the above result for the coset $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$. In many respects this is similar to what happens in the compact case: $SU(2)_k$ can be reconstructed as $\left(SU(2)_k/U(1)\times U(1)_{\sqrt{2k}}\right)/\mathbb{Z}_k$, where $\mathbb{Z}_k$ is the compact-parafermionic symmetry of the coset $SU(2)_k/U(1)$, and acts freely on the compact $U(1)_{\sqrt{2k}}$. To some extent, however, manipulations involving divergent expressions such as (3.1) can be quite formal, and require to proceed with care.

As we already pointed out, the states in $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ are those of $SL(2,\mathbb{R})$ that are annihilated by the modes J^3_n and $\bar{J}^3_n$ $n>0$, and have J^3_0 and $\bar{J}^3_0$ eigenvalues $(n-wk)/2$ and $-(n+wk)/2$. Therefore, in order to reconstruct the $SL(2,\mathbb{R})$ partition function, we need to couple the coset blocks with an appropriately chosen lattice for the Cartan generators J^3 and $\bar{J}^3$ corresponding to a free time-like boson. This coupling should mimic the $\mathbb{Z}_k$ free action that appears in the $SU(2)_k$ (see the discussion for the $SU(2)$ in [38]), in a non-compact parafermionic version, though. Since the non-compact parafermions have a $\mathbb{Z}$ symmetry, we consider here a $\mathbb{Z}$ free action. By using the conformal blocks for free bosons given in Appendix B (see Eq. (B.1)), we reach the following partition function for the *universal cover* of $SL(2,\mathbb{R})$, in the Lagrangian representation[4]:

$$
\begin{aligned}
Z_{SL(2,\mathbb{R})} &= 4\sqrt{\tau_2}(k-2)^{3/2}\int d^2s\, d^2t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau-s_2))^2}}{\left|\vartheta_1(s_1\tau-s_2|\tau)\right|^2}\times \\
&\times \sum_{m,w,m',w'\in\mathbb{Z}} \zeta\begin{bmatrix} w+s_1-t_1 \\ m+s_2-t_2\end{bmatrix}(k)\,\zeta\begin{bmatrix} w'+t_1 \\ m'+t_2\end{bmatrix}(-k)\,. \qquad (3.3)
\end{aligned}
$$

Modular invariance is manifest in this expression, since it has the structure of a freely acting orbifold (this can also be easily checked by using formulas of Appendices B and D). The extra $k-2$ factor comes along with the J^3, $\bar{J}^3$ contribution; it ensures the correct density scaling in the large-k limit, as explained in Appendix B about Eq. (B.3). We perform a Poisson resummation[5] on m and m', which are trade for n and n'. We define $n^{\pm}=n\pm n'$ and $w_{\pm}=w\pm w'$, and rewrite the partition function in the Hamiltonian

[3]This factor comes as $\int_{-\infty}^{+\infty}\frac{dxdy}{x^2+y^2}\exp -\pi(x^2+y^2)$.

[4]To find the partition function for the N-th cover, one has to replace the $\mathbb{Z}$ orbifold by a $\mathbb{Z}_{Nk}$ orbifold: $t_1\to\frac{\gamma}{Nk}$, $t_2\to\frac{\delta}{Nk}$. Then the left and right spectral flow are $w_+^{L,R}=w_+\pm N\ell$.

[5]The resummation on m' is of course performed by means of analytic continuation, as usual when dealing with a time-like direction.

representation:

$$Z_{SL(2,\mathbb{R})} = 4\sqrt{\tau_2}(k-2)^{3/2} \int d^2 s\, d^2 t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \sum_{n^\pm, w_\pm \in \mathbb{Z}} \mathrm{e}^{-i\pi\left(n^-(s_2-2t_2)+n^+s_2\right)} \times$$
$$\times\, \mathrm{e}^{-\frac{\pi\tau_2}{k}\left(n^+n^- + k^2(w_+ + s_1)(w_- + s_1 - 2t_1)\right) + i\pi\tau_1\left(n^-(w_- + s_1 - 2t_1) + n^+(w_+ + s_1)\right)}. \tag{3.4}$$

Expression (3.4) becomes more transparent by introducing light-cone directions with corresponding left and right lattice momenta:

$$P^+_{\mathrm{L,R}} = \frac{n^+}{\sqrt{2k}} \pm \sqrt{\frac{k}{2}} w_-, \tag{3.5}$$

$$P^-_{\mathrm{L,R}} = \frac{n^-}{\sqrt{2k}} \pm \sqrt{\frac{k}{2}} w_+. \tag{3.6}$$

In terms of these unshifted[6] momenta the partition function at hand reads:

$$Z_{SL(2,\mathbb{R})} = 4\sqrt{\tau_2}(k-2)^{3/2} \int d^2 s\, d^2 t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times$$
$$\times \sum_{n^\pm, w_\pm \in \mathbb{Z}} \mathrm{e}^{-i\pi\sqrt{\frac{k}{2}}\left(\left(P^+_{\mathrm{L}} + P^+_{\mathrm{R}}\right)s_2 + \left(P^-_{\mathrm{L}} + P^-_{\mathrm{R}}\right)(s_2 - 2t_2)\right)} \times$$
$$\times\, q^{\frac{1}{2}\left(P^+_{\mathrm{L}} + \sqrt{\frac{k}{2}}(s_1 - 2t_1)\right)\left(P^-_{\mathrm{L}} + \sqrt{\frac{k}{2}}s_1\right)} \bar{q}^{\frac{1}{2}\left(P^+_{\mathrm{R}} - \sqrt{\frac{k}{2}}(s_1 - 2t_1)\right)\left(P^-_{\mathrm{R}} - \sqrt{\frac{k}{2}}s_1\right)}. \tag{3.7}$$

3.2 Uncovering the spectrum

The latter expression is formally divergent. It is in fact a generalized function, which contains all the information about the spectrum. The integral over t_2 leads to the constraint $\delta_{n^-,0}$ and, due to the shift t_1, the "winding" $w_- - 2t_1$ is continuous. The momenta $P^-_{L,R}$ correspond therefore to a boson X^+ "compactified" at zero radius. Conversely, the other light-cone degree of freedom is compact. This *light-cone compactification* (for a related discussion, see [39]) is not so surprising. Indeed, the quantum numbers of a given state in the $SL(2,\mathbb{R})$ CFT, $j = \bar{j}$, $m + \bar{m}$, $m - \bar{m}$, w are respectively those of a Liouville field, a non-compact coordinate, and a compact one. Since we consider the universal cover of $SL(2,\mathbb{R})$, such that the time is non-compact, the only possibility is that one of the light-cone directions is compactified at radius $\sqrt{2k}$. Note also a particular feature of a two-dimensional lattice for two light-cone coordinates : if the radius of one light-cone coordinate shrinks to zero, the momenta and windings of the other light-cone coordinate are exchanged. This fact explains why the energy, as it appears in the partition function of $SL(2,\mathbb{R})$, is actually a (shifted) winding mode.

[6] Due to the shifted-orbifold structure of the partition function the relevant quantities are actually the shifted momenta:

$$P^{\mathrm{s}+}_{\mathrm{L,R}} = P^+_{\mathrm{L,R}} \pm \sqrt{\frac{k}{2}}(s_1 - 2t_1)\ , \quad P^{\mathrm{s}-}_{\mathrm{L,R}} = P^-_{\mathrm{L,R}} \pm \sqrt{\frac{k}{2}} s_1.$$

We will now proceed and analyze further the partition function given in Eq. (3.7). After integrating out t_2 we obtain:

$$Z_{SL(2,\mathbb{R})} = 4\sqrt{\tau_2}(k-2)^{3/2}\int d^2s \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2}\sum_{n^+,w_+\in\mathbb{Z}} \mathrm{e}^{-i\pi n^+(s_2-\tau_1(w_++s_1))}\times$$
$$\times\int_0^1 dt_1 \sum_{w_-\in\mathbb{Z}} \mathrm{e}^{-\pi\tau_2 k(w_++s_1)(w_-+s_1-2t_1)}. \tag{3.8}$$

It is straightforward to show that the large-k limit of this partition function is, up to the usual infinite-volume factor, $Z \sim k^{3/2}\pi\left(\pi\sqrt{\tau_2}\eta\bar\eta\right)^{-3}$. This was somehow built-in when writing (3.3) out of (3.2); it meets the expectations for ordinary flat-space spectrum.

It is possible to extract the spectrum of the theory at any finite k, and trace back its origin in terms of $SL(2,\mathbb{R})$ representations. We proceed along the lines of [14] and [18]. The precise derivation of the spectrum is given in Appendix C. Here we only collect the results.

Discrete representations. The discrete representations appear in the range $\frac{1}{2} < j < \frac{k-1}{2}$. Their conformal weights are the following:

$$L_0 = -\frac{j(j-1)}{k-2} + w_+\left(-\tilde m - \frac{k}{4}w_+\right) + N,$$
$$\bar L_0 = -\frac{j(j-1)}{k-2} + w_+\left(-\bar{\tilde m} - \frac{k}{4}w_+\right) + \bar N,$$

with $\tilde m + \bar{\tilde m} = -k(w - t_1)$ and $\tilde m - \bar{\tilde m} = n$.

Continuous representations. The continuous spectrum appears with the density of states:

$$\rho(s) = \frac{1}{\pi}\log\epsilon + \frac{1}{4\pi i}\frac{d}{ds}\log\frac{\Gamma\left(\frac{1}{2} - is - \tilde m\right)\Gamma\left(\frac{1}{2} - is + \bar{\tilde m}\right)}{\Gamma\left(\frac{1}{2} + is - \tilde m\right)\Gamma\left(\frac{1}{2} + is + \bar{\tilde m}\right)}$$

The weights of the continuous spectrum are

$$L_0 = \frac{s^2 + 1/4}{k-2} + w_+\left(-\tilde m - \frac{k}{4}w_+\right) + N,$$
$$\bar L_0 = \frac{s^2 + 1/4}{k-2} + w_+\left(-\bar{\tilde m} - \frac{k}{4}w_+\right) + \bar N,$$

with $\tilde m, \bar{\tilde m}$ as previously.

These results are in agreement with the unitary spectrum proposed in [13]. Here this spectrum was extracted straightforwardly from a modular-invariant partition function, constructed in the Lorentzian AdS_3 .

Our aim is now to better understand the coupling between the oscillators and the zero modes of the light-cone coordinates, as appearing in the partition function. To this end, we write the algebra $\widehat{SL}(2,\mathbb{R})_L$ by using the free-field representation of non-compact parafermions [40] [41]. The currents read:

$$J^{\pm} = -\left(\sqrt{\frac{k}{2}}\partial X \pm i\sqrt{\frac{k-2}{2}}\partial\rho\right) \mathrm{e}^{\pm i\sqrt{\frac{2}{k}}(X-T)}, \tag{3.9}$$

$$J^{3} = i\sqrt{\frac{k}{2}}\partial T, \tag{3.10}$$

with the following stress tensor:

$$T = \frac{1}{2}(\partial T)^2 - \frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial\rho)^2 + \frac{1}{\sqrt{2(k-2)}}\partial^2\rho. \tag{3.11}$$

There is a linear dilaton with background charge $Q = \sqrt{2/(k-2)}$ along the coordinate ρ. It contributes the central charge, which adds up to $c = 3 + 6/(k-2)$. The spectral flow symmetry can be realized by adding w_+ units of momentum along T:

$$J^3 \to J^3 - \frac{k}{2z}w_+ \ , \quad J^{\pm}(z) \to z^{\mp w_+} J^{\pm}(z).$$

The primary operators are those of a free-field theory with a peculiar zero-mode structure though, which is read off directly from the lattice component of the partition function, Eq. (3.4)[7]:

$$\exp\left\{\sqrt{\frac{2}{k-2}}j\rho + i\sqrt{\frac{2}{k}}\left[\frac{k}{4}w_+X^+ + \left(\tilde{m} - \frac{k}{4}w_+\right)X^- + \frac{k}{4}w_+\bar{X}^- + \left(\tilde{\bar{m}} - \frac{k}{4}w_+\right)\bar{X}^+\right]\right\}.$$

One should stress, however, that even if the theory can be represented with free fields, the descendants are constructed by acting with the modes of the affine currents: the oscillator number and the zero modes are shifted simultaneously. We are therefore lead[8] to use the the Lagrange multipliers s_1 and s_2 in the partition function (3.4) to enforce this twisting.

3.3 About the structure of the partition function

Our approach has been to build a modular-invariant partition function for $SL(2,\mathbb{R})$ starting from that of the coset model $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$. We have reached expression (3.3) or equivalently (3.7). These expressions are generalized functions which are formally divergent, as was originally the partition function for the coset, Eq. (3.2). However, the presence of a divergence is not an obstruction for uncovering the spectrum encoded in the partition functions, as shown in [18] for the coset and here for the AdS$_3$. In this section, we would like to make contact with the – not fully satisfactory – expressions found in [12] and [13], explain why the methods used previously failed, and clarify the underlying freely acting orbifold structure.

We start with Eq. (3.8) that we regulate by shifting $s_1\tau - s_2 \to s_1\tau - s_2 + \theta$ in the elliptic theta function. This breaks modular invariance unless, together with $\tau \to -1/\tau$, θ

[7]The fact that the roles of X^+ and X^- are reversed between the left-moving and the right-moving sectors will be explained in the fifth section. For the moment note that the partition function is invariant under: $\bar{X}^{\pm} \to \bar{X}^{\mp}$.

[8]This is close to the construction of gravitational waves, see [42].

transforms into $-\theta/\tau$. Then, summation over w_- and integration over t_1 can be merged into an integration over the *light-cone energy* $E = k(w_- - 2t_1)$, which is performed after analytic continuation. A Poisson resummation over n^+ finally leads to the following result:

$$\begin{aligned} Z_{SL(2,\mathbb{R})} &= 4\frac{(k-2)^{3/2}}{k\sqrt{\tau_2}} \int d^2s \sum_{m_+, w_+ \in \mathbb{Z}} \delta\left(w_+ + s_1\right) \delta\left(m_+ + s_2\right) \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2 + \theta))^2}}{\left|\vartheta_1(s_1\tau - s_2 + \theta|\tau)\right|^2} \\ &= 4\frac{(k-2)^{3/2}}{k\sqrt{\tau_2}} \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}\theta)^2}}{\left|\vartheta_1\left(\theta|\tau\right)\right|^2}. \end{aligned} \tag{3.12}$$

This result is precisely that of [12] and [13], up to an overall normalization.

In unitary conformal field theories, the partition function is usually decomposed in characters of the chiral holomorphic and anti-holomorphic algebras:

$$Z_{\mathrm{genus-one}}(\tau, \bar{\tau}) = \sum_{\mathrm{L,R}} \mathcal{N}_{\mathrm{L,R}} \chi_{\mathrm{L}}(\tau) \bar{\chi}_{\mathrm{R}}(\bar{\tau}),$$

where the summation is performed over all left-right representations present in the spectrum with multiplicities $\mathcal{N}_{\mathrm{L,R}}$, and $\chi(\tau)$ are the corresponding characters:

$$\chi(\tau) = \mathrm{Tr}_{\mathrm{rep}} q^{L_0 - \frac{c}{24}}.$$

This decomposition is very powerful for classifying models (i.e. multiplicities $\mathcal{N}_{\mathrm{L,R}}$) by following the requirements of modular invariance. From the path-integral point of view, different modular-invariant combinations correspond to different choices for boundary conditions on the fields. However, this decomposition relies on the very existence of the characters. This holds for WZW models on compact groups. It does not apply to the case of non-compact groups, unless the group is Abelian – free bosons. Then the zero-mode representations are one-dimensional, the characters of the affine algebra are well-defined, and the infinite-volume divergence decouples into an overall factor. For non-Abelian groups, unitary[9] representations of the zero-modes are infinite-dimensional, and the characters of the affine algebra diverge. This degeneracy can be lifted by switching on a source coupled to some Cartan generator:

$$\chi(\tau, \theta) = \mathrm{Tr}_{\mathrm{rep}} q^{L_0 - \frac{c}{24}} \mathrm{e}^{2i\pi\theta J_0}.$$

Such a regularization is not fully satisfactory because it alters modular-covariance and does not allow to cure the characters of the continuous part of the spectrum. Moreover, the best these generalized characters can do, is to lead (after formal manipulations) to expressions like (3.12). The information carried by the latter is quite poor: it diverges at $\theta \to 0$ and the divergence cannot be isolated as a volume factor; the large-k limit is obscure; only the discrete part of the spectrum seems to contribute. These caveats are avoided in the integral representation we have presented here (Eqs. (3.3) or (3.7)), which is closer in spirit

[9]If we give up unitarity, finite-dimensional zero-mode representations do exist, but Virasoro conditions do not eliminate all spurious states.

with the work of [14] for the thermal AdS_3. Although divergent, it is modular-invariant, contains a nice spectral decomposition and has a well-defined large-k limit in agreement with our expectations.

To close this chapter, we would like to comment on the freely acting orbifold structure implemented in Eq. (3.3). We consider for illustration the $\mathbb{Z}_N$ orbifold of a compact boson of radius R times a two-torus, presented in Appendix B. The $\mathbb{Z}_N$ acts as a twist on T^2 and as a shift on the orthogonal S^1. The partition function for this model is given by Eq. (B.12). Although it is not compatible with the symmetries of a two-dimensional lattice, we take formally the large-N limit of this expression. The first term of (B.12) vanishes while the sum over h, g in the second term becomes an integral over $s_1, s_2 \in [0,1]$. We also drop out the geometrical factor $\sin^2 \pi \frac{\Lambda(h,g)}{N}$ which is meaningless here, and find:

$$\lim_{N\to\infty} \frac{Z_{\mathbb{Z}_N}}{N} = 4 \int d^2 s \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \sum_{m,w\in\mathbb{Z}} \zeta \begin{bmatrix} w+s_1 \\ m+s_2 \end{bmatrix} (R^2) \, . \tag{3.13}$$

In order to make final contact with the partition function of the $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ (Eq. (3.2)) we must identify R^2 with k, and mod out a non-compact free boson, i.e. multiply (3.13) by $\sqrt{\tau_2}\,\eta\bar{\eta}$. We insist that we do not claim that the theory $SL(2,\mathbb{R})/U(1)_{\mathrm{A}}$ is the same as an freely acting orbifold of flat space, but only that the structure is very similar. An important difference is that the oscillators of the field X of the free-field representation (Eq. (3.9)) are twisted and its zero modes shifted simultaneously. This is not possible in flat space.

