N-particles approximation of the Vlasov equations with singular potential

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Abstract. We prove the convergence in any time interval of a point-particle approximation of the Vlasov equation by particles initially equally distributed for a force in $1/|x|^\alpha$, with $\alpha \leq 1$. We introduce discrete versions of the L^∞ norm and time averages of the force field. The core of the proof is to show that these quantities are bounded and that consequently the minimal distance between particles in the phase space never vanishes.

Key words. Derivation of kinetic equations. Particle methods. Vlasov equations.

1 Introduction

We are interested here by the validity of the modeling of a continuous media by a kinetic equation, with a density of presence in phase and speed. It others words, does the many particles follow the evolution given by the continuous media when their number is sufficiently large? This is a very general question and this paper claims to give an (partial) answer only for the mean field approach.

Let us be more precise. We study the evolution of N particles, centered at $(X_1, \ldots, X_n)$ in $\mathbb{R}^d$ with velocities $(V_1, \ldots, V_n)$ and interacting with a central force $F(x)$. The positions and velocities satisfy the following system of ODEs

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E(X_i) = \displaystyle\sum_{j \neq i} \frac{\alpha_i \, \alpha_j}{m_i} \, F(X_i - X_j), \end{cases} \tag{1.1}$$

where the initial conditions $(X_1^0, V_1^0, \ldots, X_n^0, V_n^0)$ are given. The prime example for (1.1) consists in charged particles with charges α_i and masses m_i, in which case $F(x) = -x/|x|^3$ in dimension three.

To easily derive from (1.1) a kinetic equation (at least formally), it is very convenient to assume that the particles are identical which means $\alpha_i = \alpha_j$. Moreover we will rescale system (1.1) in time and space to work with quantities of order one, which means that we may assume that

$$\frac{\alpha_i \, \alpha_j}{m_i} = \frac{1}{N}, \quad \forall i, j. \tag{1.2}$$

We write now the Vlasov equation modelling the evolution of a density f of particles interacting with a radial force in $F(x)$. This is a kinetic equation in the sense that the density depends on the position and on the velocity (and of course of the time)

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + E(x) \cdot \nabla_v f = 0, \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}^d, \ v \in \mathbb{R}^d, \\ &F(x) = \nabla\Big(\int_{x,v} \rho(t,y) \, F(x-y) \, dy\Big), \\ &\rho(t,y) = \int_v f(t,x,v) \, dv. \end{aligned} \tag{1.3}$$

Here ρ is the spatial density and the initial density f^0 is given.

When the number N of particles is large, it is obviously easier to study (or solve numerically) (1.3) than (1.1). Therefore it is a crucial point to determine whether (1.3) can be seen as a limit of (1.1).

Remark that if $(X_1, \ldots, X_N, V_1, \ldots, V_n)$ is a solution of (1.1), then the measure

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{n} \delta(x - X_i(t)) \otimes \delta(v - V_i(t))$$

is a solution of the Vlasov equation in the sense of distributions. And the question is whether a weak limit f of μ_N solves (1.3) or not. If F is smooth, then it is indeed the case as it is proved in the book by Spohn [20]. The purpose of this paper is to justify this limit if

$$|F(x)| \leq \frac{C}{|x|^\alpha}, \quad |\nabla F(x)| \leq \frac{C}{|x|^{1+\alpha}} \quad |\nabla^2 F(x)| \leq \frac{C}{|x|^{2+\alpha}}, \quad \forall x \neq 0, \tag{1.4}$$

for $\alpha < 1$, which is the first rigorous proof of the limit in a case where F is not necessarily bounded.

Before being more precise concerning our result, let us explain what is the meaning of (1.1) in view of the singularity in F. Here we assume either that we restrict ourselves to the initial configurations for which there are no collisions between particles over a time interval $[0,\ T]$ with a fixed T, independent of N. Or we assume that F is regular or regularized but that the norm $\|F\|_{W^{1,\infty}}$ may depend on N; This procedure is well presented in [1] and it is the usual one in numerical simulations (see [21] and [22]). In both cases, we have classical solutions to (1.1) but the only bound we may use is (1.4).

Other possible approaches would consist in justifying that the set of initial configurations $X_1(0), \ldots, X_N(0), V_1(0), \ldots, V_N(0)$ for which there is at least one collision, is negligible or that it is possible to define a solution (unique or not) to the dynamics even with collisions.

Finally notice that the condition $\alpha < 1$ is not unphysical. Indeed if F derives from a potential, $\alpha = 1$ is the critical exponent for which repulsive and attractive forces seem very different. In other words, this is the point where the behavior of the force when two particles are very close takes all its importance.

1.1 Important quantities

The derivation of the limit requires a control on many quantities. Although some of them are important only at the discrete level, many were already used to get the existence of strong solutions to the Vlasov-Poisson equation (we refer to [8], [9] and [15], [17] as being the closest from our method).

The first two are quite natural and are bounds on the size of the support of the initial data in space and velocity, namely we introduce

$$R(T) = \sup_{t\in[0,T],\, i=1,\ldots N} X_i(t), \quad K(T) = \sup_{t\in[0,T],\, i=1,\ldots N} V_i(t). \tag{1.5}$$

Of course R is trivially controlled by K since

$$R(T) \leq R(0) + T\,K(T). \tag{1.6}$$

Now a very important and new parameter is the discrete scale of the problem denoted ε. This quantity represents roughly the minimal distance between two particles or the minimal time interval which the discrete dynamics can see. We fix this parameter from the beginning and somehow the main part of our work is to show that it is indeed correct, so take

$$\varepsilon = \frac{R(0)}{N^{1/2d}}. \tag{1.7}$$

At the initial time, we will choose our approximation so that the minimal distance between two particles will be of order ε.

The force term cannot be bounded at every time for the discrete dynamics (a quantity like $F \star \rho_N$ is not bounded even in the case of free transport), but we can expect that its average on a short interval of time will be bounded. So we denote

$$\overline{E}(T) = \sup_{t\in[t_0,T-\varepsilon],i=1,\ldots,N} \left\{ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |E(X_i(s))|\,ds \right\}, \tag{1.8}$$

with for $T < \varepsilon$

$$\overline{E}(T) = \sup_{i=1,\ldots,N} \left\{ \frac{1}{\varepsilon} \int_0^{T} |E(X_i(s))|\,ds \right\}, \tag{1.9}$$

thus obtaining a continuous definition. Moreover we denote by E^0 the supremum over all i of $|E(X_i(0))|$.

This definition comes from the following intuition. The force is big when two particles are close together. But if their speeds are different, they won't stay close a long time. So we can expect the interaction force between these two particles to be integrable in time even if they "collide". They just remain the case of two close particles with almost the same speed. To estimate the force created by them, we need an estimate on their number. On way of obtaining it is to have a bound on

$$m(T) = \sup_{t\in[0,T],i\neq j} \frac{\varepsilon}{|X_i(t)-X_j(t)| + |V_i(t)-V_j(t)|}, \tag{1.10}$$

The control on m requires the use of a discretized derivative of E, more precisely we define for any exponent $\beta \in \,]1,\ d-\alpha[$ which we note also satisfies $\beta < 2d - 3\alpha$ ($\beta = 1$ would be enough for short time estimates)

$$\Delta\overline{E}(T) = \sup_{t\in[t_0,T-\varepsilon]} \sup_{i,j=1,\dots,N,} \left\{ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \frac{|E(X_i(s)) - E(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\, ds \right\}, \tag{1.11}$$

with as for $\overline{E}$, when $T < \varepsilon$

$$\Delta\overline{E}(T) = \sup_{i,j=1,\dots,N} \left\{ \frac{1}{\varepsilon} \int_0^{T} \frac{|E(X_i(s)) - E(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\, ds \right\}. \tag{1.12}$$

Now, we introduce what we called the discrete infinite norm of the distribution of the particle μ_N. This quantities is the supremum over all the boxes of size ε of the total mass they contains divided by the size of the box. That is, for a measure μ we denote

$$\|\mu\|_{\infty,\varepsilon} = \frac{1}{(2\varepsilon)^6} \sup_{(x,v)\in\mathbb{R}^6} \left\{ \mu(B_\infty((x,v),\varepsilon)) \right\}. \tag{1.13}$$

where $B_\infty((x,v),\varepsilon)$ is the ball of radius ε centered at (x,v) for the infinite norm. Note that we may bound $\|\mu_N(T,\cdot)\|_{\infty,\varepsilon}$ by

$$\|\mu_N(T,\cdot)\|_{\infty,\varepsilon} \leq (4\, m(T))^{2d}. \tag{1.14}$$

All the previous quantities (except for ε) will always be assumed to be bounded at the initial time $T = 0$ uniformly in N.

1.2 Main results

The main point in the derivation of the Vlasov equation is to obtain a control on the previous quantities. We first do it for a short time as given by

Theorem 1.1. *If $\alpha < 1$, there exists a time T and a constant c depending only on $R(0)$, $K(0)$, $m(0)$ but not on N such that for some $\alpha < \alpha' < 3$*

$$\begin{aligned} &R(T) \leq 2\,(1 + R(0)), \quad K(T) \leq 2\,(1 + K(0)), \quad m(T) \leq 2\, m(0), \\ &\overline{E}(T) \leq c\,(m(0))^{2\alpha'}\,(K(0))^{\alpha'}\,(R(0))^{\alpha'-\alpha}, \quad \sup_{t\leq T} \|\mu_N(t,\cdot)\|_{\infty,\varepsilon} \leq (8\, m(0))^{2d}. \end{aligned}$$

Remark
The constant 2 is of course only a matter of convenience. But the choice of a different constant is not really helpful. This result is valid only for a short time in essence and increasing the chosen constant for instance, increases only slightly the time T who in any case cannot pass a critical value.

