THE ODD–DIMENSIONAL GOLDBERG CONJECTURE

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Abstract. An odd–dimensional version of the Goldberg conjecture was formulated and proved in [3], by using an orbifold analogue of Sekigawa's arguments in [4], and an approximation argument of K–contact structures with quasi–regular ones. We provide here another proof of this result.

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1. Introduction

The celebrated Goldberg conjecture states that every compact almost Kähler Einstein manifold M is actually Kähler–Einstein. This conjecture was confirmed by Sekigawa [4] in the case when M has non–negative scalar curvature. The odd–dimensional analogues of Kähler manifolds are Sasakian manifolds, and those of almost Kähler manifolds are K–contact manifolds. In [3], Boyer and Galicki proved the following odd-dimensional analogue of Goldberg's conjecture.

Theorem 1.1. [3] *Any compact Einstein K–contact manifold* (M, g, ξ) *is Sasakian.*

Their proof goes roughly as follows. First, an Einstein K–contact manifold has prescribed (positive) Einstein constant. If the K–contact structure is quasi–regular (i.e. the orbits of the Reeb vector field ξ are closed), then the quotient of M by the flow of ξ is an almost Kähler orbifold [5] which is Einstein with positive scalar curvature by the O'Neill formulas. One then applies Sekigawa's proof to obtain that the almost Kähler structure is integrable, which in turn means that the K–contact structure is Sasakian. If the K–contact structure is not quasi–regular, the space of orbits of ξ is may not be an orbifold (and not even a tractable topological space). To overcome this difficulty, the authors of [3] provide a beautiful argument showing that the Reeb vector field ξ can be approximated (in a suitable sense) by a sequence of quasi–regular Reeb vector fields ξ_i which define K–contact structures on a sequence of (no longer Einstein) metrics g_i approaching g. Then for the sequence of orbifolds thus obtained, one can use "approximative" Sekigawa formulas and eventually show that the K–contact structure is integrable.

The aim of this note is to give another proof of Theorem 1.1, by using the following simple observation. Instead of considering the quotient of M by the Reeb flow (which can well be irregular), we consider another almost Kähler manifold naturally associated to M, namely the cone over M. It is well-known that the cone is a smooth, *non–compact*

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Ricci–flat almost Kähler manifold which is Kähler if and only if M is Sasakian. It therefore suffices to prove the integrability of the almost Kähler cone structure. A difficulty to apply directly the Sekigawa arguments in this situation is the non-compactness of the cone, which we overcome easily: we first apply a point-wise version of Sekigawa's formula on the cone manifold, and then integrate it on the level sets of the radial function (which are compact manifolds).

The use of this approach tempted us to extend the conjecture to the more general case of *contact metric structures*, when the metric is no longer bundle–like. Indeed, one could argue that the analogues of almost Kähler manifolds in odd dimensions are the contact metric structures, since they correspond to the level sets of the radial function of almost Kähler cone metrics. However, the analogue of the Goldberg conjecture in this setting, stating that *any compact Einstein contact metric manifold should be Sasakian–Einstein* turns out to be false, as it follows from an example of D. Blair which we recall in the last section.

2. Preliminaries

Let (M, g) be a Riemannian manifold. We define the cone $\bar{M} := M \times \mathbb{R}^*_+$ endowed with the metric $\bar{g} = dr^2 + r^2 g$, and denote by $\bar{\nabla}$ the covariant derivative of $\bar{g}$. It is well-known that the cone is a non–complete Riemannian manifold which can be completed at $r = 0$ if and only if M is a round sphere.

Every vector field X on M induces in a canonical way a vector field $(X, 0)$ on $\bar{M}$, which (with a slight abuse of notation) will still be denoted by X. Similarly, we denote by the same symbol the forms on M and their pull-backs to $\bar{M}$ (with respect to the projection on the first factor). Let us denote by ∂_r the vector field $\frac{\partial}{\partial r}$ on $\bar{M}$. The following formulas relate the covariant derivatives ∇ and $\bar{\nabla}$, and are immediate consequences of the definitions.

$$\bar{\nabla}_{\partial_r}\partial_r = 0; \quad \bar{\nabla}_X \partial_r = \bar{\nabla}_{\partial_r} X = \frac{1}{r}X; \quad \bar{\nabla}_X Y = \nabla_X Y - r g(X, Y)\partial_r. \tag{1}$$

Using this, we obtain for every vector X and a p–form ω on M

$$\bar{\nabla}_{\partial_r}\omega = -\frac{p}{r}\omega \quad \text{and} \quad \bar{\nabla}_X \omega = \nabla_X \omega - \frac{1}{r} dr \wedge X \lrcorner \omega, \tag{2}$$

$$\bar{\nabla}_{\partial_r} dr = 0 \quad \text{and} \quad \bar{\nabla}_X dr = r X^\flat. \tag{3}$$

The curvature tensors R and $\bar{R}$ of M and $\bar{M}$, respectively, are related by

$$\bar{R}(\partial_r, \cdot) = 0 \quad \text{and} \quad \bar{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X. \tag{4}$$