One can similarly understand the orbifold structure underlying the full $SL(2,\mathbb{R})$ model. To this end we consider the $\mathbb{Z}_N \times \mathbb{Z}_N$ model given in Appendix B. In the formal large-N limit, all but the last term vanish in the partition function (B.13); the sums over h_1, g_1 and h_2, g_2 are trade for integrals over s_1, s_2 and $t_1, t_2 \in [0,1]\times[0,1]$:

$$\begin{aligned}\lim_{N\to\infty} \frac{Z_{\mathbb{Z}_N\times\mathbb{Z}_N}}{N^2} &= 4\frac{1}{\eta\bar{\eta}} \int d^2 s\, d^2 t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times \\ &\times \sum_{m_1,w_1,m_2,w_2\in\mathbb{Z}} \zeta \begin{bmatrix} w_1+s_1-t_1 \\ m_1+s_2-t_2 \end{bmatrix} (R_1^2)\, \zeta \begin{bmatrix} w_2+t_1 \\ m_2+t_2 \end{bmatrix} (R_2^2) \, .\end{aligned} \tag{3.14}$$

Comparison with the partition function of the $SL(2,\mathbb{R})$ (Eq. (3.3)) is possible provided we identify in (B.13), R_1^2 with k, R_2^2 with $-k$, and mod out a non-compact free boson, i.e. multiply Eq. (3.14) by $\sqrt{\tau_2}\,\eta\bar{\eta}$.

The function $\vartheta_1(0,\tau)$, which is identically zero, never appears in the orbifold since its corresponds to the untwisted, unprojected sector and is replaced by the usual toroidal partition sum:

$$\frac{\Gamma_{2,2}(T,U)}{\eta^4\bar{\eta}^4}.$$

In the case of $SL(2,\mathbb{R})$ (the $N \to \infty$ limit), the sum on the sectors is replaced by an integral over s_1 and s_2. The integration over the energy picks up precisely the untwisted, unprojected sector, as it gives the constraint $\delta^{(2)}(s_1\tau - s_2)$. We can rewrite formally the integrated partition function in terms of the functional determinant for the twisted bosons:

$$Z = \frac{(k-2)^{3/2}}{k\sqrt{\tau_2}\eta\bar{\eta}} \int d^2s\ \delta^{(2)}(s_1\tau - s_2)\ \det\left|\partial + \frac{1}{2\tau_2}(s_1\bar{\tau} - s_2)\right|^{-2} = \frac{(k-2)^{3/2}}{k\sqrt{\tau_2}\eta\bar{\eta}}\ \det|\partial|^{-2}.$$

Thus we find that the partition function of $SL(2,\mathbb{R})$ is the same as a linear dilaton and two light-cone free coordinates [10].

4. Superstrings on $\mathrm{AdS}_3 \times S^3 \times T^4$

We develop in this chapter the supersymmetry tools, which are needed for studying superstrings on NS5- or NS5/F1-brane backgrounds, as well as on the deformed AdS_3 geometries interpolating between them. This includes explicit realizations of extended $N = 2$ and $N = 4$ supersymmetry algebras. In principle it is possible to construct a space–time supersymmetric string background with $SL(2,\mathbb{R}) \times SU(2) \times \mathcal{M}$, where $\mathcal{M}$ is any $N = 2$ superconformal field theory with the correct central charge $\hat{c} = 4$. To make contact with the NS5/F1 background, we can choose either T^4 or a CFT realization of $K3$. We finally present the partition function of the model on $\mathrm{AdS}_3 \times S^3 \times K3$.

4.1 Extended superconformal algebras

Since the $\mathrm{AdS}_3 \times S^3 \times T^4$ background preserves one half of the supersymmetry, the world-sheet theory should factorize into an $N = 4$ superconformal theory with $\hat{c} = 4$ and an $N = 2$ free theory with $\hat{c} = 2$ [43]. An explicit realization of the relevant extended algebras is necessary in order to prove that supersymmetry survives at the string level; it is also important for the determination of the couplings between the bosonic and the fermionic degrees of freedom. However, as we will see, the straightforward application of the rules of $N = 2$ constructions is not the correct way to implement space–time supersymmetry in $SL(2,\mathbb{R}) \times SU(2)$.

4.1.1 NS5 background

We would like here to remind the construction of the "small" $N = 4$ superconformal algebra for the wormhole background [46], which is the simplest case beyond flat space [44]; it is also relevant for discussing the supersymmetric null deformation of $SL(2,\mathbb{R})$.

The near-horizon geometry of the NS5-brane background is the target space of an exactly conformal sigma-model based on $SU(2)_k \times U(1)_Q \times U(1)^6$ [1]. Let us concentrate on the $SU(2)_k \times U(1)_Q$ factor. The full algebra of this four-dimensional internal subspace consists of the bosonic $SU(2)_k$, the Liouville coordinate, and four free fermions:

$$\begin{aligned} J^i(z)J^j(0) &\sim \frac{k}{2}\frac{\delta^{ij}}{z^2} + i\sum_{\ell=1}^{3}\frac{\epsilon^{ij\ell}J^\ell(0)}{z}\ , \quad i,j,\ell = 1,2,3, \\ \partial\rho(z)\partial\rho(0) &\sim -\frac{1}{z^2}, \\ \psi^a(z)\psi^b(0) &\sim \frac{\delta^{ab}}{z}\ , \quad a,b = 1,\dots,4. \end{aligned}$$

[10] As already stressed, there is a central charge deficit coming for the other CFT's defining the string theory which corresponds to the lowest-weight of Liouville continuous representations [2].

The "small" $N = 4$ algebra is generated by twisting the "large" $N = 4$ algebra [45] based on the affine symmetry $SU(2)_{k+1} \times SU(2)_1 \times U(1)$. The generators of that algebra read:

$$\begin{aligned}
T &= \frac{1}{k+2}\sum_{i=1}^{3} J^i J^i - \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}\sum_{a=1}^{4}\psi^a\partial\psi^a, \\
G^i &= \sqrt{\frac{2}{k+2}}\left[-J^i\psi^4 + \sum_{j,\ell=1}^{3}\epsilon^{ij\ell}\left(J^j - \psi^4\psi^j\right)\psi^\ell\right] + i\psi^i\partial\rho, \\
G^4 &= \sqrt{\frac{2}{k+2}}\sum_{i=1}^{3}\left[J^i\psi^i + \frac{1}{3}\sum_{j,\ell=1}^{3}\epsilon^{ij\ell}\psi^i\psi^j\psi^\ell\right] + i\psi^4\partial\rho, \\
S^i &= \frac{1}{2}\left(\psi^4\psi^i + \frac{1}{2}\sum_{j,\ell=1}^{3}\epsilon^{ij\ell}\psi^j\psi^\ell\right), \\
\tilde{S}^i &= \frac{1}{2}\left(\psi^4\psi^i - \frac{1}{2}\sum_{j,\ell=1}^{3}\epsilon^{ij\ell}\psi^j\psi^\ell\right) + J^i,
\end{aligned} \tag{4.1}$$

where ρ is an ordinary bosonic coordinate.

The large $N = 4$ algebra is contracted to the required small $N = 4$, provided the ρ coordinate is promoted to a Liouville field by adding a background charge Q (i.e. a linear dilaton in the corresponding direction). The effect on the algebra (4.1) is the following:

$$T \to T - Q\partial^2\rho\ , \quad G^a \to G^a - iQ\partial\psi^a.$$

The background charge Q is such that we obtain a $\hat{c} = 4$ theory: $Q = \sqrt{2/(k+2)}$. The linear dilaton background compensates the central charge deficit of the $SU(2)_k$.
We bosonize the self-dual combination of fermions:

$$i\sqrt{2}\partial H^+ = \psi^1\psi^2 + \psi^4\psi^3.$$

It defines the R-symmetry $SU(2)$ algebra *at level one* generated by the currents:

$$(S^3, S^\pm) = \left(\frac{i}{\sqrt{2}}\partial H^+, \mathrm{e}^{\pm i\sqrt{2}H^+}\right). \tag{4.2}$$

The resulting "small" ($N = 4$)-algebra generators are combined into two conjugate $SU(2)$ R-symmetry doublets:

$$G^+, \tilde{G}^- = \left[QJ^-\mathrm{e}^{\frac{i}{\sqrt{2}}H^-} + i\left(Q\left(J^3 + i\sqrt{2}\partial H^-\right) - i\partial\rho\right)\mathrm{e}^{\frac{-i}{\sqrt{2}}H^-}\right]\mathrm{e}^{\frac{\pm i}{\sqrt{2}}H^+}, \tag{4.3}$$

$$\tilde{G}^+, G^- = \left[QJ^+\mathrm{e}^{\frac{-i}{\sqrt{2}}H^-} + i\left(Q\left(J^3 + i\sqrt{2}\partial H^-\right) + i\partial\rho\right)\mathrm{e}^{\frac{i}{\sqrt{2}}H^-}\right]\mathrm{e}^{\frac{\pm i}{\sqrt{2}}H^+}. \tag{4.4}$$

4.1.2 NS5/F1 background

We now move to the $\mathrm{AdS}_3 \times S^3 \times T^4$ background, which describes the near-horizon geometry of the NS5/F1-brane system. Our focus is the six-dimensional $SL(2,\mathbb{R}) \times SU(2)$ subspace that we want to split into one $\hat{c}=4$ system with $N=4$ superconformal symmetry and one with $\hat{c}=2$ with $N=2$. The total central charge of this factor is given by:

$$\hat{c} = \frac{3k_{SL(2,\mathbb{R})}}{k_{SL(2,\mathbb{R})}-2} + \frac{3k_{SU(2)}}{k_{SU(2)}+2}. \tag{4.5}$$

Therefore to obtain a critical string background ($\hat{c}=6$) for any level, we must choose

$$k_{SL(2,\mathbb{R})} - 4 = k_{SU(2)} = k. \tag{4.6}$$

The $N=1$ algebra of the theory is generated by

$$T = \frac{1}{k+2} I_i I_j \delta^{ij} + \frac{1}{k-2} J_\alpha J_\beta \eta^{\alpha\beta} - \frac{1}{2}\psi_i \partial\psi^i - \frac{1}{2}\chi_\alpha \partial \chi^\alpha, \tag{4.7}$$

$$G = \sqrt{\frac{2}{k+2}} \left[\psi_i I^i - \frac{i}{3}\epsilon_{ij\ell}\psi^i\psi^j\psi^\ell + \chi_\alpha J^\alpha - \frac{i}{3}\epsilon_{\alpha\beta\gamma}\chi^\alpha\chi^\beta\chi^\gamma \right], \tag{4.8}$$

where I^i and J^α denote respectively the bosonic currents of $SU(2)$ and $SL(2,\mathbb{R})$, ψ^i and χ^α the corresponding fermions, and $\eta^{\alpha\beta} = (+,+,-)$[11].

The $N=2$ algebras of $SL(2,\mathbb{R})\times SU(2)$. The various currents provided by the $SU(2)$ and $SL(2,\mathbb{R})$ algebras and the associated fermions allow for extracting one $N=2$, $\hat{c}=2$ algebra generated by:

$$G_2^\pm = \frac{1}{\sqrt{2(k+2)}} \left[\left(I^3 + \psi^+\psi^-\right) \mp \left(J^3 + \chi^+\chi^-\right) \right] \left(\psi^3 \pm \chi^3\right), \tag{4.9}$$

$$J_2 = \psi^3 \chi^3. \tag{4.10}$$

We have combined the currents and the fermions as follows:

$$J^\pm = J^1 \pm i J^2 \ , \quad I^\pm = I^1 \pm i I^2 \ , \quad \psi^\pm = \frac{\psi^1 \pm i\psi^2}{\sqrt{2}} \ , \quad \chi^\pm = \frac{\chi^1 \pm i \chi^2}{\sqrt{2}}.$$

The remaining generators form another $N=2$ algebra decoupled from the first one [48]:

$$G_4^\pm = \frac{1}{\sqrt{k+2}} \left[I^\mp \psi^\pm - i J^\mp \chi^\pm \right], \tag{4.11}$$

$$S^3 = \frac{1}{2(k+2)} \left(2J^3 + (k+4)\chi^+\chi^- - 2I^3 + k\psi^+\psi^- \right). \tag{4.12}$$

The various coefficients in S^3 are such that: (i) S^3 is regular with respect to $G_2^\pm$ in order to obtain two independent algebras, and (ii) $S^3(z)G_4^\pm(0) \sim \pm\, G_4^\pm(0)/2z$. The normalization of S^3 follows from $S^3(z)S^3(0) \sim 1/2z^2$. Therefore we rewrite it in terms of a free boson as follows:

$$\begin{aligned} \frac{i}{\sqrt{2}}\partial H^+ &= \frac{1}{2}\left(\psi^+\psi^- + \chi^+\chi^-\right) + \frac{1}{k+2}\left(J^3 + \chi^+\chi^- - I^3 - \psi^+\psi^-\right) \\ &= \frac{1}{2}\left(\psi^+\psi^- + \chi^+\chi^-\right) + \frac{1}{k+2}\left(\mathcal{J}^3 - \mathcal{I}^3\right). \end{aligned} \tag{4.13}$$

[11]Indices $i,j,\ldots$ and $\alpha,\beta,\ldots$ run both over $1,2,3$, and we raise them with δ^{ij} and $\eta^{\alpha\beta}$.

We have introduced the total currents, including the fermionic part:

$$\mathcal{J}^3 = J^3 + \chi^+\chi^- \quad \text{and} \quad \mathcal{I}^3 = I^3 + \psi^+\psi^-, \tag{4.14}$$

respectively for $SL(2,\mathbb{R})$ and $SU(2)$, both at level $k+2$.

We would like to extend the superconformal symmetry to $N=4$, as we expect from the target-space supersymmetry. This is possible provided the R-symmetry current $i\partial H^+$ corresponds to a free boson compactified at the self-dual radius, which seems indeed the case here since the total currents $\mathcal{J}^3$ and $\mathcal{I}^3$ are correctly normalized in Eqs. (4.13), and can both be bosonized at radius $\sqrt{2(k+2)}$. However, in order to form the superconformal characters of $SU(2)$ and $SL(2,\mathbb{R})$, the characters of these currents are coupled to the $N=2$ coset theories:

$$\frac{SU(2)}{U(1)} \times U(1)_{R=\sqrt{\frac{k}{k+2}}} \quad \text{and} \quad \frac{SL(2,\mathbb{R})}{U(1)} \times U(1)_{R=\sqrt{\frac{k+2}{k}}}.$$

The coupling acts as a $\mathbb{Z}_{k+2}$ shift in the lattice of $\mathcal{J}^3$ and $\mathcal{I}^3$; it is similar to the discussion about the bosonic $SL(2,\mathbb{R})$. Therefore, the charges of the current S^3 are not those of a boson at self-dual radius.

Space–time supersymmetry. For a worldsheet superconformal theory with *accidental* $N=2$ superconformal symmetry, the space–time supersymmetry charges are obtained by spectral flow of the R-symmetry current [49]. However, this requires that the charges of all the physical states with respect to this $U(1)$ current are *integral*. As we have seen above, the R-symmetry current of the $\hat{c}=4$ block, whose expression is given in (4.13) does not fulfill this requirement, because its charges depend on the eigenvalues of $\mathcal{I}^3$ and $\mathcal{J}^3$. Moreover, space–time supercharges based on the above $N=2$ current lead to incompatibilities with target-space symmetries [50]. These problems arise because, even in flat space, the space–time supercharges are constructed with the fermion vertex operators at zero momentum. In the present case, the space–time momentum enters directly in the $N=2$ charges and the $SU(2)$ charges, though compact, cannot be considered as internal.

We will proceed as in [9]: we construct directly the space–time supercharges with the spin fields of the free fermions, which are BRST invariant and mutually local. This seems sensible, since the fermions in a critical string theory based on WZW models are free. The $N=2$ current of the theory:

$$J = J_2 + 2S^3 = \frac{2}{k+2}\left(\mathcal{J}^3 - \mathcal{I}^3\right) + \psi^+\psi^- + \chi^+\chi^- + \psi^3\chi^3,$$

differs from the free-fermionic one by the "null" contribution $\left(\mathcal{J}^3 - \mathcal{I}^3\right)/(k+2)$. We can of course wonder whether a more appropriate choice of $N=2$ structure exists. Another choice of complex structure does exist, and is provided by decomposing the $SL(2,\mathbb{R})$ as $SL(2,\mathbb{R})/O(1,1) \times U(1)$. It suffers, however, from the same problem.

The required projections in $SL(2,\mathbb{R}) \times SU(2) \times U(1)^4$. Here we come to the full background $\mathrm{AdS}_3 \times S^3 \times T^4$. The worldsheet fermions of the T^4 are bosonized as:

$$\lambda^1\lambda^2 = \partial H_3\ , \quad \lambda^3\lambda^4 = \partial H_4,$$

and those of $SU(2) \times SL(2,\mathbb{R})$ as:

$$\psi^+\psi^- = i\partial H_2 \ , \quad \chi^+\chi^- = i\partial H_1 \ , \quad \psi^3\chi^3 = i\partial H_0.$$

In the $-1/2$ picture, the spin fields are

$$\Theta_\varepsilon(z) = \exp\left\{-\frac{\varphi}{2} + \frac{i}{2}\sum_{\ell=0}^{4} \varepsilon_\ell H_\ell\right\}, \tag{4.15}$$

where $\mathrm{e}^{-\varphi/2}$ is the bosonized superghost ground state in the Ramond sector [51].

The standard GSO projection keeps all spin fields satisfying

$$\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1, \tag{4.16}$$

which is required by BRST invariance. In type IIB superstrings, the GSO projection is the same on the spin fields from the right-moving sector, while for type IIA it is the opposite. In the $\mathrm{AdS}_3 \times S^3 \times T^4$ background, the fields (4.15) must obey the additional relation:

$$\varepsilon_0\varepsilon_1\varepsilon_2 = 1. \tag{4.17}$$

Equivalently, for type IIB, by using the GSO projection (4.16), the restriction on the allowed spin fields can be imposed on the fermions of the four-torus:

$$\varepsilon_3\varepsilon_4 = 1. \tag{4.18}$$

Relations (4.17) or (4.18) ensure the absence of $1/z^{\frac{3}{2}}$ poles in the OPE of the spin fields with the $N = 1$ supercurrent (4.8) that would otherwise appear as a consequence of the torsion terms. Note also that in the superconformal algebra for the right-moving sector, the torsion terms come with a negative sign in the supercurrent, but the projection remains the same.

In order to preserve the $N = 1$ supercurrent, we must implement the projection (4.18) as a $\mathbb{Z}_2$ orbifold on the coordinates of the four-torus. This is why we are effectively dealing with the background $\mathrm{AdS}_3 \times S^3 \times K3$. Then, from the S-dual viewpoint, the model we are describing consists of D5-branes wrapped on a $K3$ manifold and D-strings. It is known [53] that in this case there is an *induced* D-string charge which is the opposite of the total D5-brane charge. Indeed, including gravitational corrections to the Wess–Zumino term of the D5-brane action, generates the following D1-brane charge:

$$Q_1^{\mathrm{ind}} = N_5 \int \frac{c_2(K3)}{24} = -N_5.$$

Therefore the net number of D1-branes accompanying the D5-branes is $N_1 + N_5$. This has little effect on the S-dual model under consideration here, because the number of D-strings affects the value of the string coupling but not the worldsheet CFT.