This theorem can be extended on any time interval

Theorem 1.2. *For any time* $T > 0$*, there exists a function* $\tilde{N}$ *of* $R(0)$*,* $K(0)$*,* $m(0)$*, and* T *and a constant* $C(R(0), K(0), m(0), T)$ *such that if* $N \geq \tilde{N}$ *then*

$$R(T),\ K(T),\ m(T),\ \overline{E}(T) \leq C(R(0), K(0), m(0), T).$$

¿From this last theorem, it is easy to deduce the main result of this paper which reads

Theorem 1.3. *Consider a time* T *and sequence* $\mu_N(t)$ *corresponding to solutions to* (1.1) *such that* $R(0)$*,* $K(0)$ *and* $m(0)$ *are bounded uniformly in* N*. Then any weak limit* f *of* $\mu_N(t)$ *in* $L^\infty([0,\ T],\ M^1(\mathbb{R}^{2d}))$ *belongs to* $L^\infty([0,\ T],\ L^1 \cap L^\infty(\mathbb{R}^{2d}))$*, has compact support and is a solution to* (1.3)*.*

Of course the main limitation of our results is the condition $\alpha < 1$ and the main open question is to know what happens when $\alpha \geq 1$. However this condition is not only technical and new ideas will be needed to prove something for $\alpha \geq 1$.
It would also be interesting to extend our result to more complicated forces like the ones found in the formal derivation of [12].
The derivation of kinetic equations is an important question both for numerical and theoretical aspects. We already mentioned the works of Batt [1], Spohn [20], Victory and Allen [21] and Wollmann [22] for Vlasov equations. Another interesting case concerns Boltzmann equation, for which we refer to the book by Cercignani, Illner and Pulvirenti [4] and the paper by Illner and Pulvirenti [10].
On the other hand, the derivation of macroscopic equations is usually easier and some results are already known (although not since a very long time) even in cases with singularity. In particular and that is more or less the macroscopic equivalent of our result, the convergence of the point vortex method for $2-D$ Euler equations was obtained by Goodman, Hou and

Lowengrub [7] (see also the works by Schochet [18] and [19]). The main difficulty for macroscopic systems is to control the minimal distance between two particles (which is not possible in the kinetic framework) as it is also clear in [11].
Our method of proof makes full use of the characteristics and of the procedures developed to get for the Vlasov-Poisson equation in dimension two and three. This method was introduced by Horst in [8] and [9] and was successfully used to prove the existence of strong solutions in large time in [15] and [17] and at the same time by Lions and Perthame in [13] using the moments (see also [5] for a slightly simpler proof and [14] for an application to the asymptotic behavior of the equation). These results were extended to the periodic case by Batt and Rein in [3] and to the Vlasov-Poisson-Fokker-Planck equation by Bouchut in [2]. In particular the necessity to integrate in time to control the oscillations of the force also appears in the proof of L^∞ bounds for the Vlasov-Poisson-Fokker-Planck equation by Pulvirenti and Simeoni in [16]. We refer to the book by Glassey [6] for a general discussion of the existence theory for kinetic equations.
In the rest of the paper, C will denote a generic constant, depending maybe on $R(0)$, $K(0)$, or $m(0)$ but not on N or any other quantity. We first prove Theorem 1.1, then we show a preservation of discrete L^∞ norms which proves Theorem 1.2. In the last section we explain how to deduce Theorem 1.3.

2 Proof of Theorem 1.1

The first steps are to estimate all quantities in term of themselves. Then if this is done correctly it is possible to deduce bounds for them on a short interval of time.

2.1 Estimate on $\overline{E}$

We are in fact able to estimate any time average of the force field, more precisely

Lemma 2.1. *For any α' with $\alpha < \alpha' < 3$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\begin{aligned}\overline{E}(t_0) \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\, K^{\alpha'}\, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} \\ + \varepsilon^{2d-3\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha} \overline{E}^d\, K^d),\end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$*,* R*,* K*,* m *and* $\overline{E}$ *at the time* t_0*.*

Of course if any of the above quantity is infinite then the result is obvious. This lemma could appear stupid since we control $\overline{E}(t_0)$ by itself (and with a power larger than 1 in addition). But the point is that except for the first term, the other two are very small because of the ε in front of them so that they almost do not count.

Proof. For any index i, we bound for any t_1 less than $t_0 - \varepsilon$ or $t_1 = 0$ if $t_0 < \varepsilon$

$$I^i = \frac{1}{\varepsilon} \int_{t_1}^{t_1+\varepsilon} |E(X_i(s))|\, ds,$$

which will give the desired result by taking the supremum over all i. So now let us fix t_1 and i (we choose $i = 1$ for simplicity).
We introduce the following decomposition among the particles: We define

$$C_k = \left\{ i \,\middle|\, 3\,\varepsilon\, K(t_0)\, 2^{k-1} < |X_i(t_1) - X_1(t_1)| \leq 3\,\varepsilon\, K(t_0)\, 2^k \right\}. \tag{2.1}$$

Therefore the index k varies from 1 to $k_0 = (\ln(R/4\,\varepsilon\, K(t_0)))/\ln 2$. And we denote by C_0 the rest, that is the set of indices i such that $|X_i(t_1) - X_1(t_1)| \leq 3\,\varepsilon\, K(t_0)$.
Consequently we decompose the term I^1 into two parts, $I_1 + I_{C_0}$ with

$$I_1 = \sum_{k=2}^{k_0} \sum_{i \in C_k} \frac{1}{\varepsilon} \int_{t_1}^{t_0} \frac{1}{N\,|X_1(t) - X_i(t)|^\alpha}\, dt, \tag{2.2}$$

and I_{C_0} is the sum of the same terms over the particles of C_0.

Step 1: Stability of the C_k*.* Given their definition, the C_k enjoy the following property, for any $i \in C_k$ with $k > 1$, we have for any $t \in [t_1,\ t_0]$

$$|X_1(t) - X_i(t)| \geq \varepsilon\, K(t_0)\, 2^{k-1}.$$

Indeed, we of course know that

$$\left|\frac{d}{dt}(X_i(t) - X_1(t))\right| = |V_i(t) - V_1(t)| \leq 2\,K(t_0),$$

and then

$$\begin{aligned}|X_1(t) - X_i(t)| &\geq |X_1(t_1) - X_i(t_1)| - 2\,(t_0 - t_1)\,K(t_0)\\ &\geq 3\,\varepsilon\,K(t_0)\,2^{k-1} - 2\,\varepsilon\,K(t_0),\end{aligned}$$

with the corresponding result since $k \geq 1$. Of course the same argument also shows that if $i \in C_0$ then for any $t \in [t_1,\ t_0]$,

$$|X_1(t) - X_i(t)| \leq 5\,\varepsilon\,K(t_0).$$

Step 2: Control of I_1. Using the result from the previous step, we deduce that for any $i \in C_k$ with $K \geq 1$,

$$\frac{1}{|X_i(t) - X_1(t)|^\alpha} \leq \frac{C\,2^{-\alpha k}}{\varepsilon^\alpha\,(K(t_0))^\alpha}.$$

On the other hand, we have of course $|C_k| \leq N$ and moreover $|C_k| \leq C\,\varepsilon^{-2d}\,K^{2d}\,\varepsilon^d\,2^{d\,k} \times \|\mu_N\|_{\infty,\varepsilon}$ according to the very definition of this discrete L^∞ norm (1.13). Consequently for any $\alpha' < d$, since $\varepsilon^{2d} = C/N$, interpolating between these two values, we get

$$|C_k| \leq C\,N\,(K(t_0))^{2\alpha'}\,\varepsilon^{\alpha'}\,2^{\alpha' k} \times \|\mu_N(t_0,.)\|_{\infty,\varepsilon}^{\alpha'/d}.$$

Using these last two bounds in (2.2) and summing up, we obtain

$$\begin{aligned}I_1 &\leq \sum_{k=1}^{k_0} |C_k| \times N^{-1}\,(K(t_0))^{-\alpha}\,\varepsilon^{-\alpha}\,2^{-\alpha\,k}\\ &\leq C\,\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\varepsilon^{\alpha'-\alpha}\sum_{k=1}^{k_0} 2^{(\alpha'-\alpha)k}.\end{aligned}$$

Eventually for any $\alpha < \alpha' < d$, we deduce that

$$I_1 \leq C\,\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\varepsilon^{\alpha'-\alpha}\,2^{(\alpha'-\alpha)k_0} \leq C\,\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,R^{\alpha'-\alpha}\,K^{\alpha'}, \tag{2.3}$$

the values being taken at t_0, which gives the first term in Lemma 2.1. Before dealing with the remaining term, we point out that here we have never used the condition $\alpha < 1$ and that for this term the same computation would be valid for any $\alpha < 3$.

Step 3: Redecomposition of C_0. For the force induced by the particles in C_0, we divided again this set into several parts. Set

$$Q_l = \left\{ j \in C_0 \,\Big|\, 3\,\varepsilon\,\overline{E}(t_0)\,2^{l-1} \le |V_1(t_1) - V_j(t_1)| \le 3\,\varepsilon\,\overline{E}(t_0)\,2^l \right\}, \qquad (2.4)$$

for $l \ge 1$. Remark that $Q_l = \emptyset$ if $l > l_0 = \ln(K(t_0)/(\varepsilon\,\overline{E}(t_0)))/\ln 2$. Therefore the rest Q_0 is defined by

$$Q_0 = \left\{ j \in C_0 \Big| |V_1(t_1) - V_j(t_1)| \le 3\,\varepsilon\,\overline{E}(t_0) \right\}.$$

As before we decompose I_{C_0} in a sum of I_2 and a remainder I_{Q_0} with

$$I_2 = \sum_{l=1}^{l_0} \sum_{j \in Q_l} \frac{1}{\varepsilon} \int_{t_1}^{t_0} \frac{dt}{N\,|X_j(t) - X_1(t)|}, \qquad (2.5)$$

and for I_{Q_0} the same sum but on the indices $j \in Q_0$ of course.
The idea behind this new decomposition is that although the particles in Q_l witth $l \ge 1$ are close to X_1, their speed is different from V_1. So even if they come very close to X_1 they will stay close only for a very short time. Since the singularity of the potential is not too high, we will be able to bound the force.