In conclusion, we have seen that, in order to obtain a supersymmetric spectrum consistent with the BRST symmetry, we have to project out half of the space–time spinors from the Ramond sector. By modular invariance, this projection must act as an orbifold on the fermionic characters and also on the bosonic part to be consistent with superconformal invariance. For simplicity we have chosen to act on the fermions associated with the four-torus; therefore the background is now $\mathrm{AdS}_3\times S^3\times T^4/\mathbb{Z}_2$. The $T^4/\mathbb{Z}_2$ orbifold can be replaced by another realization of $K3$ in CFT, such as a Gepner model [49]. Another way to realize the projection is to twist the fermionic characters associated with $\mathrm{AdS}_3\times S^3$. In that case, we have to act nontrivially on the $SL(2,\mathbb{R})$ and $SU(2)$ bosonic characters. Then the toroidal part of the background remains untwisted: the NS5-branes are wrapped on a T^4 rather than on a $K3$.

4.2 The partition function for superstrings on $\mathbf{AdS_3}\times S^3\times K3$

We are now in position to write the partition function for type IIB superstrings on $\mathrm{AdS}^3\times S^3\times K3$. We must combine the various conformal blocks in a modular-invariant way, and impose the left and right GSO projections together with the additional projections dictated by the presence of torsion.

The standard orbifold conformal blocks are given in Appendix B. The $SU(2)_k$ partition function is chosen to be the diagonal modular-invariant combination [54] and the $SL(2,\mathbb{R})$ factor was discussed in Sec. 3. Putting everything together, including the conformal and superconformal ghosts, we obtain:

$$\begin{aligned}
Z_{\mathrm{IIB}} =& \frac{\mathrm{Im}\tau}{\eta^2\bar\eta^2} Z_{SU(2)} Z_{SL(2,\mathbb{R})} \frac{1}{2}\sum_{h,g=0}^{1} Z^{\mathrm{twisted}}_{T^4/\mathbb{Z}_2}\begin{bmatrix}h\\g\end{bmatrix} \\
&\times \frac{1}{2}\sum_{a,b=0}^{1}(-)^{a+b}\,\vartheta^2\begin{bmatrix}a\\b\end{bmatrix}\vartheta\begin{bmatrix}a+h\\b+g\end{bmatrix}\vartheta\begin{bmatrix}a-h\\b-g\end{bmatrix} \\
&\times \frac{1}{2}\sum_{\bar a,\bar b=0}^{1}(-)^{\bar a+\bar b}\,\bar\vartheta^2\begin{bmatrix}\bar a\\\bar b\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a+h\\\bar b+g\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a-h\\\bar b-g\end{bmatrix}.
\end{aligned} \tag{4.19}$$

We can read from the latter expression the spectrum of chiral primaries with respect to the space–time superconformal algebra [9]. The vertex operators for such left-moving states in the NS sector are given in the (-1) ghost picture by (see [57]):

$$\begin{aligned}
\mathcal{V}^I_j &= \mathrm{e}^{-\varphi}\ \lambda^I\ \Phi^{SL(2,\mathbb{R})}_{j,\ m}\ \Phi^{SU(2)}_{j-1,\ m'} \\
\mathcal{W}^\pm_j &= \mathrm{e}^{-\varphi}\ \left[\chi\Phi^{SL(2,\mathbb{R})}\right]_{j\pm1,\ m}\ \Phi^{SU(2)}_{j-1,\ m'} \\
\mathcal{X}^\pm_j &= \mathrm{e}^{-\varphi}\ \Phi^{SL(2,\mathbb{R})}_{j,\ m}\ \left[\psi\Phi^{SU(2)}\right]_{j-1\pm1,\ m'}
\end{aligned}$$

where $\Phi^{SU(2)}_{j',m'}$ and $\Phi^{SL(2,\mathbb{R})}_{j,m}$ are respectively the bosonic primary fields of the holomorphic current algebras $SU(2)_k$ and $SL(2,\mathbb{R})_{k+4}$. They are combined with the worldsheet fermions into representations of $SU(2)_{k+2}$ and $SL(2,\mathbb{R})_{k+2}$.

The above states live in the five-plus-one dimensional world-volume of the NS5-branes. In order to obtain the closed string spectrum, we tensorize this left-moving spectrum with the right-moving one, and impose the $\mathbb{Z}_2$ projection on the torus. Additional states localized in one-plus-one dimensions are constructed with the twisted sectors of the orbifold:

$$\begin{aligned}\tilde{\mathcal{W}}_j^{\pm} &= \mathrm{e}^{-\varphi}\ \mathcal{V}^{\ tw} \left[\psi \Phi^{SU(2)}\right]_{j\pm1,\ m} \ \Phi^{SL(2,\mathbb{R})}_{j-1,\ m'},\\ \tilde{\mathcal{X}}_j^{\pm} &= \mathrm{e}^{-\varphi}\ \mathcal{V}^{\ tw}\ \Phi^{SU(2)}_{j,\ m} \ \left[\chi \Phi^{SL(2,\mathbb{R})}\right]_{j-1\pm1,\ m'},\end{aligned}$$

where $\mathcal{V}^{\ tw}$ are the twist fields of the NS sector.

5. Marginal deformations of $SL(2,\mathbb{R})$

The $SL(2,\mathbb{R})$ geometry in Euler (global) coordinates reads[12]:

$$\begin{aligned}ds^2 &= dr^2 - \cosh^2 r dt^2 + \sinh^2 r d\phi^2,\\ B &= \cosh^2 r d\phi \wedge dt\end{aligned}$$

and there is no dilaton. Strictly speaking the time coordinate t is 2π-periodic for the $SL(2,\mathbb{R})$, but non-compact for its universal covering (AdS_3); ϕ is 2π-periodic and $r > 0$.

Conformal deformations of this background are generated by truly marginal operators i.e. dimension-$(1,1)$ operators that survive their own perturbation. In the presence of holomorphic and anti-holomorphic current algebras, marginal operators are constructed as products $J^{\alpha}\bar{J}^{\beta}$, not all being necessarily truly marginal. In the $SU(2)$ WZW model, nine marginal operators do exist. However, they are related by $SU(2) \times SU(2)$ symmetry to one of them, say $J^3\bar{J}^3$. Hence, only one line of continuous deformations appears.

The situation is different for $SL(2,\mathbb{R})$, because here one cannot connect any two vectors by an $SL(2,\mathbb{R})$ transformation. This is intimately related to the existence of several families of conjugacy classes: the elliptic and hyperbolic, which correspond to the two different choices of *Cartan* subalgebra, and the parabolic corresponding to the *null* subalgebra. Three different truly marginal left-right-symmetric deformations are possible, leading therefore to three families of continuously connected conformal sigma models. Each of them preserves a different $U(1)_L \times U(1)_R$ subalgebra of the undeformed WZW model.

The marginal deformations of $SU(2)$ have been thoroughly investigated [29] [30] [56] with respect to: (i) the identification with the $\left(SU(2)_k/U(1) \times U(1)_{\sqrt{2k\alpha}}\right)/\mathbb{Z}_k$, where $SU(2)_k/U(1)$ is the gauging of the $SU(2)_k$ WZW model, and α is the deformation parameter; (ii) the geometrical (sigma-model) interpretation in terms of metric, torsion and dilaton backgrounds; (iii) the determination of the toroidal partition function and the spectrum as functions of R. For the $SL(2,\mathbb{R})$ deformations, the available results are less exhaustive, especially concerning the spectrum and the partition function [20] [30] [55]. Our aim is to understand the spectra – partition functions – as well as the issue of supersymmetry, when these deformations appear in a more general set up like the NS5/F1.

[12] We systematically set the AdS_3 radius to one in the expressions for the background fields. One has to keep in mind, however, that a factor k is missing in the metric. This plays a role when performing T-dualities by applying the Buscher rules [58] because it can affect the periodicity properties of some coordinates.

5.1 The $J^3\bar{J}^3$ deformation

The $J^3\bar{J}^3$ deformation of $SL(2,\mathbb{R})$ is the one that naturally appears by analytically continuing the deformed $SU(2)$. It was analyzed in [30]. Much like the latter, the metric, antisymmetric tensor and dilaton can be obtained by considering the $SL(2,\mathbb{R})\times U(1)/U(1)$ coset, where the $U(1)$ in the product is compact with a radius related to the deformation parameter, and the gauged one is a diagonal combination of the extra $U(1)$ with the "time-like" $U(1)$ in the $SL(2,\mathbb{R})$, defined by $h = \exp i\frac{\lambda}{2}\sigma^2$. The geometry corresponding to the $J^3\bar{J}^3$-deformed $SL(2,\mathbb{R})$, with deformation parameter $\alpha - 1 > 0$, is thus found to be

$$ds^2 = dr^2 + \frac{-\cosh^2 r dt^2 + \alpha \sinh^2 r d\phi^2}{\alpha\cosh^2 r - \sinh^2 r} \tag{5.1}$$

$$B = \frac{\alpha\cosh^2 r d\phi\wedge dt}{\alpha\cosh^2 r - \sinh^2 r}, \tag{5.2}$$

with dilaton

$$\mathrm{e}^{2\Phi} = \mathrm{e}^{2\Phi_0}\frac{\alpha-1}{\alpha\cosh^2 r - \sinh^2 r}. \tag{5.3}$$

The scalar curvature of this geometry is

$$R = 2\frac{\left(1-\tanh^2 r\right)\left(2\left(\alpha^2-\tanh^2 r\right) - 5\alpha\left(1-\tanh^2 r\right)\right)}{\left(\alpha-\tanh^2 r\right)^2}.$$

Notice that the background fields are usually expected to receive $1/k$ corrections; hence, they are valid semi-classically only, except when they are protected by symmetries as in the unperturbed WZW models.

At $\alpha = 1$ we recover the AdS_3 metric and antisymmetric tensor. For $\alpha \geq 1$, the geometry is everywhere regular whereas for $\alpha < 1$ the curvature diverges at $r = \mathrm{arctanh}\sqrt{\alpha}$. Similarly, the string coupling $g_\mathrm{s} = \exp\Phi$ is finite everywhere for $\alpha \geq 1$ and blows up at $r = \mathrm{arctanh}\sqrt{\alpha}$ for $\alpha < 1$. This means that the semi-classical approximation fails for $\alpha < 1$. The string theory is however well defined. It is actually related by T-duality to the range $\alpha > 1$ as will be discussed later.

The two endpoints of the deformed background are remarkable:

The $\alpha \to \infty$ limit. In this case the background fields become:

$$ds^2 = dr^2 + \tanh^2 r d\phi^2 - \frac{dt^2}{\alpha} \tag{5.4}$$

$$\mathrm{e}^{-\Phi} = \mathrm{e}^{-\Phi_0}\cosh r \tag{5.5}$$

with no antisymmetric tensor. This is the *cigar* geometry times a free time-like coordinate. The cigar – Euclidean black hole [19] – is the axial gauging ($g \to hgh$) of the $U(1)$ subgroup defined by $h = \exp i\frac{\lambda}{2}\sigma^2$. It generates time translations (see Eq. (A.7)) and acts without fixed points. The corresponding geometry is regular and the 2π-periodicity of ϕ inherited from the $SL(2,\mathbb{R})$ ensures the absence of conical singularity.

The $\alpha \to 0$ limit. Now we recover the *trumpet* plus a free time-like coordinate:

$$ds^2 = dr^2 + \coth^2 r dt^2 - \alpha d\phi^2 \tag{5.6}$$

$$\mathrm{e}^{-\Phi} = \mathrm{e}^{-\Phi_0}\sinh r. \tag{5.7}$$

The trumpet is the vector gauging ($g \to hgh^{-1}$) of the same $U(1)$, and generates now rotations around the center (see Eq. (A.7)). Throughout this gauging the time coordinate becomes space-like and vice-versa. The subgroup acts with fixed points, and this accounts for the appearance of a singularity at $r = 0$, which is present no matter the choice for the periodicity of t (in fact this coordinate is not periodic if we start from the universal covering of $SL(2,\mathbb{R})$). Notice also that the time coordinate ϕ is compact, but infinitely rescaled though.

A similar phenomenon occurs in the $J^3\bar{J}^3$ deformation line of the $SU(2)$ WZW model. The two endpoints are the axial and vector gaugings of $SU(2)$ by a $U(1)$, times a free boson, at zero or infinite radius. Axial and vector gaugings are identical in the compact case [38]. They are T-dual descriptions of the same CFT, whose background geometry is the *bell*. Therefore, in the compact case, there is a real T-duality relating large and small deformation parameters. The situation is quite different in the non-compact case.

We would like now to better understand the algebraic point of view and determine the partition function of the deformed theory at any α. To all orders in the deformation parameter, the deformation acts only on the charge lattice of the Cartan subalgebra along which the WZW theory is deformed. Therefore, as in the $SU(2)$ case, the deformation at hand corresponds to a shift of the "radius" of the $J^3, \bar{J}^3$ lattice: $\sqrt{2k} \to \sqrt{2k\alpha}$. The form (3.3) of the original $SL(2,\mathbb{R})$ partition function enables us to implement this radius shift in the time-like lattice with the modular-invariant result:

$$\begin{aligned} Z_{3\bar{3}}(\alpha) = 4\sqrt{\tau_2}(k-2)^{3/2} \int d^2s\, d^2t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{\left|\vartheta_1(s_1\tau - s_2|\tau)\right|^2} \times \\ \times \sum_{m,w,m',w' \in \mathbb{Z}} \zeta \begin{bmatrix} w+s_1-t_1 \\ m+s_2-t_2 \end{bmatrix} (k)\, \zeta \begin{bmatrix} w'+t_1 \\ m'+t_2 \end{bmatrix} (-k\alpha)\,. \end{aligned} \tag{5.8}$$

We can first expand the spectrum around the symmetric $SL(2,\mathbb{R})$ point (i.e. for $\alpha = 1+\varepsilon$, $|\varepsilon| \ll 1$). It allows to express the spectrum in terms of the $SL(2,\mathbb{R})$ quantum numbers. Using the same techniques as in Appendix C, we first perform a Poisson resummation, and integrate over t_2. We find the exponential factor:

$$\begin{aligned} \exp\Bigg\{ -\pi\tau_2\varepsilon \left[\frac{n^2}{k} - k(w_+ - (w-t_1))^2 \right] \\ -2\pi\tau_2 w_+ \left(k(w-t_1) - \frac{k}{2} w_+ \right) - 2\pi\tau_2 s_1 (q + \bar{q} + 1 + k(w-t_1)) \\ +2i\pi s_2 (q - \bar{q} - n) - (k-2)\pi\tau_2 s_1^2 + 2i\pi\tau_1 (w_+ + s_1) n \Bigg\}. \end{aligned}$$

The second and third lines are exactly the same as the undeformed $SL(2,\mathbb{R})$ and lead to the same analysis. For $\varepsilon \ll 1$, the first line gives simply a shift on the weights of the operators according to their J_0^3,$\bar{J}_0^3$ eigenvalues:

$$\begin{aligned} L_0^\varepsilon &= L_0 - \frac{\varepsilon}{k} \left(\tilde{m} + \frac{k}{2} w_+ \right) \left(\tilde{\bar{m}} + \frac{k}{2} w_+ \right), \\ \bar{L}_0^\varepsilon &= \bar{L}_0 - \frac{\varepsilon}{k} \left(\tilde{m} + \frac{k}{2} w_+ \right) \left(\tilde{\bar{m}} + \frac{k}{2} w_+ \right). \end{aligned}$$

In terms of the unflowed eigenvalues: $m = \tilde{m} + kw_+/2$, $\bar{m} = \tilde{\bar{m}} + kw_+/2$, the deformation term is

$$\delta h = \delta \bar{h} = -\varepsilon m \bar{m}/k$$

It breaks of course the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ symmetry of the CFT.

In the limits $\alpha \to 0$ or ∞, the $J^3, \bar{J}^3$ lattice decouples. We then recover a free time-like boson (of zero or infinite radius) times an $SL(2,\mathbb{R})/U(1)$ coset. By using the large/small-radius limits presented in Appendix B, we can trace the effect of these limits at the level of the partition function[13]:

$$Z_{3\bar{3}}(\alpha) \xrightarrow[\alpha\to\infty]{} \frac{k-2}{\sqrt{k\tau_2}\eta\bar{\eta}} Z_{SL(2,\mathbb{R})/U(1)_{\rm A}} \tag{5.9}$$

with $Z_{SL(2,\mathbb{R})/U(1)_{\rm A}}$ given in (3.2). This is precisely the geometrical expectation since the large-α limit of the background (5.1)–(5.3) is the cigar, Eqs. (5.4), (5.5) describing the semiclassical geometry of the axial $U(1)$ gauging of $SL(2,\mathbb{R})$. Similarly, we find

$$\begin{aligned} Z_{3\bar{3}}(\alpha) \xrightarrow[\alpha\to 0]{} \; & 4\sqrt{k\alpha}(k-2)^{3/2} \int d^2s\, d^2t \frac{{\rm e}^{\frac{2\pi}{\tau_2}({\rm Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times \\ & \times \sum_{m,w\in\mathbb{Z}} \zeta \begin{bmatrix} w+s_1-t_1 \\ m+s_2-t_2 \end{bmatrix} (k)\,. \end{aligned} \tag{5.10}$$

In terms of background geometry, this limit describes the trumpet (Eqs. (5.6)and (5.7)) times a free time-like coordinate. We can therefore read off from expression (5.10) the partition function of the vector coset:

$$\begin{aligned} Z_{SL(2,\mathbb{R})/U(1)_{\rm V}} = \; & 4(k-2)^{3/2}\sqrt{\tau_2}\eta\bar{\eta} \int d^2s\, d^2t \frac{{\rm e}^{\frac{2\pi}{\tau_2}({\rm Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times \\ & \times \sum_{m,w\in\mathbb{Z}} \zeta \begin{bmatrix} w+s_1-t_1 \\ m+s_2-t_2 \end{bmatrix} (k)\,. \end{aligned} \tag{5.11}$$

The spectrum of primary fields of this coset can be computed straightforwardly, and reads:

$$L_0^{\rm vector} = \bar{L}_0^{\rm vector} = -\frac{j(j-1)}{k-2} + \frac{\mu^2}{k} \ , \quad \text{with} \quad \mu = -\frac{k}{2}(w - t_1) \in \mathbb{R}, \tag{5.12}$$

both for continuous and discrete representations. The Gaussian variable $w - t_1$ can be integrated out, and leads to the partition function:

$$Z_{SL(2,\mathbb{R})/U(1)_{\rm V}} = 4\frac{(k-2)^{3/2}}{\sqrt{k}}\eta\bar{\eta} \int d^2s \frac{{\rm e}^{\frac{2\pi}{\tau_2}({\rm Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2}. \tag{5.13}$$

The cigar and trumpet geometries are semi-classically T-dual: they are related by Buscher duality [58]. Any two points $(\alpha, 1/\alpha)$ in the above line of deformations are in fact

[13] The $\alpha \to \infty$ limit requires analytic continuation because the lattice is time-like.

connected by an element of $O(2,2,\mathbb{R})$ [30]. However, recalling the spectrum of primaries of the axial coset:

$$L_0^{\text{axial}} = -\frac{j(j-1)}{k-2} + \frac{(\frac{n}{2} - \frac{kw}{2})^2}{k} \,, \quad \text{with} \quad \bar{L}_0^{\text{axial}} = -\frac{j(j-1)}{k-2} + \frac{(\frac{n}{2} + \frac{kw}{2})^2}{k}, \qquad (5.14)$$

we observe that they are different from the vector ones. Our previous discussion gives the explanation for this apparent "failure" of T-duality. The coordinate ϕ in the cigar metric (5.4) inherits the $2\pi\sqrt{2k}$ periodicity (in the asymptotic region $r \to \infty$) from the angular coordinate of AdS_3, irrespectively of the cover of $SL(2,\mathbb{R})$ under consideration; therefore the corresponding lattice in the partition function is compact with radius $\sqrt{2k}$. However, the t coordinate of the trumpet metric (5.6) is $2\pi N\sqrt{2k}$-periodic, for the N-th cover of $SL(2,\mathbb{R})$ (see Fig. 1). Consequently, since our original choice was the universal

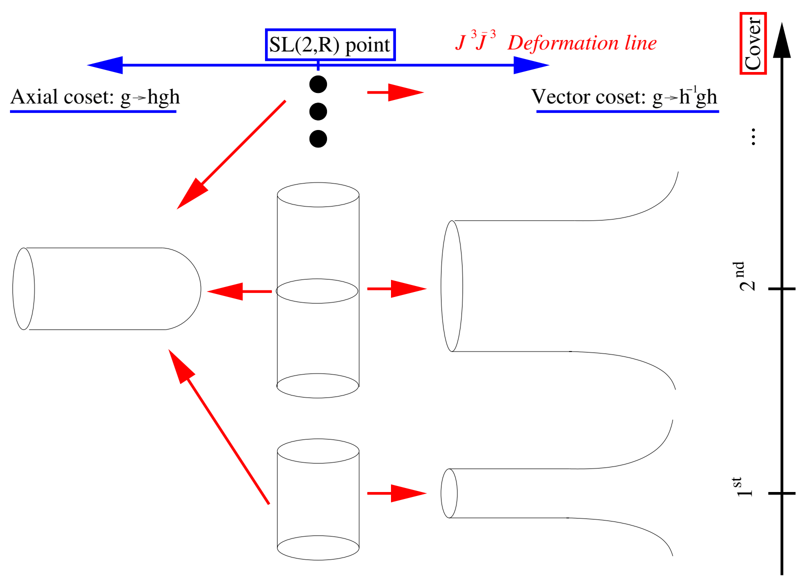

Figure 1: The T-duality for the different covers of $SL(2,\mathbb{R})$.

cover of the algebra, the spectrum we found – Eq. (5.12) – corresponds to the "universal cover" of the trumpet, i.e. with a non-compact transverse coordinate.