Step 4: Stability of the Q_l. Just as for the C_k, we may prove that for any time t in $[t_1,\ t_0]$ and any $j \in A_l$ with $l \ge 1$

$$|V_j(t) - V_1(t)| > \varepsilon\,\overline{E}(t_0)\,2^{l-1}.$$

This is again due to the fact that

$$|V_j(t) - V_j(t_1)| \le \int_{t_1}^{t_0} |E(X_j(s))|\,ds \le \varepsilon\,\overline{E}(t_0),$$

so that in fact the result is even more precise in the sense that the relative velocity $V_j(t) - V_1(t)$ remains close to $V_j(t_1) - V_1(t_1)$ up to exactly $\varepsilon\,\overline{E}(t_0)$.

Step 5: Control of I_2. Given this previous point, for any $j \in Q_l$ with $l > 0$ and any $t \in [t_1,\ t_2]$, we have, denoting by t_m the time in the interval $[t_1,\ t_0]$ where $|X_j(t) - X_1(t)|$ is minimal

$$|X_1(t) - X_j(t)| \geq \left| |X_1(t_m) - X_j(t_m)| - \frac{1}{2}(t - t_m)|V_1(t_m) - V_j(t_m)| \right|.$$

Then,

$$\begin{aligned} \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{1}{|X_1(t) - X_j(t)|^\alpha}\,dt &\leq \frac{C}{\varepsilon}\,|V_1(t_m) - V_j(t_m)|^{-\alpha}\varepsilon^{1-\alpha} \\ &\leq C\,\varepsilon^{-2\alpha}\,(\overline{E}(t_0))^{-\alpha}\,2^{-\alpha l}. \end{aligned}$$

Summing up on l, we obtain

$$|I_2| \leq C\sum_{l=1}^{l_0}|Q_l|\frac{1}{N}\,\varepsilon^{-2\alpha}\,(\overline{E}(t_0))^{-\alpha}\,2^{-\alpha l}.$$

We bound $|Q_l|$ by $|Q_l| \leq C\,\|\mu\|_{\infty,\varepsilon}\,(K(t_0)\,\varepsilon)^d(2^l\,\overline{E}(t_0)\,\varepsilon)^d$ using again the definition of the discrete L^∞ norm. It gives us

$$\begin{aligned} |I_2| &\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^{d-\alpha}\,\varepsilon^{2d-2\alpha}\,\|\mu_N\|_{\infty,\varepsilon} \times \sum_{l=2}^{l_0} 2^{(d-\alpha)l} \\ &\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,\varepsilon^{2d-2\alpha}\left(\frac{K(t_0)}{\overline{E}(t_0)\,\varepsilon}\right)^{d-\alpha} \\ &\leq C\,(K(t_0))^{2d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,\varepsilon^{d-\alpha}, \end{aligned}$$

which is indeed the second term in Lemma 2.1.

Step 6: Control on I_{Q_0}. The first point to note is that for any $j \in Q_0$ and any $t \in [t_1,\ t_0]$ by the definition (1.11) of $\Delta\overline{E}$ and the stability of C_0

$$|V_j(t) - V_1(t) - V_j(t_1) - V_1(t_1)| \leq 5\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).$$

Consequently it is logical to decompose (again) Q_0 in $Q_0' \cup Q_0''$ and I_{Q_0} in $I_{Q_0'} + I_{Q_0''}$ with

$$Q_0' = \left\{ j \in Q_0 \,\middle|\, |V_j(t_1) - V_1(t_1)| \geq 6\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0) \right\},$$

and $I_{Q_0'}$, $I_{Q_0''}$ the sums on the corresponding indices.
Then for any $j\in Q_0'$, the same computation as in the fifth step, shows that

$$\frac{1}{\varepsilon}\int_{t_1}^{t_0}\frac{dt}{N\,|X_j(t)-X_1(t)|^\alpha}\leq C\,\varepsilon^{2d-3\alpha}\,(K(t_0))^{-\alpha}\,(\Delta\overline{E}(t_0))^{-\alpha}.$$

But of course the cardinal of Q_0' is bounded by the one of Q_0 and using as always the discrete L^∞ bound

$$|Q_0'|\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^d\,\|\mu_N\|_{\infty,\varepsilon}.$$

Eventually that gives

$$I_{Q_0'}\leq C\,\varepsilon^{2d-3\alpha}\,(K(t_0))^{d-\alpha}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon},$$

since $\Delta\overline{E}(t_0)$ comes with a negative exponent and being non decreasing, may be bounded by the initial value. So $I_{Q_0'}$ corresponds to the third term in the lemma.
Let us conclude the proof with the bound on $I_{Q_0''}$. Of course if $j\in Q_0''$ then for any $t\in[t_1,\ t_0]$,

$$|V_j(t)-V_1(t)|\leq 11\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).$$

Now we use the definition (1.10) of m and the assumption in the lemma to deduce that

$$|X_j(t)-X_1(t)|\geq\frac{\varepsilon}{m(t_0)}-|V_j(t)-V_1(t)|\geq\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).$$

We bound $|Q_0''|$ by Q_0 which is the best we can do since the discrete L^∞ norm cannot see the scales smaller than ε and we obtain

$$I_{Q_0''}\leq C\,\varepsilon^{2d-2\alpha}\,(K(t_0))^{d-\alpha}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon},$$

which is dominated by the previous term and the third term in the lemma.
Before ending the proof we wish to note that the condition $\alpha<1$ was only used to get the second term in the fifth step and in the last step and the bound on m was only required in this last step. □

2.2 Estimate on $\Delta \overline{E}$

We may show the following with the same remarks as for Lemma 2.1,

Lemma 2.2. *For any α' with $\alpha < \alpha' < 3$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\Delta\overline{E}(t_0) \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\, K^{1+\alpha'}\, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha-\beta}\,\|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} \\ + \varepsilon^{2d-3\alpha-\beta}\,\|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^d\, K^d),$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

Proof. The proof follows the same procedure as for Lemma 2.1 with exactly the same decompositions. We have to bound, since as before the choice of the indices does not matter

$$\Delta I = \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{|E(X_1(t)) - E(X_2(t))|}{\varepsilon^\beta + |X_1(t) - X_i(t)|}\, dt.$$

We introduce the same decomposition as for the proof of Lemma 2.1 except that now we have two decompositions: One around X_1 denoted by a superscript 1 and another one around X_2. So we set for $\gamma = 1,\ 2$

$$C_k^\gamma = \left\{ i \,\middle|\, 3\,\varepsilon\, K(t_0)\, 2^{k-1} < |X_i(t_1) - X_\gamma(t_1)| \leq 3\,\varepsilon\, K(t_0)\, 2^k \right\},$$

with C_0^1 and C_0^2 the corresponding remaining indices. We also denote $C_0 = C_0^1 \cup C_0^2$.
Then of course

$$\Delta I = \sum_{i\neq C_0} \frac{1}{\varepsilon}\int_{t_1}^{t_0} |F(X_1 - X_i) - F(X_2 - X_i)| \times \frac{dt}{\varepsilon^\beta + |X_1(t) - X_i(t)|} + \sum_{i\in C_0} \cdots \\ = \Delta I_1 + \Delta I_{C_0}.$$

The second term is easier to bound as we simply write

$$\Delta I_2 \leq \sum_{i\in C_0} \frac{1}{\varepsilon^{1+\beta}} \int_{t_1}^{t_0} \left(\frac{1}{N\,|X_i(t) - X_1(t)|^\alpha} + \frac{1}{N\,|X_i(t) - X_2(t)|^\alpha} \right) dt.$$

Now we do exactly what we did for I_{C_0} in the proof of Lemma 2.1 and we get as a bound the two last terms in the estimate for $\overline{E}$ in this lemma, divided by ε^β, which are exactly the two last term in the estimate for $\Delta\overline{E}$ in Lemma 2.2. Note by the way that the terms where we sum on C_0^1 for instance but with $X_i - X_2$ in the denominator are in fact even easier to handle.

Now for ΔI_1, we observe that for $i \neq C_0$ then for any t

$$\begin{aligned}|F(X_1(t) - X_i(t)) - F(X_2(t) - X_i(t))| &\leq C|X_1(t) - X_2(t)| \\ &\quad \times \left(\frac{1}{N\,|X_1(t) - X_i(t)|^{\alpha+1}} + \frac{1}{N\,|X_1(t) - X_i(t)|^{\alpha+1}}\right),\end{aligned}$$

since it is always possible to find a regular path $x_t(s)$ of length less than $2\,|X_1(t) - X_2(t)|$ such that $x_t(0) = X_1(t)$, $x_t(1) = X_2(t)$ and $|x_t(s) - X_i(t)|$ is always larger than the minimum between $|X_1(t) - X_i(t)|$ and $|X_2(t) - X_i(t)|$. Therefore we are led to make exactly the same computation as for I_1 with $\alpha+1$ instead of α which gives the desired result. This is possible only because in the estimate on I_1, we never use the condition $\alpha < 1$. $\square$

2.3 Control on m and K

We prove the

Lemma 2.3. *Assume that*

$$m(t) \leq \frac{1}{\varepsilon^{\beta-1}},$$

then we also have that

$$m(t) \leq m(0) \times e^{Ct + C\varepsilon\,\Delta\overline{E}(t) + C\int_0^t \Delta\overline{E}(s)\,ds},$$

and we may eliminate the $\varepsilon\,\Delta\overline{E}(t)$ *term if* $t > \varepsilon$.

Note that we still need an assumption on m but it is a bit different (and somewhat “harder” to satisfy) than the corresponding one for Lemmas 2.1 and 2.2.