The spectrum of the above coset theories has been studied in [59]. In that work, axial and vector cosets were argued to be T-dual, but the definition used for the T-duality amounts to exchange the momenta and winding modes of the physical states. In the case of $SL(2,\mathbb{R})$, this is equivalent to the more rigorous definition of T-duality – residual discrete symmetry of a broken gauge symmetry – *only* on the *single cover* of the group manifold. Indeed, if we consider the single cover of $SL(2,\mathbb{R})$, we obtain in the limits $\alpha \to 0$ and $\alpha \to \infty$ the same spectrum of primaries for the vector and axial cosets, Eq. (5.14), but with different constraints: $\tilde{m} \pm \bar{\tilde{m}} = n$, $\tilde{m} \mp \bar{\tilde{m}} = -kw$, where plus and minus refer to the vector and axial cosets respectively.

Before closing this chapter, we would like to comment on the unitarity of the undeformed model. To this end, we consider a small deformation parameterized by ε:

$$\alpha = 1 - \varepsilon \ , \quad \varepsilon > 0.$$

Then the quadratic term in Eq. (5.9) is space-like and enables us to perform the energy integration *without* any analytic continuation (in fact, we slightly rotate the energy direction into the light-cone). In the limit $\varepsilon \to 0$, we obtain a delta-function that gives the trivial partition function previously discussed. It is possible to analyze this result from a different perspective. We decompose the characters of the discrete representations e.g., by using a non-compact generalization of the Kač-Peterson formula [60]: $\chi^{j,+}_{SL(2,R)}(\tau) = \sum_{m\in\mathbb{N}} c^j_m q^{-(j+m)^2/k}$ with the string functions $c^j_m = q^{-j(j-1)/(k-2)+(j+m)^2/k} \times$ (oscillators). Changing the Cartan radius $\sqrt{2k} \to \alpha\sqrt{2k}$, $\alpha < 1$, all characters become convergent.

5.2 The $J^2\bar{J}^2$ deformation

The operator $J^2\bar{J}^2$ is also suitable for marginal deformations of the theory. It is not equivalent to $J^3\bar{J}^3$ because it corresponds to a choice of *space-like* Cartan subalgebra instead of a time-like one. Conversely the corresponding deformation can be realized as $SL(2,\mathbb{R}) \times U(1)/U(1)$ where the gauged $U(1)$ is now a diagonal combination of the extra $U(1)$ factor with $SL(2,\mathbb{R})$ elements of the type $h = \exp -\frac{\lambda}{2}\sigma^3$.

Owing to the previous discussion, we easily determine the partition function at any value of the deformation parameter. This is realized by deforming the corresponding cycle in the Cartan torus, which amounts in shifting the radius of the space-like J^2, $\bar{J}^2$ lattice. In order to present this partition function in a form closer to the one of the $J^3\bar{J}^3$ deformation, we diagonalize J^2 instead of J^3. This is achieved by redefining the lattice variables in Eq. (3.3) as:

$$\begin{aligned} w + s_1 - t_1 &\to w - t_1 \ , \quad w' + t_1 \to w' + t_1 + s_1, \\ m + s_2 - t_2 &\to m - t_2 \ , \quad m' + t_2 \to m' + t_2 + s_2. \end{aligned}$$

Then we write the following partition function for the $J^2\bar{J}^2$ deformation:

$$\begin{aligned} Z_{2\bar{2}}(\alpha) = {} & 4\sqrt{\tau_2}(k-2)^{3/2} \int d^2s\, d^2t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{\left|\vartheta_1(s_1\tau - s_2|\tau)\right|^2} \times \\ & \times \sum_{m,w,m',w'\in\mathbb{Z}} \zeta\begin{bmatrix} w - t_1 \\ m - t_2 \end{bmatrix}(k\alpha)\, \zeta\begin{bmatrix} w' + t_1 + s_1 \\ m' + t_2 + s_2 \end{bmatrix}(-k)\,. \end{aligned} \tag{5.15}$$

Getting the effective geometry is interesting *per sei* but its systematic analysis goes beyond the scope of the present work. For extreme deformations, a space-like coordinate is factorized, and we are left with an $SL(2,\mathbb{R})/U(1)$ coset with Lorentzian target space: the Lorentzian two-dimensional black hole.

Again the $\alpha \to 0$ and $\alpha \to \infty$ limits are related by T-duality, which at the level of the semi-classical geometry describe various space–time regions of the black hole [59]. For $\alpha \to \infty$ we obtain:

$$Z_{2\bar{2}}(\alpha) \xrightarrow[\alpha\to\infty]{} \frac{1}{\sqrt{k\alpha\tau_2}\,\eta\bar{\eta}} Z_{\mathrm{BH}} \tag{5.16}$$

with

$$Z_{\rm BH} = 4\sqrt{\tau_2}(k-2)^{3/2}\,\eta\bar\eta \int d^2s \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{\left|\vartheta_1(s_1\tau - s_2|\tau)\right|^2} \sum_{m',w'\in\mathbb{Z}} \zeta\begin{bmatrix} w'+s_1 \\ m'+s_2 \end{bmatrix}(-k). \tag{5.17}$$

There is a subtlety in the latter expression compared to the ordinary Euclidean axial black hole (3.2). In the path-integral calculation of the partition function for the Euclidian coset, the oscillators were coupled to the full real momentum of the free boson. We used the periodicity of the determinant to break the zero modes into an integer part and a real compact part. The integer part was interpreted as the lattice of the zero modes of the compact boson and the real part as Lagrange multipliers which impose constraints on the Hilbert space. In the present case, we have to perform an analytic continuation in order to move to the Hamiltonian representation of the partition function,

$$Z_{\rm BH} = 4\sqrt{k}(k-2)^{3/2}\,\eta\bar\eta \int_{\mathbb{R}^2} d^2v \, \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}[\mathrm{Im}(i(v_1\tau - v_2))]^2}\,\mathrm{e}^{-\frac{k\pi}{\tau_2}|v_1\tau - v_2|^2}}{\left|\vartheta_1\left(i(v_1\tau - v_2)|\tau\right)\right|^2},$$

and read the string spectrum. Now, because the coupling is imaginary, the determinant is no longer periodic; therefore we have a *non-compact* time-like coordinate coupled to the oscillators.

5.3 The null deformation

As we already mentioned in Sec. 2.2, the $SL(2,\mathbb{R})$ WZW model allows for extra, unconventional, marginal deformations, which are not generated by Cartan left-right bilinears such as $J^2\bar J^2$ or $J^3\bar J^3$. Instead, the marginal operator we will consider is the following:

$$J\bar J \sim \left(J^1 + J^3\right)\left(\bar J^1 + \bar J^3\right),$$

(see Eqs. (2.15) and (A.8)–(A.13)).

In the holographic dual description, these null currents are the translation generators of the conformal group acting on the boundary in Poincaré coordinates [61]. We will here analyze their action from the sigma-model viewpoint and determine the spectrum and partition function of the deformed model. Supersymmetry issues will be addressed in the more complete set up of { null-deformed $SL(2,\mathbb{R})$ } $\times$ $SU(2)$, in Sec. 6.

We recall that the metric of the deformed background is (see Eqs. (2.9))

$$ds^2 = \frac{du^2}{u^2} + \frac{-dT^2 + dX^2}{u^2 + 1/M^2}.$$

The scalar curvature reads:

$$R = -2\frac{u^2\left(3u^2 - 4/M^2\right)}{\left(u^2 + 1/M^2\right)^2}.$$

This geometry is smooth everywhere for $M^2 > 0$. On the opposite, $M^2 < 0$ gives a singular geometry, which seems however interesting, and corresponds to the repulsion solution [62].

In order to study the null deformation of AdS_3, it is useful to introduce a free-field representation of $SL(2,\mathbb{R})$, in which operators J and $\bar J$ have a simple expression. We first introduce $\phi = -\log u$, $\bar{\gamma} = x^+$ and $\gamma = x^-$. The worldsheet Lagrangian (see Eq. (2.14))[14] reads:

$$\frac{2\pi}{k}\mathcal{L} = \partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma. \tag{5.18}$$

It can be represented with a (β, γ) ghost system of conformal dimensions $(1, 0)$:

$$\frac{2\pi}{k}\mathcal{L} = \partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-2\phi}. \tag{5.19}$$

Classically, β and $\bar{\beta}$ are Lagrange multipliers. They can be eliminated by using their equations of motion, and this gives back the action (5.18). At the quantum level, however, we must integrate them out; taking into account the change of the measure and the renormalization of the exponent, we obtain, after rescaling the fields:

$$2\pi\mathcal{L} = \partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-\sqrt{\frac{2}{k-2}}\phi} - \sqrt{\frac{2}{k-2}}R^{(2)}\phi. \tag{5.20}$$

The last term is the screening charge necessary to compute correlation functions in the presence of the background charge for the field ϕ ($R^{(2)}$-term). The OPE of the free fields are $\phi(z,\bar{z})\phi(0) \sim -\ln(z\bar{z})$ and $\beta(z)\gamma(0) \sim 1/z$. By using these free fields, the holomorphic currents (A.8)–(A.10) are recast, at quantum level, as:

$$\begin{aligned} J^1 + J^3 &= \beta, \\ J^2 &= -i\beta\gamma - i\sqrt{\frac{k-2}{2}}\partial\phi, \\ J^1 - J^3 &= \beta\gamma^2 + \sqrt{2(k-2)}\gamma\partial\phi + k\partial\gamma. \end{aligned}$$

They satisfy the $\widehat{SL}(2,\mathbb{R})_{\mathrm{L}}$ OPA, Eqs. (A.14). Notice finally that the holomorphic primaries of the $SL(2,\mathbb{R})$ CFT read in this basis:

$$\Phi^j_m = \gamma^{j-m}e^{\sqrt{\frac{2}{k-2}}j\phi}. \tag{5.21}$$

The conformal weight of this operator is entirely given by the Liouville primary, whereas the J^2 eigenvalue corresponds to the sum of the "ghost number" and the Liouville momentum.

We can use the above free-field representation to write the null deformation of the AdS_3 sigma model, Eq. (2.16):

$$2\pi\mathcal{L} = \partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}\left[\frac{1}{M^2} + e^{-\sqrt{\frac{2}{k-2}}\phi}\right] - \sqrt{\frac{2}{k-2}}R^{(2)}\phi.$$

For any non-zero value of the deformation parameter, the fields β and $\bar{\beta}$ can be eliminated, leading to two light-cone free coordinates. The energy–momentum tensor reads:

$$T = -\frac{1}{2}\partial\phi\partial\phi - \frac{1}{\sqrt{2(k-2)}}\partial^2\phi - M^2\partial\bar{\gamma}\partial\gamma.$$

[14] Note that the Euclidean rotation from $SL(2,\mathbb{R})$ to H_3^+ is performed by just considering γ and $\bar{\gamma}$ as complex conjugate.

We would like to determine the partition function of the model at hand. To this end we will follow the procedure which has by now become familiar: implement the deformation into the twist-shift orbifold structure of the AdS_3. The ghost number is no longer conserved because the ghost-number current $\beta\gamma$ (equal to $\gamma\partial\bar{\gamma}\exp 2\phi$, on shell) is not a conformal field (this means in particular that the fields (5.21) are now ill-defined). However, there is still a global $U(1)$symmetry: $\gamma \to \mathrm{e}^{i\alpha}\gamma$, $\bar{\gamma} \to \mathrm{e}^{-i\alpha}\bar{\gamma}$ that we can orbifoldize (this is a twist on a complex boson). However, with this representation it is not possible to implement spectral flow without adding extra fields; therefore we have to find the action of the deformation on the zero-mode structure.

Owing the above ingredients, we can construct the partition function that fulfills the following requirements:

1. The endpoint of the deformation, $M^2 \to 0$, should give the Liouville theory times a free light-cone. At $M^2 \to \infty$ one should recover the undeformed $SL(2,\mathbb{R})$.

2. The partition function must be modular invariant.

This enables us to propose the following partition function for the null-deformed model:

$$Z^{\text{null}}_{SL(2,\mathbb{R})}\left(M^2\right) = 4\sqrt{\tau_2}(k-2)^{3/2}\int d^2s\, d^2t \frac{\mathrm{e}^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{\left|\vartheta_1(s_1\tau - s_2|\tau)\right|^2} \times \tag{5.22}$$
$$\times \sum_{m,w,m',w'\in\mathbb{Z}} \zeta\begin{bmatrix} w+s_1-t_1 \\ m+s_2-t_2 \end{bmatrix}\left(k\frac{1+M^2}{M^2}\right)\zeta\begin{bmatrix} w'+t_1 \\ m'+t_2 \end{bmatrix}\left(-k\frac{1+M^2}{M^2}\right).$$

which is obtained by changing the radii of both space-like (J^1, $\bar{J}^1$)[15] and time-like (J^3, $\bar{J}^3$) lattices by the same amount. It is natural since these moduli correspond to the deformations along J^1 and J^3. In the pure $SL(2,\mathbb{R})$ theory, the spectrum is constructed by acting on the primaries with the modes of the affine currents. The difference with the linear dilaton model is that the shift of the oscillator number is linked to the shift of the zero modes. In writing (5.22), which interpolates between these two models, we have implemented that this shift should vanish at infinite deformation (i.e. $M^2 = 0$), which indeed happens provided the radius of the lattice of light-cone zero-modes becomes large. We will expand on that in the next section. In the limit of infinite deformation $M^2 \to 0$, by using the standard technology developed so far, we find

$$Z^{\text{null}}_{SL(2,\mathbb{R})}\left(M^2\right) \sim \frac{(k-2)^{3/2}M^2}{\pi^2\tau_2^{3/2}\left(\eta\bar{\eta}\right)^3}\ , \quad \text{at } M^2 \sim 0.$$

This coincides with the partition function for $U(1)_Q \times \mathbb{R}^{1,1}$.

The derivation of the spectrum goes as previously (see Appendix C). Again, we concentrate on the vicinity of the unbroken $SL(2,\mathbb{R})$ (although the spectrum is known at any M^2):

$$\frac{M^2}{1+M^2} = 1-\varepsilon\ , \quad \varepsilon \ll 1.$$

[15]or equivalently (J^2, $\bar{J}^2$) since they are exchanged by a group transformation

The deformed discrete spectrum is

$$
\begin{aligned}
L_0 &= -\frac{j(j-1)}{k-2} - w_+\left(\tilde{m} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) - \frac{k}{4}(1+\varepsilon)w_+^2 + N \\
&\quad + \frac{\varepsilon(1-2j)}{2(k-2)}\left(\tilde{m}+\tilde{\bar{m}}+\frac{k}{2(k-2)}(1-2j)\right), \\
\bar{L}_0 &= -\frac{j(j-1)}{k-2} - w_+\left(\tilde{\bar{m}} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) - \frac{k}{4}(1+\varepsilon)w_+^2 + \bar{N} \\
&\quad + \frac{\varepsilon(1-2j)}{2(k-2)}\left(\tilde{m}+\tilde{\bar{m}}+\frac{k}{2(k-2)}(1-2j)\right).
\end{aligned} \tag{5.23}
$$

The last terms of both equations are due to the displacement of the poles corresponding to the discrete representations. For the continuous spectrum, we have a similar shift on the weights:

$$
\begin{aligned}
L_0 &= \frac{s^2+1/4}{k-2} - w_+\left(\tilde{m} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) + \frac{k}{4}(1+\varepsilon)w_+^2 + N, \\
\bar{L}_0 &= \frac{s^2+1/4}{k-2} - w_+\left(\tilde{\bar{m}} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) + \frac{k}{4}(1+\varepsilon)w_+^2 + \bar{N}
\end{aligned}
$$

with a deformed density of states. We observe that the deformation term is linked to the spectral flow.

6. The supersymmetric null deformation of $SL(2,\mathbb{R}) \times SU(2)$

So far, we have been considering conformal deformations of the AdS_3 background, realized by marginal deformations of the corresponding $SL(2,\mathbb{R})$ WZW model. In supersymmetric NS5-brane configurations, the $SL(2,\mathbb{R})$ appears usually along with $SU(2)$. We will here analyze the issue of supersymmetry in presence of null deformations of the $SL(2,\mathbb{R})$ factor. We will in particular show that the requirement for the worldsheet $N=2$ superconformal symmetry to be preserved, gives very tight constraints on the allowed deformations.

6.1 The $N=2$ algebra of the deformed theory

We first rewrite the $N=2$ algebra for $SL(2,\mathbb{R}) \times SU(2)$, Eqs. (4.9), (4.10) and (4.14) in the free-field representation (Eqs. (3.9), (3.10)). We recall that I^i are the $SU(2)$ currents, ψ^i and χ^i the fermions of respectively $SU(2)$ and $SL(2,\mathbb{R})$:

$$
\begin{aligned}
\sqrt{2}G^\pm &= i\sqrt{\frac{2}{k+2}}\left(\sqrt{\frac{k+4}{2}}\partial X \mp i\sqrt{\frac{k+2}{2}}\partial\rho\right)\mathrm{e}^{\mp i\sqrt{\frac{2}{k+4}}(X-T)}\chi^\pm \\
&\quad + \sqrt{\frac{2}{k+2}}\left[\mathcal{I}^3 \mp \left(\sqrt{\frac{k+4}{2}}i\partial T + \chi^+\chi^-\right)\right]\frac{\psi^3 \pm \chi^3}{2} + \sqrt{\frac{2}{k+2}}I^\mp\psi^\pm,
\end{aligned} \tag{6.1}
$$

$$
J = \psi^3\chi^3 + \chi^+\chi^- + \psi^+\psi^- + \frac{2}{k+2}\left[\sqrt{\frac{k+4}{2}}i\partial T + \chi^+\chi^- - \mathcal{I}^3\right]. \tag{6.2}
$$

The fermions χ^+, χ^- are bosonized as $\chi^+\chi^- = i\partial H_1$. Note also a shift $k \to k+4$ with respect to the formulas for pure AdS_3; k is the level of $\widehat{SU}(2)$, and this shift ensures that the total bosonic central charge equals six (see (4.5) and (4.6)).

In [39], a map was given between the free-field representation of the superconformal algebra for $SL(2,\mathbb{R})$ and the algebra for $N=2$ Liouville times two free coordinates. As a first step towards the supersymmetrization of the null deformation studied previously, we will show that there exists a *one-parameter family of* $N=2$ *algebras* interpolating between $SL(2,\mathbb{R}) \times SU(2)$ and $U(1)_Q \times \mathbb{R}^{1,1} \times SU(2)$.

The $N=2$ generators (with a non-standard complex structure) are, for $U(1)_Q \times \mathbb{R}^{1,1} \times SU(2)$,

$$\begin{aligned}
\sqrt{2}\hat{G}^{\pm} &= \left(i\partial\hat{X} - \sqrt{\frac{2}{k+2}} i\partial\hat{H}_1 \pm \partial\rho \right) \mathrm{e}^{\pm i\hat{H}_1} \\
&+ \left(\sqrt{\frac{2}{k+2}} \mathcal{I}^3 \mp i\partial\hat{T} \right) \frac{\psi^3 \pm \chi^3}{\sqrt{2}} + \sqrt{\frac{2}{k+2}} I^{\mp}\psi^{\pm}, \\
\hat{J} &= i\partial\hat{H}_1 + \psi^3\chi^3 + \chi^+\chi^- + \frac{2}{k+2}\left[\sqrt{\frac{k+2}{2}} i\partial\hat{X} - \mathcal{I}^3 \right],
\end{aligned}$$

where $\hat{T}$, $\hat{X}$, are the light-cone coordinates. This is not the usual $N=2$ subalgebra of the $N=4$ superconformal algebra (Eqs. (4.3) and (4.4)) of the $U(1)_Q \times SU(2)$ SCFT.

We now perform the following $SO(2,1)$ rotation, which leaves unchanged the OPE's:

$$\begin{aligned}
\hat{H}_1 &= H_1 - tX^-, \\
\hat{T} &= cT + sH_1, \\
\hat{X} &= cX + s(H_1 - tX^-),
\end{aligned}$$

where $X^- = X - T$, and we have introduced:

$$c = \cosh\sigma\ , \quad t = \tanh\sigma\ , \quad s = \sinh\sigma = s_0/p\ , \quad \text{with } s_0 = \sqrt{\frac{2}{k+2}} \quad \text{and } p \geq 1.$$

For $p=1$, the rotated $N=2$ algebra corresponds exactly to the superconformal algebra of $SL(2,\mathbb{R}) \times SU(2)$, Eqs. (6.1) and (6.2). Furthermore, one can check that the $N=2$ superconformal structure is preserved for any p.

Coming back to the null deformation of $SL(2,\mathbb{R}) \times SU(2)$, we conclude that the above one-parameter family of supercurrents can be implemented along the line of deformation. The $N=2$ R-symmetry current of the deformed theory takes the form:

$$J = \psi^3\chi^3 + \chi^+\chi^- + i\partial H_1 + ps^2\left(\frac{1}{t} i\partial T + i\partial H_1\right) - s_0^2\, \mathcal{I}^3 + (p-1)t\ i\partial X^-,$$

and the deformed supercurrents read:

$$\begin{aligned}
\sqrt{2}G_2^{\pm} &= \left(s_0\mathcal{I}^3 \mp i(c\ \partial T + s\ \partial H_1)\right)\frac{\psi^3 \pm \chi^3}{\sqrt{2}} + s_0 I^{\mp}\psi^{\pm} \\
&+ i\left[c\ \partial X - (s_0 - s)\left(\partial H_1 - t\ \partial X^-\right) \mp i\partial\rho\right] \mathrm{e}^{\pm i\left(H_1 - tX^-\right)}.
\end{aligned} \tag{6.3}$$

Clearly, the background contains both torsion (in the first line of (6.3)) and a background charge (in the second line). The field H_1 still corresponds to a free complex fermion, provided we change the radius of the compact light-cone direction:

$$\hat{R}_+ = \frac{2}{t} = \sqrt{2\left(p^2(k+2)+2\right)}.$$

The latter statement holds because the deformation is null. It therefore confirms our assumption (Sec. 5.3) that the null deformation corresponds to a change of the radius of the compact light-cone coordinate. The relation between this parameterization and the mass scale introduced in Sec. 2 is, up to $1/k$ corrections:

$$\frac{1}{M^2} = \frac{k+2}{k+4}\left(p^2-1\right). \tag{6.4}$$

The holomorphic primary fields for the deformed $SL(2,\mathbb{R})$ are recast as:

$$\Phi^{w_+,\ \mathrm{def}}_{j\ m} = \exp\left\{\sqrt{\frac{2}{k+2}}j\phi + it\left[\left(\tilde{m} - \frac{1}{2t^2}w_+\right)X^- + \frac{1}{2t^2}w_+(X+T)\right]\right\},$$

where, as in the undeformed theory, $\tilde{m} - \tilde{\bar{m}} \in \mathbb{Z}$ and $\tilde{m} + \tilde{\bar{m}} - 2w_+/t^2$ is the energy. This gives the following $N=2$ charges:

$$\mathcal{Q}_R = q_1 + q_2 + q_0 - w_+ + \frac{2}{p(k+2)}\left[\tilde{m} + \ q_1 \ - w_+\right] - \frac{2}{k+2}\left[m_{SU(2)} \ + \ q_2\right]$$

with q_i the fermionic charges.

The worldsheet supersymmetry of the deformed theory works similarly to the undeformed one ($p=1$). In fact, the spectral-flow charge w_+ in the first bracket has to be compensated by a shift of q_1, because the spectral-flow symmetry must act on the total current [52]. We are left with well-normalized charges for $\mathcal{I}^3$ and deformed $\mathcal{J}^3$.

6.2 Space–time supersymmetry

The supersymmetry generators of the original $SL(2,\mathbb{R}) \times SU(2)$ model are given in Eq. (4.15) with a restriction on the allowed charges captured in (4.18), on top of the usual GSO projection.

In the deformed theory, these operators are no longer physical with respect to the supercurrent $G = G^+ + G^-$ given in Eq. (6.3). The physical spin fields are instead

$$\Theta^{\mathrm{def}}_{\varepsilon}(z) = \exp\left\{-\frac{\varphi}{2} + \frac{i}{2}\sum_{\ell=0}^{4}\varepsilon_\ell H_\ell + \frac{i}{2}\varepsilon_1(p-1)t\,X^-\right\}. \tag{6.5}$$

Since the only modification resides in the X^- term, it changes neither the conformal dimensions of these fields nor their mutual locality. By using the same projection as before, we obtain a set of well-defined physical spin operators. Acting with one of these operators on a left-moving vertex operator of the NS sector ($q_\ell \in \mathbb{Z}$),

$$V(z') \sim \mathrm{e}^{-\varphi}\Phi^{w_+,\ \mathrm{def}}_{j\,m} \exp i\sum_{\ell=0}^{4} q_\ell\, H_\ell(z'),$$

gives a leading term behaving like

$$(z - z')^{\frac{1}{2}\left\{-1+\sum_{\ell=0}^{4}\varepsilon_\ell q_\ell+\varepsilon_1(p-1)w^+\right\}}.$$

We conclude that the locality condition of space–time supercharges with respect to the states requires *the deformation parameter be an odd integer:* $p \in 2\mathbb{Z}+1$. Note that since this quantization condition originates from the massive states (those with $w_+ \neq 0$), it is not visible in the supergravity analysis. In the limit of infinite deformation, this choice generates modified space–time supercharges, constructed with $H_1 - s_0 X^-$. This choice is not the one obtained from the $N = 4$ algebra (see Eqs. (4.3) and (4.4)), because the complex structure for the fermions is different. An appropriate choice, though, avoids infinite shifts in the spin fields, and allows for reaching a well-defined theory at the limit of infinite deformation.

At this point we want to discuss the issue of supersymmetry breaking. In type IIB, the first projection performed in the undeformed theory keeps the spinors $(+,\mathbf{2}',\mathbf{2}')$ and $(-,\mathbf{2},\mathbf{2}')$ of $SO(1,1)\times SO(4)\times SO(4)_T$ for both supersymmetry generators. When the AdS_3 factor of the background is deformed in the null direction, the gravitino must be right-moving in space–time (hence, it depends on X^-), which picks up the spinor $(-,\mathbf{2},\mathbf{2}')$. So, although the number of covariantly constant spinors is reduced by a factor of two in the deformed background, the number of *transverse* fermionic degrees of freedom appearing in the spectrum is the same. A subtlety comes, however, while dealing with the right-moving sector of the theory. As already mentioned, in WZW models, the right superconformal algebra is written with a torsion term of opposite sign. The correct torsion for the right movers in the $SL(2,\mathbb{R})$ factor demands to rotate the fields of $SU(2)_R \times U(1)_Q \times \mathbb{R}^{1,1}$ as:

$$\begin{aligned}
\hat{\bar{H}}_1 &= \bar{H}_1 - t\bar{X}^+,\\
\hat{\bar{T}} &= c\bar{T} - s\bar{H}_1,\\
\hat{\bar{X}} &= c\bar{X} + s(H_1 + t\bar{X}^+).
\end{aligned}$$

Therefore, the right algebra is written with $\bar{X}^+$ rather than with $\bar{X}^-$. For the undeformed model this is irrelevant since the two free-field representations are isomorphic. However, for non-zero deformation, it makes a difference because the right-moving spin fields will be corrected with $\bar{X}^+$ rather than $\bar{X}^-$:

$$\bar{\Theta}^{\text{def}}_{\bar{\varepsilon}}(z) = \exp\left\{-\frac{\varphi}{2} + \frac{i}{2}\sum_{\ell=0}^{4}\bar{\varepsilon}_\ell \bar{H}_\ell + \bar{\varepsilon}_1(p-1)t\,\bar{X}^+\right\}.$$

As a consequence, gravitinos from the right- and left-moving sectors of the worldsheet CFT propagate in opposite light-cone directions. They give space–time transverse supercharges $(\mathbf{2},\mathbf{2}')$ from the left and $(\mathbf{2}',\mathbf{2}')$ from the right. In type IIA, the same reasoning leads to the representations $(\mathbf{2},\mathbf{2}')$ for both generators. The conclusion is that the deformation *flips the chirality of space–time fermions from the right sector.*

6.3 The partition function for superstrings on deformed $SL(2,\mathbb{R})\times SU(2)$

As we have seen in the previous analysis, the only necessary modification on the fermionic part is the flip of chirality for the right-moving fermions. It is implemented by inserting

$(-)^{F_{\mathrm{R}}}$, where F_{R} is the space–time fermion number for right-movers. This is an orbifold, that projects out the Ramond states from the untwisted sector, while the twisted sector restores the Ramond states, with opposite chirality though. The fermionic vertex operators are thus constructed with $\exp \pm i \left(H_1 + (p-1)t\ X^-\right)/2$. However, since this modification has no effect on the conformal weights of the spin fields, it does not alter the fermionic characters in the partition function. The remaining parts of the partition function have been discussed in previous sections. Putting everything together, we find for the {null-deformed $SL(2,\mathbb{R})$} $\times SU(2) \times T^4/\mathbb{Z}_2$:

$$Z_{\mathrm{IIB}}(p) = \frac{\mathrm{Im}\tau}{\eta^2\bar{\eta}^2} Z_{SU(2)} Z^{\text{null}}_{SL(2,\mathbb{R})}(p) \frac{1}{2}\sum_{h,g=0}^{1} Z^{\text{twisted}}_{T^4/\mathbb{Z}_2}\begin{bmatrix}h\\g\end{bmatrix}$$
$$\times \frac{1}{2}\sum_{a,b=0}^{1} (-)^{a+b}\ \vartheta^2\begin{bmatrix}a\\b\end{bmatrix}\vartheta\begin{bmatrix}a+h\\b+g\end{bmatrix}\vartheta\begin{bmatrix}a-h\\b-g\end{bmatrix}$$
$$\times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1} (-)^{\bar{a}+\bar{b}}\ \frac{1}{2}\sum_{h',g'=0}^{1} (-)^{(1-\delta_{p,1})\left[\bar{a}g'+\bar{b}h'+h'g'\right]}\ \bar{\vartheta}^2\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\bar{a}+h\\\bar{b}+g\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\bar{a}-h\\\bar{b}-g\end{bmatrix},$$

where $p \in 2\mathbb{Z}+1$ and $Z^{\text{null}}_{SL(2,\mathbb{R})}(p)$ is given in (5.22) with $k \to k+4$; the relation between p and M^2 has been given by Eq. (6.4). The sum over h' and g' flips the chirality of the right-moving fermionic representation for any $p \neq 1$, according to the left-right asymmetry discussed in the text.

This is the *simplest* modular invariant combination of the various ingredients, with the correct projections dictated by the superconformal invariance. It should, by no means, be considered as unique, and many other models do exist, which are equally acceptable.

7. Some comments about holography

In this last section of the paper, we will give some remarks about the holographic dual of string theory in the background (2.9). Since this string theory is exactly solvable and perturbative everywhere, we can use the gauge/gravity correspondence beyond the supergravity approximation. The purpose here is only to identify the non-gravitational dual of the setup and to explain its relevance to study little string theory. Although the space–time studied in this paper is constructed as a deformation of AdS_3, the holographic interpretation is different from the undeformed case.

7.1 The D1/D5 setup and AdS/CFT

Let first review briefly the usual holographic dual of the theory of D1/D5-branes [23]. Starting with the supergravity solution of Eqs. (2.1), (2.2) and (2.3), we would like to take a limit where the theory on the branes is decoupled from the bulk modes. This is obtained by the low-energy limit:

$$\alpha' \to 0 \quad , \quad U = r/\alpha' \ \ \text{fixed}$$

so that the gravitational coupling constant goes to zero. All open-string modes become infinitely massive and only the zero modes survive; the dual theory is a field theory.

In the above limit, the dimensionless volume of the four-torus has a fixed, finite value:

$$\hat{v}(T^4) = N_1/N_5.$$

Therefore, the dual conformal theory is (1 + 1)-dimensional. In the low-energy limit, the D1-branes are trapped inside the D5-brane world-volume and can be considered as string-like instantons of the six-dimensional $U(N_5)$ gauge theory of the D5-branes. The $\mathcal{N} = (4,4)$ superconformal field theory dual to the near-horizon limit of the background is the Higgs branch of a sigma-model on the moduli space of these instantons, of central charge $c = 6(N_1 N_5 + 1)$ [23].

In regions of the moduli space where the string coupling is large, the theory is appropriately described in the S-dual frame. The string background is then given by the WZW theory: $SU(2)_k \times SL(2,\mathbb{R})_{k+4}$, where $k = N_5$. Now the constant value of the six-dimensional string coupling is $g_6 = N_5/N_1$. Note that, although the number of fundamental strings do not appear in the worldsheet CFT description, the string theory is weakly coupled only for large values of N_1.

An interesting feature of this theory is that it is possible to construct directly out of the worldsheet currents the generators of the space–time Virasoro algebra [9], which act on the boundary of AdS_3.

7.2 The null deformation of AdS_3

The null deformation of AdS_3 cannot be small; regardless the value of the deformation parameter, the causal structure of the space–time is completely changed. There is no longer conformal boundary, but rather an asymptotic flat geometry with a linear dilaton. Anyway, the deformation parameter can always been scaled to one (if positive) by rescaling the non-compact coordinates of the light-cone.

From the holographic point of view, the null deformation corresponds to adding an (infra-red) irrelevant operator in the Lagrangian. It is therefore more appropriate for holography to start with the holographic dual of the linear dilaton background and perturb it in the infra-red by a relevant operator.

Our decoupling limit (2.4) is quite different from the standard one. We send the ten-dimensional string coupling to infinity. Therefore in order to study this part of the moduli space of the brane theory we have to perform an S-duality. In the dual variables, the limit under consideration is

$$\tilde{g}_s \to 0 \quad , \quad \tilde{\alpha}' \text{ fixed} \quad , \quad r/\tilde{g}_s \text{ fixed.} \tag{7.1}$$

This looks the same as the little-string-theory limit. However, in the case at hand, the background contains fundamental strings, which affect the geometry in the vicinity of the branes. In the limit (7.1) the energy of the D1-branes stretched between the NS5-branes is kept fixed: $E_{D1} \sim r/(\tilde{\alpha}'\tilde{g}_s)$. If instead we keep the energy of the fundamental strings fixed, $E_{F1} \sim r/\tilde{\alpha}'$, the contribution of the fundamental strings disappears and we are left with the "pure" NS5-brane background (see [9]). Another important difference between the standard near-horizon limit and the partial near-horizon limit is that the later involves the decompactification of the torus, in the D-brane picture (we send the asymptotic value of the dimensionless volume v to infinity). Therefore the dual "gauge" theory should be six-dimensional; this is again the same as little string theory.

7.3 A little review of little string theory

The little string theory is the decoupled theory living on the world-volume of the NS5-branes (for a review, see [63]). The world-volume theory is decoupled from the bulk by taking the limit:

$$\tilde{g}_s \to 0 \quad , \quad \tilde{\alpha}' \text{ fixed.}$$

This is not a low-energy limit, unlike in the AdS/CFT case. The resulting theory is interacting and non-local, since it exhibits the T-duality symmetry. In the case of type IIB string theory, the low-energy limit of LST is a $U(N_5)$ gauge theory with $N = (1,1)$ supersymmetry and a bare gauge coupling:

$$g_{YM}^2 = \tilde{\alpha}'.$$

The coupling grows at large energy, and additional degrees of freedom, which are identified as string-like instantons in the low- energy theory, appear at energies of order $\tilde{\alpha}'^{-1/2}$. These instantons are identified with fundamental strings attached to the NS5-branes. In the infra-red this theory flows to a free fixed point. In the case of type IIA string theory, the infra-red limit is an $N = (2,0)$ interacting superconformal theory. It contains tensionless strings.