Proof. We consider any two indices $i \neq j$. Then we write

$$\frac{d}{ds}\left(\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\right)=\frac{\varepsilon}{(|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|)^2}$$
$$\times\Big(\frac{X_i-X_j}{|X_i-X_j|}\cdot(V_i-V_j)+\frac{V_i-V_j}{|V_i-V_j|}\cdot(E(X_i)-E(X_j))\Big)$$
$$\leq\frac{\varepsilon\,(|V_i(s)-V_j(s)|+|E(X_i(s))-E(X_j(s))|)}{(|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|)^2}.$$

Since $m(t)\leq\varepsilon^{1-\beta}$, the same is true of $m(s)$ and at least one of the quantities $|X_i(s)-X_j(s)|$ and $|V_i(s)-V_j(s)|$ is larger than $\varepsilon^\beta/2$, therefore

$$\frac{d}{ds}\left(\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\right)\leq\frac{C\,\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}$$
$$\times\left(1+\frac{|E(X_i(s))-E_(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\right).$$

But by the definition of $\Delta\overline{E}$, see (1.11), we know that for $t>\varepsilon$

$$\int_\varepsilon^t\frac{|E(X_i(s))-E_(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\,ds\leq\int_0^t\Delta\overline{E}(s)\,ds,$$

and of course for $t<\varepsilon$

$$\int_0^t\frac{|E(X_i(s))-E_(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\,ds\leq\varepsilon\Delta\overline{E}(t).$$

Hence, integrating in time, we find

$$\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\leq\frac{\varepsilon}{|X_i(0)-X_j(0)|+|V_i(0)-V_j(0)|}$$
$$\times\,e^{Ct+C\varepsilon\,\Delta\overline{E}(t)+C\int_0^t\Delta\overline{E}(s)\,ds},$$

wich after taking the supremum in i and j is precisely the lemma. □

As for K, using the equation that $\dot{V}_i(t)=E_i(X_i(t))$, we may prove by the same method which we do not repeat, the result

Lemma 2.4. *We have that for any* t

$$K(t)\leq K(0)+Ct+C\varepsilon\,\overline{E}(t)+C\int_0^t\overline{E}(s)\,ds.$$

2.4 Conclusion on the proof of Theorem 1.1

Here (but only in this subsection) for a question of clarity, we keep the notation C for the constants appearing in Lemmas 2.1, 2.2, 2.3 and 2.4 and we denote by $\tilde{C}$ any other constant depending only on $R(0)$, $K(0)$ and $m(0)$. We assume that on a time interval $[0,\ T]$, we have (for a given α')

$$\begin{aligned} &m(t) \le 2\, m(0), \quad \overline{E}(t) \le 2C\, 2^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha}, \\ &K(t) \le 2\,(1+K(0)), \quad R(0) \le 2\,(1+R(0)), \quad \forall\, t \in [0,\ T], \end{aligned} \tag{2.6}$$

which we may always do since all these quantities are continuous in time (although they may a priori increase very fast).
Then we show that if T is too small we have in fact the same inequalities but with a $3/2$ constant instead of 2. By contradiction this of course shows that we can bound T from below in terms of only $R(0)$, $K(0)$ and $m(0)$ and it proves Theorem 1.1 with $c = C \times 2^{8\alpha'-\alpha+1}$.

First of all, we note that since $m(t) \le 2m(0)$, we may apply Lemmas 2.1, 2.2, and 2.3. Furthermore we immediately know from (1.14) that

$$\|\mu_N(t,.)\|_{\infty,\varepsilon} \le (8\, m(0))^{2d}.$$

Let us start with Lemma 2.1, using the assumption (2.6) we deduce that for any $t \in [0,\ T]$,

$$\overline{E}(t) \le C\, 2^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha} + \tilde{C}\, \varepsilon^{d-a} + \tilde{C}\, \varepsilon^{2d-3\alpha}.$$

For ε small enough this proves that

$$\overline{E}(t) \le \frac{3\, C}{2}\, 2^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha},$$

which is the first point.

Next applying Lemma 2.2, we deduce that for any $t \in [0,\ T]$

$$\Delta\overline{E}(t) \le \tilde{C}.$$

From Lemma 2.3, we obtain that

$$m(t) \le m(0) \times e^{\tilde{C}T},$$

so if T is such that $\tilde{C}\,T < \ln(3/2)$ then we get

$$m(t) \le \frac{3}{2}\, m(0).$$

Lemma 2.4 implies that for $t \in [0,\ T]$

$$K(t) \le K(0) + \tilde{C}\,T,$$

so that again for T small enough

$$K(t) \le \frac{3}{2}\,(1 + K(0)).$$

Eventually thanks to relation (1.6), we know that for $t \in [0,\ T]$

$$R(t) \le R(0) + T\,K(t) \le R(0) + \tilde{C}\,T,$$

hence the corresponding estimate for R provided $\tilde{C}\,T \le 3/2$.
In conclusion we have shown that if (2.6) holds and if T is smaller than a given time depending only on $R(0)$, $K(0)$ and $m(0)$ then the same inequalities are true with 3/2 instead of 2. By the continuity of R, K, m and $\overline{E}$ this has for consequence that (2.6) is indeed valid at least on this time interval thus proving Theorem 1.1.

3 Preservation of $\|\mu_N\|_{\infty,\eta}$

From the form of the estimate on m in Lemma 2.3, it is clear that with this estimate we will never get a result for a long time. Indeed, even assuming that we have bounded before K and R, we would have the equivalent of $\dot{m} \le m \times \Delta\overline{E} \le C\, m \times m^{2+2\alpha'}$.
On the other hand this is somewhat strange since, in the limit, the L^∞ norm is conserved. And this preservation is very useful in the proof of the existence and uniqueness of the solution of the Vlasov equation, see for instance [13]. But, how to obtain the analog of this in the discrete case? At this time, we just have a bound on $\|\mu_N\|_{\infty,\varepsilon}$ on a small time, and the bound is too huge to allows us to prove convergence results for long time. Of course, this norm is not preserved at all because we are looking at the scheme at the scale of the the discretization. And in our calculation we do not use the fact that the

flow is divergence free, a property that is the key for the preservation of the L^∞ norm.

So what else can we do? One of the solution is to look at a scale $\eta > \varepsilon$, with ε/η going to zero as ε goes to zero. At this scale, we have many more particles in a cell and we will be able to obtain the asymptotical preservation of this norm. This will be very useful because it will allow us to sharpen our estimate on E and δE. And with this we will obtain long time convergence results.

Now, we will try to give roughly the idea of the proof in dimension 1 before beginning the genuine calculations. We choose a time t and a particle i, and look at the square of size η in the phase space centered on the particle i. We called it $S_t = \{(x,v) | |x - X_i(t)| < \eta, |v - V(t)| < \eta\}$. We want an estimation of the number of particles in this square at the time t. For this, we first wanted to know where were these particles at the time $t - \varepsilon$. During the interval of time $[t - \varepsilon, t]$, the particle j has moved of

$$X_j(t) - X_j(t-\varepsilon) \approx \varepsilon V_j(t) \tag{3.1}$$

$$V_j(t) - V_j(t-\varepsilon) \approx \varepsilon E(t, X_j(t)) \tag{3.2}$$

To make it correct, we will have to replace the second left hand side by an average on an interval of size ε in time, as usual, but we keep this for explanation. If we take the values in the right hand side for exact, the particles we are interested in were in the following domain $S_{t-\varepsilon}$.

$$S_{t-\varepsilon} = \{(x,v) | |x - X_i(t) - \varepsilon v| \leq \eta \quad \text{and} \quad |v - V_i(t) - \varepsilon E(t, x - \varepsilon v)| \leq \eta\}$$

There is two steps to obtain $S_{t-\varepsilon}$ from S_t. First, translate for a fixed $\bar{v}$, the set $S_t \cap \{v = \bar{v}\}$ of $\bar{v}$, for all v. We call the resulting set S'_t. Then, for a fixed $\bar{x}$, translate $S'_t \cap \{x = \bar{x}\}$ of $-\overline{E}(x)$, for all x. With these steps, we see that S_t and $S_{t-\varepsilon}$ have the same volume (see figure 1). But, if we keep the set S'_t, and do this another time, we will obtain a set with a more strange shape and so one. So, we will approximate $S_{t-\varepsilon}$ by a parallelogram and iterate this step from the parallelogram. We will choose the parallelogram

$$\begin{aligned} \bar{S}_{t-\varepsilon} = \{(x,v) \mid & |x - X_i(t) - \varepsilon v| < \eta \\ \text{and} \quad & |v - V_i(t) - \varepsilon(x - X_i(t))\nabla E_\varepsilon(t, X_i(t)) - \varepsilon \bar{E}(X_i(t))| \leq \eta\}, \end{aligned} \tag{3.3}$$

where E_ε is an approximation at ε of the field, defined by:

$$E_\varepsilon(x) = \frac{1}{N} \sum_{i \neq j} \frac{x - X_i(t)}{(|x - X_i(t)| + \varepsilon)^{(\alpha+1)}}.$$

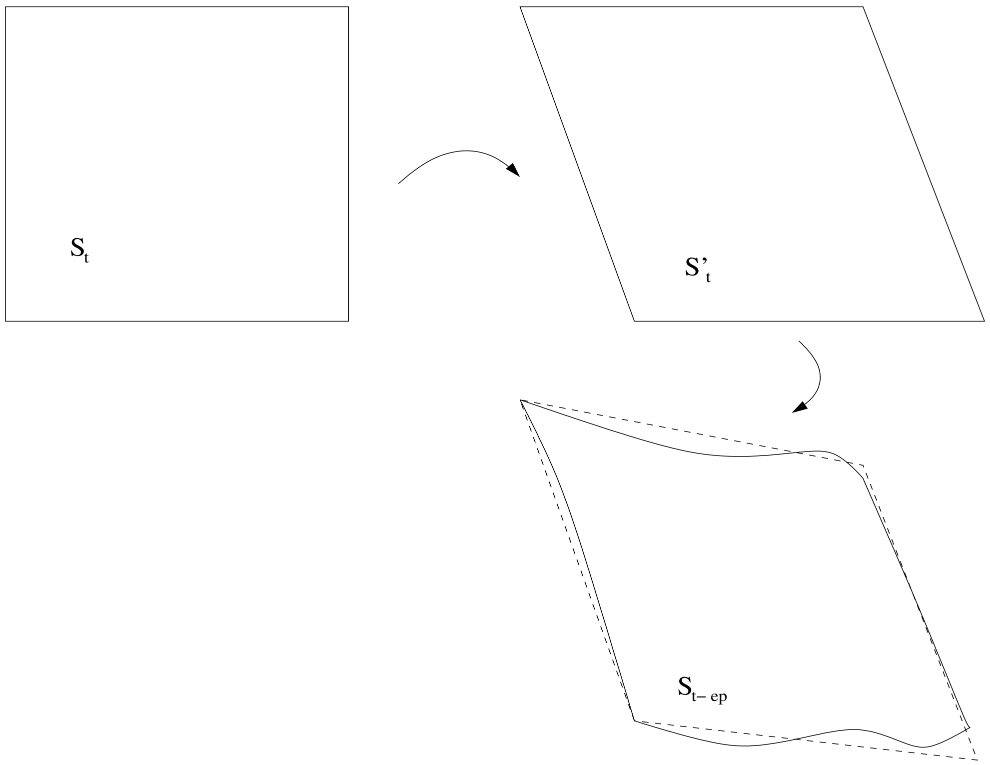

Figure 1: Evolution of S_t.