The conjectured holography of [25] states that this theory is dual to string theory in the NS5-brane background in the near-horizon limit [1]:

$$\begin{aligned} ds^2 &= dx^\mu dx^\nu \eta_{\mu\nu} + \tilde{\alpha}' N_5 \left(d\rho^2 + d{\Omega_3}^2 \right), \\ \Phi &= \Phi_0 - \rho, \\ H &= 2\tilde{\alpha}' N_5 \; \epsilon(\Omega_3). \end{aligned}$$

In this limit, $r/\tilde{g}_s$ – the energy of the D1-branes stretched between the NS5-branes – is kept fixed. This geometry corresponds to the exact conformal field theory $SU(2)_k \times U(1)_Q$, with $k+2 = N_5$ that was discussed in Sec. (4.1.1). This holographic description breaks down in the region $\Phi \to -\infty$, near the branes, because the string coupling blows up. In the type IIB case, this is related to the fact that the infra-red fixed point is free. In the type IIA theory, this strong-coupling region is resolved by lifting the background to M-theory [25]; we obtain the background $\text{AdS}_7 \times S^4$ of eleven-dimensional supergravity in the vicinity of the M5-branes (distributed on a circle in the eleventh dimension).

As in the AdS/CFT correspondence, on-shell correlators of non-normalizable states in string theory corresponds to off-shell Green functions of observables in LST. The non-normalizable states of the NS5-brane background are constructed with the discrete representations of the linear dilaton:

$$V = \left(\psi \bar{\psi} \Phi_j \bar{\Phi}_j \right)_{j+1} \mathrm{e}^{\frac{2j}{\sqrt{2(k+2)}} \rho}, \tag{7.2}$$

where the primary operator of spin j of $SU(2)$ and the fermions are combined into an operator of spin $j+1$. From the worldsheet point of view, the operators (7.2) are necessary to balance the background charge in the correlation functions. The theory contains also delta-function-normalizable states, from the continuous representations:

$$V \sim \mathrm{e}^{\left(-\frac{1}{\sqrt{2(k+2)}}+is\right)\rho},$$

but their holographic interpretation is less clear. In the bulk they are propagating fields in the linear dilaton background. There is a third class of operators from the discrete representations, normalizable in the ultraviolet but not in the infra-red:

$$V \sim \mathrm{e}^{\frac{-2(j+1)}{\sqrt{2(k+2)}}\rho}\Phi_j\bar{\Phi}_j.$$

These operators correspond to states localized on the five-branes. When we deform the theory towards $SL(2,\mathbb{R})$, they become the discrete representations of the unitary spectrum of $SL(2,\mathbb{R})$.

7.4 Low energy limit, strong coupling and fundamental strings

The decoupling limit of the LST (7.2) does not really make sense for type IIB superstrings in the deep infra-red region. In fact, as the string coupling blows up here, sending its asymptotic value to zero does not ensure that the theory living on the branes decouples from the bulk at very low energies (see [64] for a related discussion). Since the bulk theory is non-perturbative in this region, the resolution of this puzzle is quite conjectural.

Our background, viewed from the LST side, provides a possible mechanism to better describe the situation. We first take the worldsheet description of the bulk physics; the bosonic Lagrangian of the theory is

$$2\pi\mathcal{L} = \partial\rho\bar{\partial}\rho - \sqrt{\frac{2}{k+2}}R^{(2)}\rho + \partial X^+\bar{\partial}X^- + 2\pi\mathcal{L}^{WZW}_{SU(2)_k} + \sum_{i=6}^{9}\partial X^i\bar{\partial}X^i.$$

Starting from this extremity of the line of marginal deformations, we add at first order the following $(1,1)$ operator to the Lagrangian:

$$\delta\mathcal{L} \sim M^2\mathrm{e}^{-\sqrt{\frac{2}{k+2}}\rho}\ \partial X^+\bar{\partial}X^-.$$

It provides a Liouville potential for the linear dilaton and thus regulate the strong-coupling region. This potential adds a non-trivial "electric" NSNS flux in the background. We know from our previous analysis that deforming the theory in this way corresponds to adding fundamental strings in the infra-red region of the background. This operator is a singlet of $SU(2)_L \times SU(2)_R$. The brane picture is that, as we go down into the throat (the strong-coupling region) macroscopic fundamental strings condense in the world-volume of the NS five-branes.

Now consider the dual, non-gravitational theory living on the D5-branes. The low-energy bosonic Euclidean action is

$$S_{\mathrm{gauge}} = \int \frac{1}{\alpha' g_{\mathrm{s}}}\mathrm{Tr}\ F\wedge *F + \frac{i}{\alpha'}C_0\ dt\wedge dx\wedge \mathrm{Tr}\ F\wedge F + \frac{1}{\alpha' g_{\mathrm{s}}}\left\{DX^i \wedge *DX^i + [X^i, X^j]^2 * \mathbb{I}\right\}.$$

Where C_0 is the flux generated by the D1-branes. It is well known that for gauge theories in dimensions higher than four, field configurations of non-zero energy (instanton-like) have an infinite action and therefore do not contribute to the path integral. The conclusion is different in the presence of the D1-brane flux. Starting with any instanton solution in four dimensions,

$$*_4 F = F,$$

we can lift it to a field configuration in six dimensions which obey a generalized self-duality condition:

$$*_6 F = F \wedge dt \wedge dx,$$

Such a solution is not really an instanton, since it is invariant under time translations. This configuration is one-half BPS, because it imposes the following fermionic projection on supersymmetry generators:

$$\frac{1}{2}\left(1 - \gamma^{6789}\right)\eta = 0,$$

where $6, 7, 8, 9$ are the coordinates of the 4-torus in our conventions. The gauge action for such a solution is

$$S_{\text{gauge}} = \frac{1}{\alpha'}\left[\frac{1}{g_{\text{s}}} + iC_0\right]\int dt \wedge dx \wedge \text{Tr} F \wedge F.$$

In the infra-red, the RR 2-form behaves as:

$$C_0 = -\frac{i}{g_{\text{s}}}\left[\frac{\alpha' v U^2}{g_{\text{s}} N_1} - 1\right] \underset{U \to 0}{\to} \frac{i}{g_{\text{s}}}.$$

Therefore, the classical action for such a configuration vanish. The conclusion is that, in presence of the D1-brane flux, an imaginary "theta-like" topological term is added to the SYM action. Now stringy instanton solutions of the gauge theory are minima of the action and contribute to the path integral; in the infra-red the theory is not the free SYM fixed point, but rather the $(1+1)$-dimensional dynamics of these objects.

At this point one can wonder if there are other natural infra-red completions of the perturbative string background. Another proposal, called "double-scaled little string theory" has been made [4]. The idea is to describe the Higgs phase of the little string theory, where the NS5-branes are distributed on a circle of radius r_0. The double scaling limit is defined by: $\tilde{g}_s \to 0$, $r_0/\sqrt{\tilde{\alpha}'} \to 0$. In this limit, the worldsheet dual background is described by an orbifold of an $N = 2$ Liouville theory with a potential tensorized with the coset $SU(2)/U(1)$. By using a duality discussed recently in [65], this theory is argued to be equivalent to

$$\left(\frac{SU(2)}{U(1)} \times \frac{SL(2,\mathbb{R})}{U(1)}\right) \Big/ \mathbb{Z}_{N_5}.$$

In this model the strong-coupling throat region of the NS5-branes background is replaced by the tip of the cigar geometry, so the dilaton value is bounded from above. However, because of the duality, this model is *not* continuously connected to the pure NS5-brane background by a marginal deformation. The holographic interpretation of this mirror symmetry needs further investigation.

8. Discussion and conclusions

Let us summarize the main conclusions of this paper. We have shown that a new, interesting decoupling limit for the D1/D5-brane theory (or NS5/F1 in the S-dual description) exists. It captures not only the infra-red dynamics, but also the full renormalization-group flow. Furthermore, this theory is free of strong-coupling problems in the bulk, in contrast with the little-string-theory limit in the moduli space. We have studied this background mainly from the string worldsheet point of view, since the bulk background in the NS5/F1 picture is an exactly solvable worldsheet conformal field theory: $\{$null-deformed $SL(2,\mathbb{R})\} \times SU(2) \times (T^4$ or $K3)$.

We have first analyzed the undeformed $SL(2,\mathbb{R})$ theory, for which our achievement is the construction of the partition function for the (Lorentzian) AdS_3, including both discrete and continuous representations, in all the sectors of spectral flow. Our procedure is to start from the coset theory $SL(2,\mathbb{R})/U(1)_A$, and to reconstruct the $SL(2,\mathbb{R})$ partition function by coupling the former with a lattice corresponding to the time direction. The partition function is thus obtained in a linearized form, where the energy integration is manifest. Although formally divergent, the expression of the partition function contains all the information about the full spectrum, and behaves as expected in the large-k limit. Upon integrating the energy, we recover the partition function of [12] [13] [17]. An important feature of our partition function is that one light-cone direction is compact, whereas the other is non-compact. This allows for a natural definition of the light-cone Hamiltonian.

The $SL(2,\mathbb{R})$ WZW model is a building block for physically interesting backgrounds, such as $\mathrm{AdS}_3 \times S^3 \times T^4$, which preserve supersymmetry and have a brane origin. We have written the extended worldsheet superconformal algebra for this theory, although, as discussed in [9] [50], it is somehow problematic to define the space–time supersymmetry in that way. In order to implement the projection that leaves unbroken one half of flat-space supersymmetry, we have chosen to consider superstrings on the $T^4/\mathbb{Z}_2$ orbifold point of $K3$. This implies only minor modifications in the brane interpretation of the background: the topological sector of the orbifold corresponds to additional D1-branes with negative charge (this is clear from the D-brane gravitational couplings), while the microscopic degrees of freedom living on the branes are different from the T^4 case. We could instead have insisted on keeping the T^4 as the internal manifold, but then the orbifold action acting on $SU(2) \times SL(2,\mathbb{R})$ – necessary for cutting half of the supersymmetries – would have been more complicated. The lattice interpretation of the $SL(2,\mathbb{R})$ partition function is a very powerful tool in the process of understanding marginal deformations in AdS_3 backgrounds. We have considered left-right symmetric bilinears in the currents. For $SL(2,\mathbb{R})$ this amounts to three different deformations: two marginal deformations corresponding to the two different choices of Cartan subalgebra, and one marginal deformation along a "null" direction.

The deformation with respect to the time-like generator J^3 relates the theory to the Euclidean black hole $SL(2,\mathbb{R})/U(1)_A$, on one side, and to its T-dual $SL(2,\mathbb{R})/U(1)_V$ on the other side. We have found that the spectra of the two cosets do not match exactly. This is related to the fact that we have started with a theory on the universal cover of $SL(2,\mathbb{R})$. The conclusion is different if we take the single cover instead, but then string theory does not make sense except at the coset points, because of the presence of closed time-like curves. The J^3 deformation gives a geometry that also interpolates between the AdS_3 geometry (in global coordinates) and the linear dilaton background. In the case at hand, however, the brane picture – if any – remains to be understood.

Similar considerations hold for marginal deformations driven by the space-like choice of Cartan generator, namely J^2. In the limit of infinite deformation, we obtain now the Lorentzian two-dimensional black hole.

The case of null deformation of $SL(2,\mathbb{R})$ has attracted most of our attention, because of its brane interpretation and the underlying decoupling limit. We have reached a modular-invariant partition function for the purely bosonic case, and extended the whole set up to the supersymmetric background. We have in particular shown that the physical spin fields, which give the space–time supercharges, are modified asymmetrically by the background, and are restricted by the same projection as in the absence of any deformation. The locality condition for this charges with respect to the string states, however, gives an extra quantization condition, on the deformation parameter. Therefore, as a *superconformal worldsheet theory*, the line of deformation is *continuous*, but *space–time supersymmetry* further selects a *discrete* subset of deformation points. We observe that, at least for k large, these special points are such that the $O(2,2,\mathbb{R})$ transformations of Eq. (2.19) that give the null-deformed model belong to $O(2,2,\mathbb{Z})$, i.e. become a discrete line of *dualities*.

The decoupling limit of the D1/D5-brane configuration that we have presented here, calls for further holography investigation. In the present paper, the analysis of the holographic picture of this gravitational background has been very superficial: it provides a natural infra-red regularization of little string theory, by imposing an upper bound on the string coupling constant, without changing the asymptotic ultraviolet geometry. More work is needed to understand it.

There are many other issues that remain open, as for example:

- Study in more detail other realizations of the worldsheet supersymmetry that do not involve the orbifold of the four-torus.

- Give a complete picture of the $\mathrm{AdS}_3 \times S^3$ landscape by means of a systematic analysis of other supersymmetry-preserving marginal deformations e.g. cosets or limiting gravitational-wave backgrounds.

- Interpret these backgrounds in terms of brane set-ups.

- Put holography at work; the explicit calculation of correlation functions in the deformed theory seems at first sight rather difficult, and needs further investigation.

Finally it is worth stressing that connecting the near-horizon geometry of NS5/F1-branes ($SL(2,\mathbb{R}) \times SU(2) \times U(1)^4$) with the near-horizon limit for the NS5-branes alone ($\mathbb{R}^{1,1} \times U(1)_Q \times SU(2) \times U(1)^4$) is a step towards the search of an exact CFT description of a background which is $SL(2,\mathbb{R})$ in some region of space–time and asymptotically flat in another one.

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A. The $SL(2,\mathbb{R})$ WZW model: a reminder

We collect in this appendix some well-known facts about the $SL(2,\mathbb{R})$ WZW model, within a consistent set of conventions. The commutation relations for the generators of the $SL(2,\mathbb{R})$ algebra are

$$\left[J^1, J^2\right] = -iJ^3 \ , \quad \left[J^2, J^3\right] = iJ^1 \ , \quad \left[J^3, J^1\right] = iJ^2. \tag{A.1}$$

The sign in the first relation is the only difference with respect to the $SU(2)$. Introducing

$$J^{\pm} = iJ^1 \mp J^2,$$

yields[16]

$$\left[J^3, J^{\pm}\right] = \pm J^{\pm} \ , \quad \left[J^+, J^-\right] = 2J^3, \tag{A.2}$$

which are also valid for $SU(2)$. This is the $s\ell(2)$ algebra. Its representations are the same for both $SL(2,\mathbb{R})$ and $SU(2)$; only their unitarity properties are different (see e.g. [66]).

The quadratic Casimir for $SL(2,\mathbb{R})$ is defined as:

$$C_2 = \left(J^1\right)^2 + \left(J^2\right)^2 - \left(J^3\right)^2 = -\frac{1}{2}\left(J^+J^- + J^-J^+\right) - \left(J^3\right)^2, \tag{A.3}$$

and its eigenvalues parametrized by[17] $C_2 = j(1-j)$.

Irreducible representations of the above algebra are essentially of two kinds: discrete $\mathcal{D}^{\mp}(j)$ or continuous principal $\mathcal{C}_{\mathrm{p}}(b,a)$ and continuous supplementary $\mathcal{C}_{\mathrm{s}}(j,a)$. The discrete ones have highest ($\mathcal{D}^-$) or lowest ($\mathcal{D}^+$) weight, whereas the continuous ones do not. The spin j of the discrete representations is real[18], and their states are labelled by $|jm\rangle$, $m = \mp j, \mp j \mp 1, \mp j \mp 2, \ldots$ For the principal continuous ones, $j = \frac{1}{2} + ib,\ b > 0$, and the magnetic number is $m = a, a \pm 1, a \pm 2, \ldots,\ -\frac{1}{2} \leq a < \frac{1}{2},\ a, b \in \mathbb{R}$; for the supplementary continuous ones, $0 < j \leq \frac{1}{2}$ and $-\frac{1}{2} \leq a < \frac{1}{2}$, with the constraint $\left|j - \frac{1}{2}\right| < \frac{1}{2} - |a|$, $a, j \in \mathbb{R}$. These representations are unitary and infinite-dimensional; $\mathcal{D}^{\pm}(j)$ become finite-dimensional when j is a negative integer or half-integer, and are non-unitary for any negative j. Notice finally that the quadratic Casimir C_2 is positive for both continuous series; for the discrete ones it is positive or negative when $0 < j < 1$ or $1 < j$, respectively.

[16] In some conventions $J^{\pm} = J^1 \pm iJ^2$, as for $SU(2)$.

[17] There is an arbitrariness in the sign of C_2, as well as on that of j. The ones we consider here are the most popular in the community. However, the most efficient are the opposite ones both for C_2 and j, since they allow for a unified presentation of $SU(2)$ and $SL(2,\mathbb{R})$ representations.

[18] In order to avoid closed time-like curves, we are considering the universal covering of $SL(2,\mathbb{R})$. Therefore, j is not quantized.