We use this approximation to obtain a usable value of ∇E. This set has almost the same volume that S_t. And if we begin with a parallelogram, we still obtain a parallelogram. What we have to check is that this approximation make sense, that means that we forgot or added a non relevant set of particles. For this, we need that the parallelogram do not become too stretched, in other words that the two sides do not become parallel on the figure in dimension one. Remember that we are not exactly interested by the volume of the set S_0 (the set that we obtain iterating the process till we reached the initial time), but by the number of vertices of the initial networking $\varepsilon\mathbb{Z}^{2d}$ inside the parallelogram. The volume of the parallelogram is a good approximation of this if its width is big with respect to ε.

We avoid this if the slope of the sides of the parallelogram are distinct. This is ensured if t and $\varepsilon\sum_{k\leq t/\varepsilon}|\nabla E_\varepsilon(kt/\varepsilon, X_i(kt/\varepsilon))|$ are smaller than $1/2$, because this two quantities are respectively the tangent and the cotangent of the angle between the sides of the parallelogram and the x-axis. Remark that the sum is bounded by $t\Delta\overline{E}(t)$.

This limits us in time, so we will obtain the conservation of the $\|\mu_N\|_{\infty,\eta}$ only on a short time. But this is not a problem. Call this time T'. After this, we choose a new scale η' sufficiently large, by instance $\sqrt{\eta}$. Since we have proved the asymptotic preservation of $\|\mu_N\|_{\infty,\eta}$, we can do the same think again, replacing ε by η and η by η'. And obtain the preservation of $\|\mu_N\|_{\infty,\eta'}$ on a new small interval of time and so on.

Now, this is time to do the calculation for the proof of what we claimed before. We just need to remark that we draw our pictures in dimension one. In this case, we deal with true parallelogram. But we will only prove result for $d\geq 2$ and in this case we will not use true parallelograms. We will deal with sets defined by

$$S=\left\{(x,v)|\left\|M\left(\begin{array}{c}x\\v\end{array}\right)\right\|\leq\eta\right\}$$

where M is a matrix of $\mathcal{M}_{2d}(\mathbb{R})$, and the norm $\|\cdot\|$ is defined by

$$\|(x,v)\|=\max(|x|,|v|)$$

For convenience, we will often decompose the matrix M in four blocks like below

$$M=\left(\begin{array}{cc}A&B\\C&D\end{array}\right)$$

Proposition 3.1. *Choose* $(X_0, V_0) \in \mathbb{R}^{2dn}$, *and A,B,C,D four square matrices in* $\mathcal{M}_n(\mathbb{R})$ *satisfying the following conditions that will be called the norm conditions in the rest of the paper*

$$\max(\|A - Id\|, \|D - Id\|) \leq 1/2 \quad \textit{and} \quad \max(\|C\|, \|D\|) \leq 1/2, \qquad (3.4)$$

with the dual norm on the matrix. We define M *as above and denote by* N_t *the number of particles in*

$$S_t = \left\{ (x, v) | \left\| M \cdot \left(\begin{array}{c} x - X_0 \\ v - V_0 \end{array} \right) \right\| \leq \eta \right\}$$

We assume as in all the preceding results, that

$$m(t_0) \leq \frac{1}{12\varepsilon\, K(t) \Delta \overline{E}(t)}$$

Then, there exist a constant $C = C(R, K, \overline{E}, \|\mu\|_{\infty,\varepsilon})$, *there exists a position* X' *a speed* V', *four matrices* A',B',C',D' *and then a matrix* M' *defined with this four blocks such that if we denote* $N_{t-\varepsilon}$ *the number of particles in*

$$S_{t-\varepsilon} = \left\{ (x, v) | \left\| M' \cdot \left(\begin{array}{c} x - X_0' \\ v - V_0' \end{array} \right) \right\| \leq \eta + C\varepsilon(\eta^\beta + \varepsilon) \right\}$$

we have the following inequality

$$N_t \leq N_{t-\varepsilon}.$$

Moreover, we have

$$\max(\|D - D'\|, \|D - D'\|, \|C - C'\|, \|D - D'\|) \leq C\varepsilon.$$

Proof. We divide it in two steps.
Step 1: Estimate on $V_j(t - \varepsilon) - v - \varepsilon E(X_j(t))$. Choose $j \in \{1, \ldots, N\}$ such that $X_j(t) \in S_t$. We will first work on the velocities and then integrate our estimation to obtain what we need on the positions. We have

$$\begin{aligned} V_j(t - \varepsilon) - V_j(t) &= \varepsilon \int_0^1 E(X_j(t - s\varepsilon))\, ds \\ &= \varepsilon \int_0^1 E(X_j(t - s\varepsilon)) - E_\varepsilon(X_j(t - s\varepsilon))\, ds \\ &\qquad + \varepsilon \int_0^1 E_\varepsilon(X_j(t - s\varepsilon))\, ds \\ &= I \quad + \quad II. \end{aligned}$$

We need to bound the first term I. The approximation error is

$$E(X_j(t-s\varepsilon)) - E_\varepsilon(X_j(t-s\varepsilon)) = \frac{1}{N}\sum_{k\neq j}\Big(\frac{1}{|X_j(t')-X_k(t')|^{1+\alpha}} - \frac{1}{(|X_j(t')-X_k(t')|+\varepsilon)^{1+\alpha}}\Big)(X_j(t')-X_k(t')).$$

To compute this term, we use again the same decomposition of the phase space as in lemma 2.2. If k is such that $|X_k - X_j| \geq \varepsilon$, we can bound a difference in the sum by $2^{1+\alpha}\varepsilon/(|X_k - X_j|)^{1+\alpha}$ and then compute it dividing the phase space in diadic subset. We obtain that

$$\sum_{|X_k-X_j|\geq\varepsilon} 2^{1+\alpha}\varepsilon/(|X_k-X_j|)^{\alpha} \leq C\varepsilon.$$

The constant C that we obtain here depends of R, K, $\overline{E}$, $\|\mu_N\|_{\infty,\varepsilon}$ exactly as in lemma 2.2. For the others terms, those were $|X_k - X_j| \leq \varepsilon$, we bound the difference by the sum of the norm of the two term and do the same estimation as in (2.2). It is possible because we do a mean over a interval of time of size ε. We obtain

$$|I| \leq C\varepsilon^2 + C'\varepsilon^{d+1-\alpha}.$$

again with the same dependence for C'. In the following, we will assume to simplify the presentation that $d \geq 2$. In that case, we may replace it by $I \leq C\varepsilon^2$ and this bound will be enough. But all is still true if $d = 1$. In that case, we replace the last estimate by $I \leq C\varepsilon^{2-alpha}$. Since $-\alpha > 1$, all our estimates will remain valid.

Now, we approximate the second term II by the same expression where we replace E_ε by its first order linearization near X_0. We defined E^{lin}_ε by

$$E^{lin}_\varepsilon(t,x) = E_\varepsilon(t,X_0) - \nabla E_\varepsilon(t,X_0)\cdot(x-X_0).$$

We have to estimate the difference between E_ε and E^{lin}_ε. For this, we need to bound a sum of terms of the following type

$$\frac{x-X_k}{(|x-X_k|+\varepsilon)^{1+\alpha}} - \frac{X_0-X_k}{(|X_0-X_k|+\varepsilon)^{1+\alpha}} - \nabla\left(\frac{x-X_k}{(|x-X_k|+\varepsilon)^{1+\alpha}}\right)(t,X_0)\cdot(x-X_0). \quad (3.5)$$

For this, we find a path $I(s)$ between x and X_0 (in others words $I(0) = x$ and $I(1) = X_0$) of length smaller than $4|x - X_0|$ and such that $|I(t) - X_k| \geq \min(|x - X_k|, |X_0 - X_k|)$. The term to estimate may be rewritten

$$\int_0^1 \Big(\nabla\Big(\frac{x - X_k}{(|x - X_k| + \varepsilon)^{1+\alpha}}\Big)(t, I(s)) \\ - \nabla\left(\frac{x - X_k}{(|x - X_k| + \varepsilon)^{1+\alpha}}\right)(t, X_0)\Big) \cdot I'(s)\, ds,$$

and each difference (without the multiplication by $I'(s)$) may be bounded by

$$C\frac{|I(t) - X_0|}{\min(|x - X_k|, |X_0 - X_k|)^{2+\alpha}},$$

where C is just a numerical constant and also by

$$\frac{C}{\min(|x - X_k|, |X_0 - X_k|)^{1+\alpha}},$$

where the constant C is again independent of the problem. Doing a classical interpolation between this two terms, we may bound it for every $\gamma \in (0, 1)$ by

$$C\frac{|I(t) - X_0|^{1+\gamma}}{\min(|x - X_k|, |X_0 - X_k|)^{1+\alpha+\gamma}}.$$

Thus, we may bound the approximation error by

$$\frac{C}{N}\sum_{k \neq j} \frac{|x - X_k|^{1+\gamma}}{\min(|x - X_k|, |X_0 - X_k|)^{1+\alpha+\gamma}}.$$

Now, we do the same computation as in the estimation of $\Delta\bar{E}$. We obtain, as in lemma (2.2), that for every $\gamma = \beta - 1$, there exists a constant C depending always on R, K, $\overline{E}$, $\|\mu_N\|_{\infty,\varepsilon}$ such that

$$|\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} E_\varepsilon(s, X_j(t - s\varepsilon)) - E_\varepsilon^{lin}(s, X_j(t - s\varepsilon))\, ds| \leq C(|X_j j(t) - X_0|^\beta + \varepsilon).$$

We approximate the term II by

$$|II - \varepsilon\int_0^1 E_\varepsilon^{lin}(s, X_j(t - s\varepsilon))|\, ds \leq C\varepsilon(|X_j(t - s\varepsilon) - X_0|^\beta + \varepsilon).$$

Moreover, we know that the approximated force field is Lipschitz with a constant depending on R, K, $\overline{E}$ and $\|\mu_N\|_{\infty,\varepsilon}$ on our small interval of time. This can be seen just by doing the same calculation that gives the bound on $\Delta\bar{E}$. Indeed, it is the same because regularization at order ε or estimation on ΔE at order ε are similar. Moreover, we may replace the value of E_ε^{lin} taken at $X_j(t-s\varepsilon)$ by those taken at $X_j(t)$. Because of the bound on the speed, this will only introduce an error of order ε. Actually, if we denote $\tilde{E} = 1/\varepsilon \int_{t-\varepsilon}^t E_\varepsilon(s,X_0)\,ds$ and $\nabla\tilde{E} = 1/\varepsilon \int_{t-\varepsilon}^t \nabla E_\varepsilon(s,X_0)\,ds$, we have the following inequalities

$$|V_j(t-\varepsilon) - V_j(t) - \varepsilon(\tilde{E} + \nabla\tilde{E}\cdot X_j(t-\varepsilon))| \leq C\varepsilon(|X_j(t) - X_0| + \varepsilon), \tag{3.6}$$

$$|X_j(t-\varepsilon) - X_j(t) - \varepsilon V_j(t-\varepsilon)| \leq C\varepsilon^2, \tag{3.7}$$

where the constant C has the same dependance as before.

Step 2: The new parallelogram. Now, we obtain an approximated parallelogram that contains at time $t-\varepsilon$ almost all the particles that are in S_t at time t. We look first at the two first blocks of the matrix and define the following linear mappings

$$A' = A + \varepsilon B \cdot \nabla E_\varepsilon,$$

$$B' = B + \varepsilon A.$$

The center of our new parallelogram is given by

$$X_0' = X_0 - \varepsilon V_0 \quad , \quad V_0' = V_0 - \int_{t-\varepsilon}^t E_\varepsilon(s,X_0)\,ds.$$

The previous calculation tells us that

$$|A'\cdot(X_j(t-\varepsilon) - X_0') - B'\cdot(V_j(t-\varepsilon) - V_0')| \leq \eta + C\varepsilon(|X_j(t-\varepsilon) - X_0'|^\beta + \varepsilon).$$

Of course we obtain the same for the second line, and if we define also $C' = C + \varepsilon D\cdot\nabla E_\varepsilon$, $D' = D + \varepsilon C$ and the associated M', we obtain

$$\left\| M' \begin{pmatrix} X_j(t-\varepsilon) - X_0' \\ V_j(t-\varepsilon) - V_0' \end{pmatrix} \right\| \leq C\varepsilon(\|X_j(t) - X_0, V_j(t) - V_0\|^\beta + \varepsilon).$$

To conclude, we just have to prove that $|X_j(t) - X_0|$ is of order η. But, this is true because our initial parallelogram is not too stretched thanks to the conditions on the norm of A, B, C, D. The following lemma is more precise.

Lemma 3.1. *Assume that the four matrices A, B, C, D satisfy* $\max(\|A-Id\|, \|D-Id\|, \|C\|, \|D\|) \le 1/2$. *We use here the dual norm on matrices, $\|A\| = \sup_{|X|\le 1} |A\cdot X|/|X|$. We denote as usual by M the matrix composed of the four blocks A, B, C, D, and*

$$S = \{(x,v) | \|M\cdot (x,v)^T\| \le \eta\}.$$

Then

$$S \subset B(0, 2\eta).$$

Proof of the lemma. Choose an $x \in S$. With the definition of our norm $\|\cdot\|$ it means that

$$\begin{array}{rcl} |A\,x + B\,v| & \le & \eta, \\ |D\,v + C\,x| & \le & \eta. \end{array}$$

This implies

$$\begin{array}{rcl} |A\,x| - |B\,v| & \le & \eta, \\ |D\,v| - |C\,x| & \le & \eta. \end{array}$$

With the assumption on the matrix, we obtained

$$\begin{array}{rcl} |x| - \dfrac{1}{2}(|x| + |v|) & \le & \eta, \\ |v| - \dfrac{1}{2}(|x| + |v|) & \le & \eta. \end{array}$$

We add this two lines and obtain

$$|x| + |v| \le 2\eta.$$

□

There only remains to prove the estimate on the determinant of M'. But,

$$M' - = \left(\begin{array}{cc} A + \varepsilon B\,\nabla\tilde{E} & B + \varepsilon A \\ C + \varepsilon D\cdot\nabla\tilde{E} & D + \varepsilon C \end{array} \right).$$

Then,

$$\begin{aligned}
\det(M') &= \det\begin{pmatrix} A+\varepsilon B\,\nabla\tilde{E} & B+\varepsilon A \\ C+\varepsilon D\nabla\tilde{E} & D+\varepsilon C \end{pmatrix} \qquad (3.8)\\
&= \det\begin{pmatrix} A-\varepsilon^2 A\nabla\tilde{E} & B+\varepsilon A \\ C-\varepsilon^2 C\nabla\tilde{E} & D+\varepsilon C \end{pmatrix} \qquad (3.9)\\
& \qquad (3.10)
\end{aligned}$$

To obtain the second line, we substract the second column multiplied by $\nabla\tilde{E}$ to the first. Then, we see that the new determinant is ε^2 close from the preceding. This gives us the expected bound on the determinant of M'. $\square$

With the help of the proposition, we prove the almost preservation of the L^∞ norm at the scale η, which is stated in the theorem below.

Theorem 3.1. *There exist a time T', a constant C both depending on $R(T')$, $K(T')$, $\overline{E}(T')$, $sup_{t\leq T'}\|\mu_N(t)\|_{\infty,\varepsilon}$) such that if $t \leq T'$ and*

$$m(T') \leq \frac{1}{12\,\varepsilon\,K(T')\,\Delta\overline{E}(T')},$$

the following inequality holds:

$$\|\mu_N(t)\|_{\infty,\eta} \leq \|\mu\|_{\infty,\varepsilon} + C\,(\eta^\beta + \varepsilon/\eta).$$

Proof. Let us choose a box $S_t = \{(x,v) \mid \|x-X_0, v-V_0\| \leq \eta\}$. We can find a parallelogram containing at time $t-\varepsilon$ the particles that are in S_t at time t. And we can iterate this process till the parallelogram do not become too much stretched. In others words, till the conditions 3.4 are satisfied by the matrix M. As at each step, the matrix move from at worst $C'\varepsilon$, we may iterate the process on an interval of time of length $1/2C'$. So we obtain a parallelogram S_0 not too stretched, if $t \leq 1/2C'$. We made $([t/\varepsilon]+1)$ steps to reach the time 0. The last is necessary of length less than ε, but as in the proof of the theorem one, this raises no difficulty. So, we know that S_0 is not too stretched, but we have to check that this parallelogram S_0 is not too big.

Step 1: The size of S_0 We define real functions g_ε by $g_\varepsilon(\eta) = \eta + C\varepsilon(\varepsilon+\eta^\beta)$. We introduce it because if $S_{t'}$ is defined by

$$S_{t'} = \left\{(x,v) \middle| \left\| M \cdot \begin{pmatrix} x-X_0 \\ v-V_0 \end{pmatrix} \right\| \leq \eta \right\}.$$

Then $S_{t'-\varepsilon}$ is defined by a similar condition with new M, X_0,V_0 where the right hand side of the inequality is replaced by $g_\varepsilon(\eta)$. And, so S_0 is defined by a similar conditions, where the right hand side η is replaced by

$$\eta' = g_\varepsilon \circ \cdots \circ g_\varepsilon(\eta),$$

where the dots means $[t/\varepsilon]+1$ times. We need to control this quantity. We define $r_n = g_\varepsilon^n(\eta)$ where the exponent means composed n times. This r_n is the quantity in the right hand side of the definition of $S_{t-n\varepsilon}$. From the formula $r_{n+1} = g(r_n)$ we expect that $r_n \approx \eta + Cn\varepsilon(\varepsilon+\eta^\beta)$. To prove this rigourously, we introduced α_n defined by $r_n = \eta + Cn\varepsilon(\varepsilon+\eta^\beta)+\alpha_n$. Provided $\alpha_n < \eta$, it satisfies the following relation:

$$\alpha_{n+1} \le (1+C'\varepsilon\eta^{\beta-1})\alpha_n + C\,C'\varepsilon^2 n\eta^{\beta-1}(\varepsilon+\eta^\beta),$$

where C' is a numerical constant. We can bound $n\varepsilon$ by t since we will only iterate the process till $n = [t/\varepsilon]+1$. The previous inequality becomes

$$\alpha_{n+1} \le (1+C'\varepsilon\eta^{\beta-1})\alpha_n + C\,C'\varepsilon\,\eta^{\beta-1}(\varepsilon+\eta^\beta).$$

Since $\alpha_0 = 0$, we obtain that

$$\alpha_n \le ((1+C'\varepsilon\eta^{\beta-1})^n - 1)C(\varepsilon+\eta^\beta).$$

This gives us

$$\begin{aligned} \eta' &\le \eta + (1+e^{C'\eta^{\beta-1}}-1)C(\varepsilon+\eta^\beta) \qquad (3.11)\\ &\le \eta + C(\varepsilon+\eta^\beta), \qquad (3.12) \end{aligned}$$

with a new constant for the last line.

Step 2: Covering of S_0 by balls of radius $\varepsilon/2$.