The three-dimensional anti-de Sitter space is the universal covering of the $SL(2,\mathbb{R})$ group manifold. The latter can be embedded in Minkowski space with signature $(-,+,+,-)$ and coordinates (x^0,x^1,x^2,x^3) – we set the radius to one:

$$g = \begin{pmatrix} x^0+x^2 & x^1+x^3 \\ x^1-x^3 & x^0-x^2 \end{pmatrix} . \tag{A.4}$$

The Poincaré patch introduced in the Gauss decomposition, $(u, x^{\pm}) \in \mathbb{R}^3$ covers exactly once the $SL(2,\mathbb{R})$. Comparing Eqs. (2.13) and (A.4) yields

$$x^0+x^2 = \frac{1}{u} \ , \quad x^1 \pm x^3 = \frac{x^{\pm}}{u} \ , \quad x^0 - x^2 = u + \frac{x^+x^-}{u}.$$

The metric and antisymmetric tensor read:

$$ds^2 = \frac{du^2 + dx^+dx^-}{u^2} \ , \quad H = dB = \frac{du \wedge dx^+ \wedge dx^-}{u^3}. \tag{A.5}$$

The isometry group of the $SL(2,\mathbb{R})$ group manifold is generated by left or right actions on g: $g \to hg$ or $g \to gh$ $\forall h \in SL(2,\mathbb{R})$. From the four-dimensional point of view, it is generated by the Lorentz boosts or rotations $\zeta_{ab} = i\left(x_a\partial_b - x_b\partial_a\right)$ with $x_a = \eta_{ab}x^b$. We list here explicitly the six generators in the Poincaré coordinates, as well as the action they correspond to:

$$\begin{aligned}
L_1 &= \frac{1}{2}\left(\zeta_{32}-\zeta_{01}\right) = -i\left(\frac{x^-}{2}u\partial_u + \frac{1}{2}\left((x^-)^2-1\right)\partial_- - \frac{u^2}{2}\partial_+\right) \ , \quad g \to \mathrm{e}^{-\frac{\lambda}{2}\sigma^1}g, \\
L_2 &= -\frac{1}{2}\left(\zeta_{02}-\zeta_{31}\right) = -i\left(\frac{1}{2}u\partial_u + x^-\partial_-\right) \ , \quad g \to \mathrm{e}^{-\frac{\lambda}{2}\sigma^3}g \\
L_3 &= \frac{1}{2}\left(\zeta_{03}-\zeta_{12}\right) = i\left(\frac{x^-}{2}u\partial_u + \frac{1}{2}\left((x^-)^2+1\right)\partial_- - \frac{u^2}{2}\partial_+\right) \ , \quad g \to \mathrm{e}^{i\frac{\lambda}{2}\sigma^2}g, \\
R_1 &= \frac{1}{2}\left(\zeta_{01}+\zeta_{32}\right) = i\left(\frac{x^+}{2}u\partial_u + \frac{1}{2}\left((x^+)^2-1\right)\partial_+ - \frac{u^2}{2}\partial_-\right) \ , \quad g \to g\mathrm{e}^{\frac{\lambda}{2}\sigma^1}, \\
R_2 &= \frac{1}{2}\left(\zeta_{31}-\zeta_{02}\right) = -i\left(\frac{1}{2}u\partial_u + x^+\partial_+\right) \ , \quad g \to g\mathrm{e}^{-\frac{\lambda}{2}\sigma^3} \\
R_3 &= \frac{1}{2}\left(\zeta_{03}+\zeta_{12}\right) = -i\left(\frac{x^+}{2}u\partial_u + \frac{1}{2}\left((x^+)^2+1\right)\partial_+ - \frac{u^2}{2}\partial_-\right) \ , \quad g \to g\mathrm{e}^{i\frac{\lambda}{2}\sigma^2}.
\end{aligned}$$

Both sets satisfy the algebra (A.1). Notice also that in terms of Euler angles defined by

$$g = \mathrm{e}^{i(t+\phi)\sigma_2/2}\mathrm{e}^{r\sigma_1}\mathrm{e}^{i(t-\phi)\sigma_2/2}, \tag{A.6}$$

L_3 and R_3 simplify considerably:

$$L_3 + R_3 = -i\partial_t \ , \quad L_3 - R_3 = -i\partial_\phi; \tag{A.7}$$

these generate time translations and rotations around the center.

We will now focus on the WZW model on $SL(2,\mathbb{R})$. The above isometries turn into symmetries of the action displayed in Eq. (2.12), leading thereby to conserved currents. In writing Eq. (2.14), we have chosen a gauge for the B field:

$$B = -\frac{1}{2u^2} dx^+ \wedge dx^- .$$

The two-form is not invariant under $R_{1,3}$ and $L_{1,3}$, and the action (2.14) leads correspondingly to boundary terms which must be properly taken into account in order to reach the conserved currents. The latter can be put in an improved-Noether form, in which they have only holomorphic (for L_i's) or anti-holomorphic (for R_j's) components. These are called $J^i(z)$ and $\bar{J}^j(\bar{z})$ respectively. Their expressions are the following:

$$J^1(z) \pm J^3(z) = -\frac{k}{8\pi} \mathrm{Tr}\left(\sigma^1 \mp i\sigma^2\right) \partial g\, g^{-1},\ J^2(z) = -\frac{k}{8\pi} \mathrm{Tr} \sigma^3 \partial g\, g^{-1},$$
$$\bar{J}^1(\bar{z}) \pm \bar{J}^3(\bar{z}) = \frac{k}{8\pi} \mathrm{Tr}\left(\sigma^1 \pm i\sigma^2\right) g^{-1} \bar{\partial} g,\ \bar{J}^2(\bar{z}) = -\frac{k}{8\pi} \mathrm{Tr} \sigma^3 g^{-1} \bar{\partial} g.$$

These yield in Poincaré coordinates:

$$J^1 + J^3 = -\frac{k}{4\pi} \frac{\partial x^+}{u^2} = -\frac{k}{4\pi} J \tag{A.8}$$
$$J^1 - J^3 = \frac{k}{4\pi} \left(2x^- \frac{\partial u}{u} - \partial x^- + (x^-)^2 \frac{\partial x^+}{u^2} \right) \tag{A.9}$$
$$J^2 = \frac{k}{4\pi} \left(\frac{\partial u}{u} + x^- \frac{\partial x^+}{u^2} \right), \tag{A.10}$$
$$\bar{J}^1 + \bar{J}^3 = \frac{k}{4\pi} \frac{\bar{\partial} x^-}{u^2} = \frac{k}{4\pi} \bar{J} \tag{A.11}$$
$$\bar{J}^1 - \bar{J}^3 = \frac{k}{4\pi} \left(-2x^+ \frac{\bar{\partial} u}{u} + \bar{\partial} x^+ - (x^+)^2 \frac{\bar{\partial} x^-}{u^2} \right) \tag{A.12}$$
$$\bar{J}^2 = \frac{k}{4\pi} \left(\frac{\bar{\partial} u}{u} + x^+ \frac{\bar{\partial} x^-}{u^2} \right), \tag{A.13}$$

where J and $\bar{J}$ are the null currents introduced in Eq. (2.15).

At the quantum level, these currents, when properly normalized, satisfy the following $\widehat{SL}(2,\mathbb{R})_{\mathrm{L}} \times \widehat{SL}(2,\mathbb{R})_{\mathrm{R}}$ OPA[19]:

$$\begin{aligned} J^3(z) J^3(0) &\sim -\frac{k}{2z^2}, \\ J^3(z) J^{\pm}(0) &\sim \pm \frac{J^{\pm}}{z}, \\ J^+(z) J^-(0) &\sim \frac{2J^3}{z} - \frac{k}{z^2}, \end{aligned} \tag{A.14}$$

[19]In some conventions the level is $x = -k$. This allows to unify commutation relations for the affine $\widehat{SL}(2,\mathbb{R})_x$ and $\widehat{SU}(2)_x$ algebras. Unitarity demands $x < -2$ for the former and $0 < x$ with integer x for the latter.

and similarly for the right movers. Equivalently on the modes of these currents generate the affine Lie algebra:

$$\begin{aligned}
\left[J^3_n, J^3_m\right] &= -\frac{k}{2} n \delta_{m,-n}, \\
\left[J^3_n, J^\pm_m\right] &= \pm J^\pm_{n+m}, \\
\left[J^+_n, J^-_m\right] &= 2J^3_{n+m} - kn\delta_{m,-n}.
\end{aligned}$$

The Virasoro algebra generators of the conformal field theory are built out of these currents:

$$\begin{aligned}
L_0 &= \frac{-1}{k-2}\left[\frac{1}{2}\left(J^+_0 J^-_0 + J^-_0 J^+_0\right) + (J^3_0)^2 + \sum_{m=1}^{\infty}\left(J^+_{-m}J^-_m + J^-_{-m}J^+_m + 2J^3_{-m}J^3_m\right)\right] \\
L_n &= \frac{-1}{k-2}\sum_{m=1}^{\infty}\left(J^+_{n-m}J^-_m + J^-_{n-m}J^+_m + 2J^3_{n-m}J^3_m\right).
\end{aligned}$$

The central charge is $c = 3 + 6/(k-2)$.

Lowest-weight representation of this CFT can be constructed by using the standard rule: start with a set of primary states annihilated by the operators J^i_n with $n \geq 1$; these ground states fall into representations of the global algebra generated by the zero modes $J^{\pm,3}_0$. The module is then constructed by acting with the creation operators J^i_{-n} $(n \geq 1)$.

Because the metric of the algebra is indefinite, the representations of the affine algebra will contain negative norm states, and the CFT is not unitary. However, by using the Virasoro constraints it is possible to construct a unitary string theory containing the $SL(2,\mathbb{R})$ CFT. Since the level 0 generators commute with the Virasoro algebra, the spectrum of the string theory must be constructed out of unitary representations of $SL(2,\mathbb{R})$. The unitary representations relevant here are: the *discrete representations* $\mathcal{D}^\pm(j)$ with $j > 0$ and the *principal continuous representations* $\mathcal{C}_p(b,a)$. The second step in the proof of the unitarity of the spectrum is to show that the negative norm states obtained with the creation operators are removed at each level by the Virasoro constraints. For the discrete representations, this is true only if the spin of the allowed representations is bounded: $0 < j < k/2$. This is not consistent with the general structure of string theory ; in fact assuming that the internal CFT contributes positively to L_0, this restriction on the spin puts an *absolute upper bound* on the level of string excitations: $N \leq 1 + k/4$. The case of continous representations is worse: the only allowed states are tachyons.

From the representations given above, it is possible to construct new ones by acting with an automorphism of the affine algebra called *spectral flow* $(w \in \mathbb{Z})$:

$$\begin{aligned}
\tilde{J}^3_n &= J^3_n - \frac{k}{2} w\ \delta_{n,0}, \\
\tilde{J}^\pm_n &= J^\pm_{n\pm w}.
\end{aligned}$$

This solves the above consistency problem. The eigenvalues of the states are then shifted according to:

$$\tilde{m} = m - \frac{k}{2}w\ ,\quad \tilde{\bar{m}} = \bar{m} - \frac{k}{2}w,$$

and the Virasoro generators as:

$$\tilde{L}_n = L_n + wJ_n^3 - \frac{k}{4}w^2\delta_{n,0} = L_n + w\tilde{J}_n^3 + \frac{k}{4}w^2\delta_{n,0}.$$

The flowed representations obtained from the lowest-weight representations constructed above are generically not bounded from below ; after imposing the Virasoro constraints, one can show that the physical spectrum of the string theory still contains only positive norm states.

B. Free-boson conformal blocks

The generic conformal blocks for a free compactified boson are the $U(1)$ characters,

$$\zeta\begin{bmatrix}\omega\\ \mu\end{bmatrix}\left(R^2\right) = \frac{R}{\sqrt{\tau_2}}\exp - \frac{\pi R^2}{\tau_2}\left|\omega\tau - \mu\right|^2, \tag{B.1}$$

where R is the compactification radius (imaginary for a time-like boson), and ω, μ need not be integers.

The partition function for an ordinary, free compactified boson reads (in the Lagrangian representation):

$$Z(R) = \frac{\Gamma_{1,1}(R)}{\eta\bar{\eta}} = \frac{1}{\eta\bar{\eta}}\sum_{m,w\in\mathbb{Z}}\zeta\begin{bmatrix}w\\ m\end{bmatrix}\left(R^2\right). \tag{B.2}$$

Notice that for $(\omega,\mu)\in\mathbb{R}^2$ in (B.1), a modular-invariant combination is provided by

$$\begin{aligned}\tilde{Z}(R) &= \frac{1}{\eta\bar{\eta}}\int_{\mathbb{R}^2} d\omega\, d\mu\, R^2\,\zeta\begin{bmatrix}\omega\\ \mu\end{bmatrix}\left(R^2\right) &\text{(B.3)}\\ &= \frac{R}{\sqrt{\tau_2}\eta\bar{\eta}}, &\text{(B.4)}\end{aligned}$$

which is the partition function of a decompactified free boson (the measure $d\omega\, d\mu\, R^2$ ensures the correct scaling of with R).

Bosons can also be twisted or shifted. This corresponds to ordinary or freely acting orbifolds. We will first focus on $\mathbb{Z}_N$ shifts, whose spectra are also captured in Eq. (B.1). The shifted-partition-function sectors read in this case:

$$Z\begin{bmatrix}h\\ g\end{bmatrix}(R) = \frac{\Gamma_{1,1}\left[{h \atop g}\right](R)}{\eta\bar{\eta}} = \frac{1}{\eta\bar{\eta}}\sum_{m,w\in\mathbb{Z}}\zeta\begin{bmatrix}w+h/N\\ m+g/N\end{bmatrix}\left(R^2\right)\ ,\quad h,g\in\{0,\ldots,N-1\}, \tag{B.5}$$

and satisfy the periodicity conditions

$$Z\begin{bmatrix}h\\ g\end{bmatrix}(R) = Z\begin{bmatrix}h+N\\ g\end{bmatrix}(R) = Z\begin{bmatrix}h\\ g+N\end{bmatrix}(R). \tag{B.6}$$

The basic properties of the quantities introduced so far are summarized as follows:

$$\begin{aligned}\tau\to\tau+1 &: \quad \zeta\begin{bmatrix}\omega\\ \mu\end{bmatrix}\left(R^2\right)\to\zeta\begin{bmatrix}\omega\\ \mu-\omega\end{bmatrix}\left(R^2\right),\\ \tau\to-\frac{1}{\tau} &: \quad \zeta\begin{bmatrix}\omega\\ \mu\end{bmatrix}\left(R^2\right)\to|\tau|\zeta\begin{bmatrix}\mu\\ -\omega\end{bmatrix}\left(R^2\right),\end{aligned}$$

and

$$\frac{1}{N}\sum_{h,g=0}^{N-1} Z\begin{bmatrix} h \\ g \end{bmatrix}(R) = Z\left(\frac{R}{N}\right). \tag{B.7}$$

Notice finally that the duality symmetry of the partition function (B.2), namely $Z(R) = Z\left(R^{-1}\right)$, does not survive the $\mathbb{Z}_N$ shift. This becomes clear in the following identity, obtained by double Poisson resummation:

$$\sum_{m,w\in\mathbb{Z}} \zeta\begin{bmatrix} w+h/N \\ m+g/N \end{bmatrix}\left(R^{-2}\right) = \sum_{y,n\in\mathbb{Z}} \mathrm{e}^{\frac{2i\pi}{N}(ng-yh)}\zeta\begin{bmatrix} n \\ y \end{bmatrix}\left(R^{2}\right).$$

As a consequence, the two limits ($R \to 0$ or ∞) of (B.5) are distinct:

$$Z\begin{bmatrix} h \\ g \end{bmatrix}(R) \xrightarrow[R\to\infty]{} \begin{cases} R/\sqrt{\tau_2}\eta\bar{\eta} & \text{for } h=g=0, \\ 0 & \text{otherwise}, \end{cases} \tag{B.8}$$

whereas

$$Z\begin{bmatrix} h \\ g \end{bmatrix}(R) \xrightarrow[R\to 0]{} \frac{1}{R\sqrt{\tau_2}\eta\bar{\eta}} \quad \forall h, g, \tag{B.9}$$

up to exponentially suppressed terms[20].

We now consider $\mathbb{Z}_N$ twists of a two-torus, for $N \leq 4$. The corresponding sums read:

$$\begin{aligned} Z_{2,2}\begin{bmatrix} 2h/N \\ 2g/N \end{bmatrix} &= \frac{\Gamma_{2,2}(T,U)}{\eta^2\bar{\eta}^2} \quad \text{for} \quad h=g=0, \\ &= 4\frac{\eta\bar{\eta}}{\left|\vartheta\left[{}^{1+2h/N}_{1-2g/N}\right](0|\tau)\right|^2}\sin^2\pi\frac{\Lambda(h,g)}{N} \text{ otherwise}, \end{aligned} \tag{B.10}$$

where T, U are the usual T^2-compactification moduli. Here $\Lambda(h,g)$ is an integer which is correlated to the number of fixed points of the torus, depending of the twisted sector under consideration. In the case of a T^4, the $\mathbb{Z}_N$ twists give rise to twisted sectors which are the square of the those given in Eq. (B.10). The $T^2/\mathbb{Z}_N$ twisted partition function reads:

$$Z^{\text{twisted}}_{T^2/\mathbb{Z}_N}(T,U) = \frac{1}{N}\sum_{h,g=0}^{N-1} Z_{2,2}\begin{bmatrix} 2h/N \\ 2g/N \end{bmatrix}. \tag{B.11}$$

Shifts and twists can be combined. We will consider here two cases, which happen to play a role in the analysis of the $SL(2,\mathbb{R})$ and of its cosets. The first is a $\mathbb{Z}_N$ orbifold of a compact boson of radius R times a two-torus. The $\mathbb{Z}_N$ acts as a twist on the T^2 and as a shift on the orthogonal S^1. The order of the orbifold is restricted to $N \leq 4$ by the

[20] When h/N and g/N become continuous variables, δ-functions appear, which must be carefully normalized.

symmetries of the lattice of the two-torus. The partition function for this model reads:

$$
\begin{aligned}
Z_{\mathbb{Z}_N} &= \frac{1}{N}\sum_{h,g=0}^{N-1} \frac{\Gamma_{1,1}\left[{h \atop g}\right](R)}{\eta\bar\eta} Z_{2,2}\begin{bmatrix}2h/N\\2g/N\end{bmatrix} \\
&= \frac{1}{N}\frac{\Gamma_{1,1}(R)\,\Gamma_{2,2}(T,U)}{\eta^3\bar\eta^3} \\
&+ \frac{1}{N}\sum_{(h,g)\neq(0,0)} 4\sin^2\pi\frac{\Lambda(h,g)}{N}\frac{\mathrm{e}^{2\pi\tau_2 h^2/N^2}}{\left|\vartheta_1\left(\frac{h\tau-g}{N}|\tau\right)\right|^2}\sum_{m,w\in\mathbb{Z}}\zeta\begin{bmatrix}w+h/N\\m+g/N\end{bmatrix}\left(R^2\right). \qquad \text{(B.12)}
\end{aligned}
$$

Similarly, we can consider a $\mathbb{Z}_N\times\mathbb{Z}_N$ orbifold of four free bosons. The first $\mathbb{Z}_N$ acts as a shift on a compact boson of radius R_1 and as a twist on a T^2; the second $\mathbb{Z}_N$ acts as a shift on the compact boson of radius R_1 and similarly on another compact boson of radius R_2. The partition function now reads:

$$
\begin{aligned}
Z_{\mathbb{Z}_N\times\mathbb{Z}_N} &= \frac{1}{N^2}\sum_{h_1,g_1=0}^{N-1}\sum_{h_2,g_2=0}^{N-1}\frac{\Gamma_{1,1}\left[{h_1-h_2 \atop g_1-g_2}\right](R_1)}{\eta\bar\eta}\frac{\Gamma_{1,1}\left[{h_2 \atop g_2}\right](R_2)}{\eta\bar\eta}Z_{2,2}\begin{bmatrix}2h_1/N\\2g_1/N\end{bmatrix} \\
&= \frac{1}{N^2}\frac{\Gamma_{1,1}(R_1)\,\Gamma_{1,1}(R_2)\,\Gamma_{2,2}(T,U)}{\eta^4\bar\eta^4} \\
&+ \frac{1}{N^2}\sum_{(h_1,g_1)\neq(0,0)}\frac{\Gamma_{1,1}\left[{h_1 \atop g_1}\right](R_1)\,\Gamma_{1,1}(R_2)}{\eta^2\bar\eta^2}Z_{2,2}\begin{bmatrix}2h_1/N\\2g_1/N\end{bmatrix} \\
&+ \frac{1}{N^2}\sum_{(h_2,g_2)\neq(0,0)}\frac{\Gamma_{1,1}\left[{-h_2 \atop -g_2}\right](R_1)\,\Gamma_{1,1}\left[{h_2 \atop g_2}\right](R_2)\,\Gamma_{2,2}(T,U)}{\eta^4\bar\eta^4} \\
&+ \frac{1}{N^2}\frac{1}{\eta\bar\eta}\sum_{(h_1,g_1)\neq(0,0)}\sum_{(h_2,g_2)\neq(0,0)}4\sin^2\pi\frac{\Lambda(h_1,g_1)}{N}\frac{\mathrm{e}^{2\pi\tau_2 h_1^2/N^2}}{\left|\vartheta_1\left(\frac{h_1\tau-g_1}{N}|\tau\right)\right|^2}\times \\
&\times\sum_{m_1,w_1,m_2,w_2\in\mathbb{Z}}\zeta\begin{bmatrix}w_1+(h_1-h_2)/N\\m_1+(g_1-g_2)/N\end{bmatrix}\left(R_1^2\right)\,\zeta\begin{bmatrix}w_2+h_2/N\\m_1+g_2/N\end{bmatrix}\left(R_2^2\right). \qquad \text{(B.13)}
\end{aligned}
$$

Models based on freely acting orbifolds exhibit rich decompactification properties [67] [68] [69], which are due to the breaking of the duality symmetries. There are two limits of interest[21] here:

$$
Z_{\mathbb{Z}_N\times\mathbb{Z}_N}\xrightarrow[R_2\to 0]{}\frac{1}{R_2\sqrt{\tau_2}\eta\bar\eta}Z\left(\frac{R_1}{N}\right)Z^{\text{twisted}}_{T^2/\mathbb{Z}_N}(T,U) \qquad \text{(B.14)}
$$

and

$$
Z_{\mathbb{Z}_N\times\mathbb{Z}_N}\xrightarrow[R_2\to\infty]{}\frac{R_2}{N\sqrt{\tau_2}\eta\bar\eta}Z_{\mathbb{Z}_N}(R_1,T,U) \qquad \text{(B.15)}
$$

obtained by using the above equations. In the first limit the two circles decouple from the T^2, and the only reminiscence of the second $\mathbb{Z}_N$ shift is the rescaling R_1/N. In the second limit, only the decompactifying circle decouples.