Lemma 3.2. *let S be a parallelogram defined as above by*

$$S = \left\{(x,v)| \left\| M \cdot \begin{pmatrix} x - X_0 \\ v - V_0 \end{pmatrix} \right\| \le \eta \right\}$$

with M composed of the four blocks A, B, C, D satisfying the assumption 3.4 and also $\det(M) \le 1 + C\varepsilon$. *Then, there exists a constant C' depending only on C such that S can be covered by $[\varepsilon^{-2d}(Vol(S)+C\varepsilon\eta^{2d-1})]$ balls of size ε. In others words, there exists a finite set P of cardinal less than $\varepsilon^{-2d}(Vol(S)+C'\varepsilon\eta^{2d-1})$ such that $S \subset \bigcup_{p\in P} B(p,\varepsilon)$*

Proof of the lemma. We define

$$S^+_{2\varepsilon} = \left\{(x,v)| \left\| M \cdot \begin{pmatrix} x - X_0 \\ v - V_0 \end{pmatrix} \right\| \leq \eta + 2\varepsilon \right\} ,$$

$$S^+_{4\varepsilon} = \left\{(x,v)| \left\| M \cdot \begin{pmatrix} x - X_0 \\ v - V_0 \end{pmatrix} \right\| \leq \eta + 4\varepsilon \right\},$$

and $P = \varepsilon\mathbb{Z} \cap S^+_{2\varepsilon}$. We look at the set P_ε consisting of the union of all the balls (for the norm $\max(|x|,|v|)$) of size ε centered at points of P, that is $P_\varepsilon = P + B(0,\varepsilon)$. We will show that this set is included in $S^+_{4\varepsilon}$. For this, we choose $(x,v) \in P_\varepsilon$. We associate to this point the couple (m,n) such that $\|(x - \varepsilon m, v - \varepsilon n)\| \leq \varepsilon$. Then,

$$\begin{aligned} \left\| M \cdot \begin{pmatrix} x \\ v \end{pmatrix} \right\| &\leq \left\| M \cdot \begin{pmatrix} x - \varepsilon m \\ v - \varepsilon n \end{pmatrix} \right\| + \left\| M \cdot \begin{pmatrix} \varepsilon m \\ \varepsilon n \end{pmatrix} \right\| \\ &\leq \|M\|\varepsilon + \eta + 2\varepsilon \\ &\leq \eta + 4\varepsilon. \end{aligned}$$

In the last line, we use $\|M\| \leq 2$. This inequalities comes from the condition 3.4. Therefore we have the inclusion $P_\varepsilon \subset S^+_{4\varepsilon}$.
Moreover, if we choose a point $(x,v) \in S$, we can find a point $(\varepsilon m, \varepsilon n)$of $\varepsilon\mathbb{Z}^{2d}$ such that $\|(x - \varepsilon m, v - \varepsilon n)\| \leq \varepsilon$. As above, we have

$$\left\| M \cdot \begin{pmatrix} \varepsilon m \\ \varepsilon n \end{pmatrix} \right\| \leq \eta + 2\varepsilon.$$

Thus, $\varepsilon(m,n) \in P$. That proves that $S \subset P_\varepsilon$. So, we have the inclusions

$$S \subset P_\varepsilon \subset S^+_{4\varepsilon}.$$

The first is the recovering we want. The second gives us an estimate on the cardinal of P. Comparing the volume of P_ε and $S^+_{4\varepsilon}$ we obtain

$$(2\varepsilon)^{2d}|P| \leq det(M)(\eta + 4\varepsilon)^{2d}.$$

If M satisfies $\det(M) \leq 1 + C\varepsilon$, that gives us

$$|P| \leq \varepsilon^{-2d}\eta^{2d-1}(\eta + C'\varepsilon).$$

□

Step 3: Conclusion of the proof. We choose t as in the previous step and a box S_t in the phase space of size η. We define the parallelogram S_0 as above. It exits and is not too stretched because we choose $t \leq T'$. From Step 1, we know that

$$\begin{aligned} Vol(S_0) &\leq (1 + C\varepsilon)(\eta + C(\varepsilon + \eta^\beta))^{2d} && (3.13) \\ &\leq \eta^{2d}(1 + C(\eta^\beta + \varepsilon)). && (3.14) \end{aligned}$$

And since it is not too stretched, we may used lemma 3.2 and cover it by less than $\varepsilon^{-2d}(Vol(S_0) + C'\varepsilon\eta^{2d-1})$ balls of radius ε. Actually, that means that

$$N_t \leq N_0 \leq \|\mu_N(0)\|_{\varepsilon,\infty}\eta^{2d}(1 + C(\eta^\beta + \varepsilon),$$

and dividing by η^{2d} we obtain

$$\|\mu_N(t)\|_{\eta,\infty} \leq \|\mu_N(0)\|_{\varepsilon,\infty}(1 + C(\eta^\beta + \varepsilon)).$$

□

3.1 New estimates on $\overline{E}$ and $\Delta\overline{E}$

The almost preservation of the $\|\mu_N\|_{\infty,\eta}$ norms will enable us to prove a new estimate on $\overline{E}$. We can obtain it by doing the same separation of the position space in dyadic cells, but we begin with cells $\tilde{C}_k$ satisfying

$$C_k = \left\{ i \,\middle|\, 3\,\eta\, K(t_0)\, 2^{k-1} < |X_i(t_1) - X_1(t_1)| \leq 3\,\eta\, K(t_0)\, 2^k \right\}.$$

That gives a first term in $\|\mu_N\|_{\infty,\eta}K^{\alpha'}R^{\alpha'-\alpha}$. Next we decompose $\tilde{C}_0$ in a union of C_k (the "true" cells with size ε); Since the index k goes only up to $\log(\eta/\varepsilon)/\log 2$, the corresponding term is less than $\|\mu_N\|_{\infty,\varepsilon}\, K^{2\alpha'-\alpha}\, \eta^{\alpha'-\alpha}$. The remainder term coming for C_0 is dealt just as in Lemma 2.1 and it yields the two same terms.
That gives us the following lemma

Lemma 3.3. *For any α' with $\alpha < \alpha' < 3$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\begin{aligned}\overline{E}(t_0) \le C\,(&\|\mu_N\|_{\infty,\eta}^{\alpha'/d}\,K^{\alpha'}\,R^{\alpha'-\alpha} + \|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\eta^{\alpha'-\alpha}\\ &+ \varepsilon^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d),\end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$*,* R*,* K*,* m *and* $\overline{E}$ *at the time* t_0*.*

The only non-negligable term in this estimate is sub-linear if α' is chosen sufficiently close to α.
Of course we can perform the same changes in the proof of $\Delta\overline{E}$ to get

Lemma 3.4. *For any* α' *with* $\alpha < \alpha' < 3$*, assume that*

$$m(t_0) \le \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\begin{aligned}\Delta\overline{E}(t_0) \le C\,(&\|\mu_N\|_{\infty,\eta}^{(1+\alpha')/d}\,K^{1+\alpha'}\,R^{\alpha'-\alpha} + \|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\,K^{1+2\alpha'-\alpha}\,\eta^{\alpha'-\alpha}\\ &+ \varepsilon^{d-\alpha-\beta}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} + \varepsilon^{2d-3\alpha-\beta}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d),\end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$*,* R*,* K*,* m *and* $\overline{E}$ *at the time* t_0*.*

3.2 Proof of Theorem 1.2

Let us fix any time $T > 0$. The aim is to show that we have bounds for R, K, $\overline{E}$ and m, uniform in N on $[0,\ T]$.
Next we choose $\eta_0 = \varepsilon^{1/2}$ for instance and $\eta' = \varepsilon^{1/4}$.
Since for any N the quantities R, K, $\overline{E}$ and m are continuous in time, we may define $T_N < T$ as the first time t (if it exists) such that one of the following inequality at least is not true for some integer M to be chosen after

$$\begin{aligned}&T' = T(R(t), K(t), \overline{E}(t), \sup_{s\le t}\|\mu_N\|_{\infty,\varepsilon}) \ge \frac{T}{M},\\ &m(t) \le \frac{1}{12\,\varepsilon\,K(t)\,\Delta\overline{E}(t)}, \quad C(R(t), K(t), \overline{E}(t), \sup_{s\le t}\|\mu_N\|_{\infty,\varepsilon}) \le \varepsilon^{-1/8M},\\ &\varepsilon^{d-\alpha}(m(t))^{-2d}\,(K(t))^{2d-\alpha} \le \varepsilon^{\beta}, \quad \varepsilon^{2d-3\alpha}(m(t))^{-2d}\,(\overline{E}(t))^{d}\,(K(t))^{d-\alpha} \le \varepsilon^{\beta}.\end{aligned} \tag{3.15}$$

The quantity T' and C are the time and constant defined in Theorem 3.1. Therefore on $[0,\ T_N]$ all inequalities (3.15) are true and we may apply both Theorem 3.1 and Lemma 3.3.
We define $t_i = i\,T'$ and $\eta_i = \eta_0 \times r^i$ with $r = \varepsilon^{-1/4M}$ so that $\eta_M = \eta'$. We are going to apply M times Theorem 3.1, once on every interval $[t_{i-1},\ t_i]$ (of size less than T') and with $\eta = \eta_i$ and ε replaced by η_{i-1}. That gives

$$\sup_{t\in[t_{i-1},t_i]} \|\mu_N(t)\|_{\infty,\eta_i} \leq \|\mu_N\|_{\infty,\eta_{i-1}} + C(\overline{E}(t_i),\Delta\overline{E}(t_i))\,(\eta_i^\gamma + \varepsilon^{1/4M}),$$

and consequently thanks to (3.15)

$$\sup_{t\leq T_N} \|\mu_N(t)\|_{\infty,\eta'} \leq \|\mu_N\|_{\infty,\varepsilon} + C(\overline{E}(T_N),\Delta\overline{E}(T_N))\,M\,\varepsilon^{1/4M} \leq 2\,\|\mu_N^0\|_{\infty,\varepsilon}, \tag{3.16}$$

independently of N (and T_N). Now we apply Lemma 3.3 at time T_N and because of (3.15), we obtain

$$\begin{aligned}\overline{E}(T_N) &\leq C\,\|\mu_N(T_N)\|_{\infty,\eta}^{\alpha'/d}\,(K(T_N))^{\alpha'}\,(R(T_N))^{\alpha'-\alpha}\\ &\leq C\,(K(T_N))^{\alpha'}\,(R(T_N))^{\alpha'-\alpha},\end{aligned} \tag{3.17}$$

using (3.16). As $T_N > \varepsilon$, Lemma 2.4 implies that

$$K(T_N) \leq K(0) + C\int_0^{T_N} \overline{E}(t)\,dt \leq K(0) + C\,T_N\,\overline{E}(T_N).$$

From this inequality, we immediately deduce that

$$R(T_N) \leq R(0) + T_N\,K(0) + C\,T_N^2\,\overline{E}(T_N) \leq C\,T + C\,T^2\,\overline{E}(T_N).$$

Inserting these last two inequalities in (3.17), we find

$$\overline{E}(T_N) \leq C\,T + C\,T^2\,(\overline{E}(T_N))^{2\alpha'-\alpha}.$$

Since $2\alpha' - \alpha < 1$, there exists a constant $C(T)$ depending only on T and the initial distribution such that

$$\overline{E}(T_N) \leq C(T), \quad K(T_N) \leq C(T), \quad R(T_N) \leq C(T). \tag{3.18}$$

We are almost ready to conclude, we only need to apply once Lemma 3.4 and by (3.15), (3.16) and (3.18)

$$\Delta\overline{E}(T_N) \leq C(T). \tag{3.19}$$

Inserting (3.19) in Lemma 2.3, we eventually get

$$m(T_N) \leq C(T). \tag{3.20}$$

Together (3.18), (3.19) and (3.20) imply that all the inequalities of (3.15) are true with a factor 1/2 at time T_N, provided N and M are large enough. Therefore (3.15) is still true on at least a short time interval after T_N and that means that necessarily $T_N = T$. The consequence is that (3.18), (3.19) and (3.20) are true on any time interval $[0,\ T]$ which is exactly Theorem 1.2. Finally note that we have implicitly used the short time result when we said that $T_N > \varepsilon$.

4 Convergence of the density in the approximation

The existence of the bound on R, K, $\overline{E}$, $\Delta\overline{E}$ and $\|\mu_N\|_{\infty,\eta}$ implies the weak convergence of the distribution μ_N to a weak solution of the Vlasov equation and Theorem 1.3 is only a consequence of Theorem 1.2 and the following proposition

Proposition 4.1. *Let μ_N be the distributions associated with the solutions to* (1.1)*. We assume that the initial conditions μ_N^0 converges weakly in $M^1(\mathbb{R}^{2d})$ to some $f_0 \in L^1 \cap L^\infty(\mathbb{R}^{2d})$. We choose a time $T > 0$. Assume furthermore that there exists a constant $C(T)$ independent of N such that*

$$\sup_{\varepsilon>0}(R(T),\ K(T),\ \overline{E}(T),\ \Delta\overline{E}(T),\ \|\mu_{\infty,\eta}\|) < +\infty\ ,$$

where η depends on ε and N and goes to zero when ε goes to zero. Then, $\mu_N(t)$ converges weakly to $f(t)$, a solution to the Vlasov equation with initial conditions f^0.

Proof. We recall that the distribution of the particles satisfies the Vlasov equation in the sense of distribution. Moreover, the sequence μ_N is bounded in $C([0,T], M^1(\mathbb{R}^{3d}))$. Up to an extraction, we may assume that μ_N converges weakly to some $f \in L^\infty([0,T], M^1(\mathbb{R}^{2d}))$. Moreover, the fact that $\|\mu_N\|_{\infty,\eta}$ is bounded implies that $f \in L^\infty$. To see this, we choose a regular test function ϕ with compact support. We have

$$\langle \mu_N, \phi \rangle = \frac{1}{N}\sum_{i=1}^{N} \phi(X_i(t), V_i(t)).$$

Now, we define $\rho_\eta(x,v) = \chi_C(x/\eta, v/\eta)$ where χ_C is the characteristic function of the set $C = \{(x,v) | \|(x,v)\| \leq 1\}$ and we write

$$\begin{aligned}\langle \mu_N, \Phi\rangle = \frac{1}{N}\sum_{i=1}^{N} \Phi * \rho_\eta(X_i(t), V_i(t)) \\ + \frac{1}{N}\sum_{i=1}^{N} (\Phi(X_i(t), V_i(t)) - \Phi * \rho_\eta(X_i(t), V_i(t))).\end{aligned}$$

The first term is $\int \phi * \rho_\eta(x,v)\, d\mu_N(x,v) = \int \phi(\mu_N * \rho_\eta)\, dxdv$. So it is bounded by $\|\phi\|_1 \|\phi\mu_N * \rho_\eta\|_\infty$. But $\|\phi\mu_N * \rho_\eta\|_\infty$ is exactly $\|\mu_N\|_{\infty,\eta}$. The second term is easily bounded by $\eta\|\nabla\Phi\|_\infty$. Putting all together, we obtain that

$$\langle \mu_N, \Phi\rangle \leq \|\mu_N\|_{\infty,\eta}\|\Phi\|_1 + \eta\|\nabla\Phi\|_\infty.$$

At the limit,

$$\langle f, \Phi\rangle \leq \liminf_{N\to\infty} \|\mu_N\|_{\infty,\eta}\|\Phi\|_1,$$

which means that $f \in L^\infty$ and that $\|f\|_\infty \leq \liminf_{N\to\infty} \|\mu_N\|_{\infty,\eta}$.

The passage to the limit in the linear part of the equation does not raise any difficulty. For the term in $F \cdot \nabla_v f$, we need a strong convergence in the force. We denote by F_∞ the force induced by f and by F_N the force induced by μ_N

$$\begin{aligned} F_\infty(x) &= \int \frac{x-y}{|x-y|^{1+\alpha}}\, dydw, \\ F_N(x) &= \frac{1}{N}\sum_{i=1}^{N} \frac{x - X_i(t)}{|x - X_i(t)|^{1+\alpha}}. \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} F_N(X_i(t)) - F_\infty(X_i(t))\, dt &= I_1 + I_2 + I_3 \\ &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \int_{|y-X_i(s)|\geq r} \frac{y - X_i(s)}{|y - X_i(s)|^{\alpha+1}}\, d(\mu_N - f)(y)\, ds \\ &\quad + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \int_{|y-X_i(t)|\leq r} \frac{y - X_i(s)}{|y - X_i(s)|^{\alpha+1}}\, d\mu_N(y)\, ds \\ &\quad - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \int_{|y-X_i(t)|\leq r} \frac{y - X_i(s)}{|y - X_i(s)|^{\alpha+1}}\, df(y)\, ds, \end{aligned}$$

for all $r > 0$. The first term I_1 in the right hand side always goes to zero because μ_N converges weakly to f. The second term is dominated by $\|f\|_\infty \int_{B(0,R)} dy/|y|^\alpha$, a quantity which is less than $C\|f\|_\infty r^{d-\alpha}$. The last one is the field created by the close particles in the discrete case. To estimate it we will use the lemma 2.1, replacing $R(T)$ by r. This gives

$$\begin{aligned} I_3 \leq C\, (\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\, K^{\alpha'}\, r^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} \\ + \varepsilon^{2d-3\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^d\, K^d) \leq C\, r^{\alpha'-\alpha}. \end{aligned}$$

And these bounds are independent of N or i.
Then, letting ε going to 0 and then r, we find that

$$\sup_{i,t} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |F_N(X_i(s)) - F_\infty(X_i(s))|\, ds \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.1}$$

With this strong convergence, we are able to prove the convergence of the term $F_N \cdot \nabla_v \mu_N\, ds$ towards $F_\infty \cdot \nabla_v f$ in the sense of distributions. We choose a test smooth test function ϕ with compact support and compute

$$\begin{aligned} J = \int_0^T \Big(\int_{x,v} F_\infty(t,x) \cdot \nabla_v \phi(t,x,v) f(t,x,v)\, dxdv \\ - \sum_{i=1}^N F_N(t, X_i(t), V_i(t)) \cdot \nabla_v \phi(t, X_i(t), V_i(t))\Big)\, dt \end{aligned} \tag{4.2}$$

We separate J in $J_1 + J_2$, with

$$J_1 = \int_0^T \int_{x,v} F_\infty(t,x) \cdot \nabla_v \phi(t,x,v) d(f - \mu_N)(,x,v)\, dt,$$

and

$$J_2 = \int_0^T \Big(\sum_{i=1}^N F_\infty(t, X_i(t), V_i(t)) - F_N(t, X_i(t), V_i(t)) \cdot \nabla_v \phi(t, X_i(t), V_i(t)) \Big)\, dt$$

Because of the continuity of F_∞, J_1 vanishes as ε goes to zero. To show that J_2 vanishes as well, we decompose it in $M = [T/\varepsilon] + 1$ integrals on M intervals of time with length ε. The last interval is of length less than ε, but that does not any difficulty and we do as if it were of length ε. We obtain,

$$\begin{aligned} J_2 &= \sum_{k=1}^{M} \int_{k\varepsilon}^{(k+1)\varepsilon} \Big(\sum_{i=1}^{N} \big(F_\infty(t, X_i(t), V_i(t)) - F_N(t, X_i(t), V_i(t)) \big) \\ &\qquad \cdot \nabla_v \phi(t, X_i(t), V_i(t)) \Big)\, dt \qquad (4.3) \\ &\le C \sum_{k=1}^{M} \int_{k\varepsilon}^{(k+1)\varepsilon} \Big(\sum_{i=1}^{N} \big| F_\infty(t, X_i(t), V_i(t)) - F_N(t, X_i(t), V_i(t)) \big| \Big)\, dt. \end{aligned}$$

This sum may be bounded by

$$CT \sup_{i,t} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |F_N(X_i(s)) - F_\infty(X_i(s))|\, ds,$$

a quantity which goes to zero according to (4.1). Thus, J goes to zero when ε goes to zero and the proof is done. □

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