[21]Similarly, for $R_1\to 0$ or ∞, we obtain respectively (B.14) or (B.15), with $R_1\leftrightarrow R_2$.

C. Derivation of the spectrum

C.1 The spectrum of $SL(2,\mathbb{R})$

In this appendix we solve for the constraints s_1 and s_2 to obtain the spectrum of $SL(2,\mathbb{R})$, following the lines of [14] and [18]. The total exponential factor in (3.4) after expanding the ϑ_1 function and integrating t_2 is

$$\exp\Bigg\{ -\pi\tau_2 k(w_+ + s_1)(w_- - 2t_1 + s_1) - 2i\pi\tau_1 n(w_+ + s_1) \\ +2i\pi n s_2 - 4\pi\tau_2 \frac{1}{4(k-2)} + 2\pi\tau_2 s_1^2 - 2\pi\tau_2\left(q+\bar{q}+1\right)s_1 \\ +2i\pi\tau_1\left(q-\bar{q}\right)s_1 - 2i\pi\left(q-\bar{q}\right)s_2 - 2\pi\tau_2\left(N+\bar{N}\right) + 2i\pi\tau_1\left(N-\bar{N}\right)\Bigg\},$$

where q is the number of $J^+_{n<0}$ minus $J^-_{n<0}$ operators acting on the ground state, and similarly for $\bar{q}$. The integration over s_2 gives simply the constraint: $q-\bar{q}=n$. The remaining integration over s_1 reads:

$$\int_0^1 ds_1 \exp\left\{-2\pi\tau_2 s_1\left(k(w-t_1)+q+\bar{q}+1\right)-(k-2)\pi\tau_2 s_1^2\right\}.$$

As in [18], we introduce an auxiliary variable s in order to integrate the constraint:

$$\exp\left\{-2\pi\tau_2 s_1\left(k(w-t_1)+q+\bar{q}+1\right)-(k-2)\pi\tau_2 s_1^2\right\} \\ = 2\sqrt{\frac{\tau_2}{k-2}}\int_{-\infty}^{+\infty} ds\ \exp\left\{-4\pi\tau_2\frac{s^2}{k-2}-2\pi\tau_2\left(2is+q+\bar{q}+1+k(w-t_1)\right)s_1\right\}.$$

We would like to emphasize that s is not just an auxiliary variable but stands for the momentum of the Liouville field coupled to the other degrees of freedom of the theory. At this stage, the interpretation of the partition function is clear in the free-field representation (see Sec. 3.2). In this case, the BRST constraint relates the Liouville momentum, the oscillator number and the lattice of the light-cone coordinates. The integration over s_1 gives finally

$$\frac{\mathrm{e}^{-4\pi\tau_2\frac{s^2}{k-2}}}{\pi\sqrt{\tau_2(k-2)}}\left[\frac{1}{2is+q+\bar{q}+1+k(w-t_1)}-\frac{\mathrm{e}^{-2\pi\tau_2(2is+q+\bar{q}+1+k(w-t_1))}}{2is+q+\bar{q}+1+k(w-t_1)}\right].$$

We complete the square in the second term by shifting the integration contour of s in the complex plane: $s\to s-i(k-2)/2$. We thus pick up residues corresponding to the discrete representations, for

$$\mathrm{Im}s=\frac{q+\bar{q}+1+k(w-t_1)}{2}.$$

These poles are located in the strip

$$-(k-2)<q+\bar{q}+1+k(w-t_1)<0,$$

and correspond to the states of the discrete spectrum, obeying the constraint

$$\tilde{m} + \tilde{\bar{m}} = 2j + q + \bar{q} = -k(w - t_1).$$

Hence, we obtain the discrete representations in the correct range given in [13]:

$$\frac{1}{2} < j < \frac{k-1}{2}.$$

We note here that due to the continuous shift t_1 the spin is not quantized by the constraint.

Putting all factors together, we obtain the following weights for the discrete spectrum:

$$L_0 = -\frac{j(j-1)}{k-2} + w_+ \left(-\tilde{m} - \frac{k}{4} w_+ \right) + N,$$
$$\bar{L}_0 = -\frac{j(j-1)}{k-2} + w_+ \left(-\tilde{\bar{m}} - \frac{k}{4} w_+ \right) + \bar{N}.$$

The remaining part of the partition function reads:

$$\sum_{w,w_+} \int ds \left[\frac{1}{2is + q + \bar{q} + 1 + k(w - t_1)} - \frac{\mathrm{e}^{-2\pi\tau_2(q+\bar{q}+k(w-t_1+1/2))}}{2is + q + \bar{q} - 1 + k(w - t_1 + 1)} \right] \times$$
$$\times \exp\left\{ -2\pi\tau_2 \left(2\frac{s^2}{k-2} + kw_+ \left(w - t_1 - \frac{w_+}{2} \right) + N + \bar{N} - 2 \right) + 2i\pi\tau_1 \left(N - \bar{N} - nw_+ \right) \right\}.$$

In order to identify the continuous part of the spectrum, we note that the exponent of the second term, namely

$$-2\pi\tau_2 \left(2\frac{s^2}{k-2} + 2(w_+ + 1) \left(k(w - t_1) - \frac{k}{2} - \frac{k}{4}(w_+ + 1) \right) + N + q + \bar{N} + \bar{q} - 2 \right),$$

can be seen as the spectral flow by one unit of the w_+ sector of the theory: $w_+ \to w_+ + 1$, $m \to m - k/2$, $\bar{m} \to \bar{m} - k/2$, $N \to N + q$, $\bar{N} \to \bar{N} + \bar{q}$. We can then combine the first term from the w_+ sector and the second term from the flowed $w_+ - 1$ sector to obtain:

$$\sum_{w,w_+} \int ds \left[\frac{1}{2is + q + \bar{q} + 1 + k(w - t_1)} - \frac{1}{2is + q + \bar{q} - 1 + k(w - t_1)} \right] \times$$
$$\times \exp\left\{ -2\pi\tau_2 \left(\frac{s^2}{k-2} + kw_+ \left(w - t_1 - \frac{w_+}{2} \right) + N + \bar{N} - 2 \right) + 2i\pi\tau_1 \left(N - \bar{N} - nw_+ \right) \right\}.$$

The second line represents the density of long-string states, and gives a divergence while summing over q. By regularizing the sum as explained in [14],

$$\sum_{r=0}^{\infty} \frac{1}{A+r} \mathrm{e}^{-r\epsilon} = \log\epsilon - \frac{d}{dA} \log\Gamma(A),$$

we obtain the density of states of the continuous spectrum:

$$\rho(s) = \frac{1}{\pi} \log\epsilon + \frac{1}{4\pi i} \frac{d}{ds} \log \frac{\Gamma\left(\frac{1}{2} - is - \tilde{m}\right) \Gamma\left(\frac{1}{2} - is + \tilde{\bar{m}}\right)}{\Gamma\left(\frac{1}{2} + is - \tilde{m}\right) \Gamma\left(\frac{1}{2} + is + \tilde{\bar{m}}\right)}.$$

The weights of the continuous spectrum are

$$
\begin{aligned}
L_0 &= \frac{s^2+1/4}{k-2} + w_+\left(-\tilde{m} - \frac{k}{4}w_+\right) + N,\\
\bar{L}_0 &= \frac{s^2+1/4}{k-2} + w_+\left(-\tilde{\bar{m}} - \frac{k}{4}w_+\right) + \bar{N},
\end{aligned}
$$

with $\tilde{m} + \tilde{\bar{m}} = -k(w - t_1)$ and $\tilde{m} - \tilde{\bar{m}} = n$. Therefore we have identified both types of representations, including the sectors obtained by spectral flow.

C.2 The spectrum of the null-deformed $SL(2,\mathbb{R})$

We now derive the first-order spectrum of the null-deformed $SL(2,\mathbb{R})$ theory, whose partition function is given in (5.22).We expand, as previously, all oscillator terms and obtain the overall exponential factor

$$
\exp\Bigg\{ -2\pi\tau_2\alpha k w_+(w - t_1 - w_+/2) - 4\pi\tau_2\frac{s^2+1/4}{k-2} - 2i\pi\tau_1 n w_+
$$
$$
+2i\pi s_2(n - q + \bar{q}) - 2\pi\tau_2\left(2i\sqrt{\frac{\alpha k-2}{k-2}}s + q + \bar{q} + 1 + \alpha k(w - t_1)\right)s_1\Bigg\},
$$

where

$$
\alpha = \frac{M^2+1}{M^2}.
$$

After integrating over s_1 we are left with

$$
\frac{\exp\left\{-4\pi\tau_2\frac{s^2+1/4}{k-2}\right\}}{2\pi\tau_2\left(2i\sqrt{\frac{\alpha k-2}{k-2}}s + q + \bar{q} + 1 + \alpha k(w - t_1)\right)}
- \frac{\exp\left\{-2\pi\tau_2\left(2\frac{s^2+1/4}{k-2} + 2i\sqrt{\frac{\alpha k-2}{k-2}}s + q + \bar{q} + 1 + \alpha k(w - t_1)\right)\right\}}{2\pi\tau_2\left(2i\sqrt{\frac{\alpha k-2}{k-2}}s + q + \bar{q} + 1 + \alpha k(w - t_1)\right)}.
$$

As previously, we complete the square in the second term by shifting the Liouville momentum:

$$
s \to s - \frac{i}{2}\sqrt{(\alpha k-2)(k-2)}.
$$

The poles corresponding to the discrete representations are now

$$
\mathrm{Im}(s) = \frac{1}{2}\sqrt{\frac{k-2}{\alpha k-2}}\left[q + \bar{q} + 1 + \alpha k(w - t_1)\right];
$$

they are located in the strip:

$$
-\sqrt{(\alpha k-2)(k-2)} < q + \bar{q} + 1 + \alpha k(w - t_1) < 0.
$$

The deformed discrete spectrum, for $\alpha = 1+\varepsilon$, $\varepsilon \ll 1$, reads:

$$\begin{aligned}
L_0 &= -\frac{j(j-1)}{k-2} - w_+\left(\tilde{m} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) - \frac{k}{4}(1+\varepsilon)w_+^2 + N \\
&\quad + \frac{\varepsilon(1-2j)}{2(k-2)}\left(\tilde{m} + \tilde{\bar{m}} + \frac{k}{2(k-2)}(1-2j)\right), \\
\bar{L}_0 &= -\frac{j(j-1)}{k-2} - w_+\left(\tilde{\bar{m}} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) - \frac{k}{4}(1+\varepsilon)w_+^2 + \bar{N} \\
&\quad + \frac{\varepsilon(1-2j)}{2(k-2)}\left(\tilde{m} + \tilde{\bar{m}} + \frac{k}{2(k-2)}(1-2j)\right).
\end{aligned} \tag{C.1}$$

The last terms of both equations are due to the displacement of the poles corresponding to the discrete representations. By using the parameterization considered in the supersymmetric model, $\alpha k = p^2(k-2)+2$, the expressions are simpler: the poles of the discrete representations are now located at

$$\mathrm{Im}(s) = \frac{1}{2p}\left[q + \bar{q} + 1 + \left(2 + (k-2)p^2\right)(w - t_1)\right],$$

inside the strip

$$-p(k-2) < q + \bar{q} + 1 + \left(2 + (k-2)p^2\right)(w - t_1) < 0.$$

If we define, by analogy with the undeformed theory:

$$2\mathrm{Im}(s) \equiv 1 - 2j,$$

we find the discrete spectrum:

$$\begin{aligned}
L_0 &= -\frac{j(j-1)}{k-2} - w_+\left(j + q + \frac{1-p}{2}(1-2j)\right) - \frac{p^2(k-2)+2}{4}w_+^2 + N, \\
\bar{L}_0 &= -\frac{j(j-1)}{k-2} - w_+\left(j + \bar{q} + \frac{1-p}{2}(1-2j)\right) - \frac{p^2(k-2)+2}{4}w_+^2 + \bar{N}.
\end{aligned}$$

For the continuous spectrum, we use the spectral flow as in the undeformed case:

$$\begin{aligned}
&\left\{\frac{1}{2i\sqrt{\frac{\alpha k-2}{k-2}}s + q + \bar{q} + 1 + \alpha k(w-t_1)} - \frac{1}{2i\sqrt{\frac{\alpha k-2}{k-2}}s + q + \bar{q} - 1 + \alpha k(w - t_1 + 1)}\right\} \times \\
&\exp\left\{-2\pi\tau_2\left(2\frac{s^2+1/4}{k-2} - w_+\alpha k(w-t_1) - \frac{\alpha k}{2}w_+^2 + N + \bar{N}\right) + 2i\pi\tau_1\left(N - \bar{N} - nw_+\right)\right\}.
\end{aligned}$$

This gives the deformed continuous spectrum at first order:

$$\begin{aligned}
L_0 &= \frac{s^2+1/4}{k-2} - w_+\left(\tilde{m} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) + \frac{k}{4}(1+\varepsilon)w_+^2 + N, \\
\bar{L}_0 &= \frac{s^2+1/4}{k-2} - w_+\left(\tilde{\bar{m}} + \frac{\varepsilon}{2}(\tilde{m}+\tilde{\bar{m}})\right) + \frac{k}{4}(1+\varepsilon)w_+^2 + \bar{N},
\end{aligned}$$

with the density of long-string states:

$$\rho(s) = \frac{1}{\pi}\log\epsilon + \frac{1}{4\pi i}\frac{d}{ds}\log\frac{\Gamma\left(\frac{1}{2} - i\left(1+\frac{k\epsilon}{2(k-2)}\right)s - \tilde{m} - \frac{\tilde{m}+\tilde{\bar{m}}}{2}\epsilon\right)}{\Gamma\left(\frac{1}{2} + i\left(1+\frac{k\epsilon}{2(k-2)}\right)s - \tilde{m} - \frac{\tilde{m}+\tilde{\bar{m}}}{2}\epsilon\right)} \times$$
$$\times \quad \frac{\Gamma\left(\frac{1}{2} - i\left(1+\frac{k\epsilon}{2(k-2)}\right)s + \tilde{\bar{m}} + \frac{\tilde{m}+\tilde{\bar{m}}}{2}\epsilon\right)}{\Gamma\left(\frac{1}{2} + i\left(1+\frac{k\epsilon}{2(k-2)}\right)s + \tilde{\bar{m}} + \frac{\tilde{m}+\tilde{\bar{m}}}{2}\epsilon\right)}.$$

D. Theta functions

We recall here the basic properties of Jacobi functions. Our conventions are

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v|\tau) = \sum_{p\in\mathbb{Z}} \mathrm{e}^{\pi i\tau\left(p+\frac{a}{2}\right)^2 + 2\pi i\left(v+\frac{b}{2}\right)\left(p+\frac{a}{2}\right)}$$

$a, b \in \mathbb{R}$, so that

$$\vartheta_1 = \vartheta\begin{bmatrix}1\\1\end{bmatrix}, \quad \vartheta_2 = \vartheta\begin{bmatrix}1\\0\end{bmatrix}, \quad \vartheta_3 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}, \quad \vartheta_4 = \vartheta\begin{bmatrix}0\\1\end{bmatrix}.$$

We also recall that

$$\vartheta_1(v|\tau) = -2q^{1/8}\sin\pi v \prod_{m=1}^{\infty}\left(1-\mathrm{e}^{2i\pi v}q^m\right)\left(1-q^m\right)\left(1-\mathrm{e}^{-2i\pi v}q^m\right),$$
$$\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m),$$

and

$$\vartheta_1' = -2\pi\eta^3 = -\pi\vartheta_2\vartheta_3\vartheta_4,$$

where the prime stands for $\partial_v|_{v=0}$. Notice that

$$|\vartheta_1(a\tau+b|\tau)|^2 = \mathrm{e}^{2\pi\tau_2 a^2}\left|\vartheta\begin{bmatrix}1+2a\\1+2b\end{bmatrix}(0|\tau)\right|^2,$$

which leads in particular to the following:

$$\left|\vartheta\begin{bmatrix}1+2h/N\\1-2g/N\end{bmatrix}(0|\tau)\right| = \mathrm{e}^{-\pi\tau_2 h^2/N^2}\left|\vartheta_1\left(\frac{h\tau-g}{N}|\tau\right)\right|. \tag{D.1}$$

Finally, the Riemann identity[22] reads:

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+\mu ab}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v_0)\prod_{j=1}^{3}\vartheta\begin{bmatrix}a+h_j\\b+g_j\end{bmatrix}(v_j) = \tag{D.2}$$
$$= \vartheta\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{(-)^{\mu+1}v_0 - \sum_j v_j}{2}\right)\prod_{j=1}^{3}\vartheta\begin{bmatrix}1-h_j\\1-g_j\end{bmatrix}\left(\frac{(-)^{\mu+1}v_0 + \cdots - v_j + \cdots)}{2}\right),$$

where the parameter $\mu = 0$ or 1, and $\sum_j h_j = \sum_j g_j = 0$.

[22]We use the short-hand notation $\vartheta\left[{a \atop b}\right](v)$ for $\vartheta\left[{a \atop b}\right](v|\tau)$.

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