Definition 2.1. *A* contact metric structure *on a Riemannian manifold M is a unit length vector field ξ such that the 1–form $\eta := \langle \xi, \cdot \rangle$ and the endomorphism φ associated to $\frac{1}{2}d\eta$ are interrelated by*

$$\varphi^2 = -1 + \eta \otimes \xi. \tag{5}$$

Since $\varphi^2(\xi)=0$*, we get* $|\varphi(\xi)|^2=-\langle\xi,\varphi^2(\xi)\rangle=0$*, so* $\varphi(\xi)=0$*. In other words,* φ *defines a complex structure on the distribution orthogonal to* ξ*.*

A contact metric structure (M,g,ξ,φ,η) *is called* K–contact *if* ξ *is Killing. The K–contact structure* (M,g,ξ,φ,η) *is called* Sasakian *if*

$$\nabla.\nabla\xi=\xi\wedge\cdot. \tag{6}$$

Given a contact metric manifold (M,g,ξ,φ,η), we construct a 2–form Ω on $\bar M$, defined by

$$\Omega=rdr\wedge\eta+\frac{r^2}{2}d\eta. \tag{7}$$

This 2–form is clearly compatible with $\bar g$, and therefore defines an almost complex structure J on $\bar M$ by $\Omega(\cdot,\cdot)=\bar g(J\cdot,\cdot)$. Moreover, Ω is obviously closed, meaning that $(\bar M,J)$ is almost Kähler. It is well–known that Ω is parallel (i.e. $(\bar M,J)$ is Kähler) if and only if the contact structure ξ is Sasakian.

We close this section with the following

Lemma 2.2. (i) *The codifferentials on* M *and* $\bar M$ *are related by*

$$\delta^{\bar M}(r^k\sigma)=r^{k-2}\delta^M\sigma,\quad \forall\sigma\in\Lambda^1M. \tag{8}$$

(ii) *The Laplacians on* M *and* $\bar M$ *are related by*

$$\Delta^{\bar M}(r^kf)=r^{k-2}(\Delta^Mf-k(2n+k)f),\quad \forall f\in C^\infty(M). \tag{9}$$

Proof. (i) If (e_i) denotes a local orthonormal base on M, we have

$$\begin{aligned}
\delta^{\bar M}(r^k\sigma) &= \sum_i(-\frac{e_i}{r}(r^k\sigma(\frac{e_i}{r}))+r^k\sigma(\bar\nabla_{\frac{e_i}{r}}\frac{e_i}{r}))-\partial_r(r^k\sigma(\partial_r))+r^k\sigma(\bar\nabla_{\partial_r}\partial_r)\\
&= \sum_i -r^{k-2}e_i(\sigma(e_i))+r^{k-2}\sigma(\nabla_{e_i}e_i)-r\partial_r)=r^{k-2}\delta^M\sigma.
\end{aligned}$$

(ii) Similarly,

$$\begin{aligned}
\Delta^{\bar M}(r^kf) &= \sum_i(-\frac{e_i}{r}(\frac{e_i}{r}(r^kf))+\bar\nabla_{\frac{e_i}{r}}\frac{e_i}{r}(r^kf))-\partial_r(\partial_r(r^kf))\\
&= \sum_i(-r^{k-2}e_i(e_i(f))+\frac{1}{r^2}(\nabla_{e_i}e_i-r\partial_r)(r^kf))-k(k-1)r^{k-2}f\\
&= r^{k-2}\Delta^Mf-k(2n+1)r^{k-2}f-k(k-1)r^{k-2}f\\
&= r^{k-2}(\Delta^Mf-k(2n+k)f).
\end{aligned}$$

∎

3. Proof of Theorem 1.1

Let (M^{2n+1}, g, ξ) be a compact K–contact Einstein manifold. By a result of Blair ([2], Theorem 7.1), a contact metric manifold is K-contact if and only if $\mathrm{Ric}(\xi, \xi) = 2n$; thus, the Einstein constant in our case must be $2n$.

Consider now the cone $\bar{M}$, which is an almost Kähler manifold. We use the following Weitzenböck–type formula, taken from [1, Prop.2.1].

Proposition 3.1. *For any almost Kähler manifold $(\bar{M}, \bar{g}, J, \Omega)$ with covariant derivative denoted by $\bar{\nabla}$ and curvature tensor $\bar{R}$, the following point-wise relation holds:*

$$\begin{aligned}(10)\qquad \Delta(s^* - s) &= -4\delta(J\delta^{\bar{\nabla}}(J\bar{\mathrm{Ric}}")) + 8\delta(\langle\bar{\rho}^*, \bar{\nabla}.\ \Omega\rangle) + 2|\bar{Ric}''|^2\\ &\quad -8|\bar{R}''|^2 - |\bar{\nabla}^*\bar{\nabla}\Omega|^2 - |\phi|^2 + 4\langle\rho, \phi\rangle - 4\langle\rho, \bar{\nabla}^*\bar{\nabla}\Omega\rangle\ ,\end{aligned}$$

where: s and s^* are respectively the scalar and $*$–scalar curvature, $\phi(X, Y) = \langle\bar{\nabla}_{JX}\Omega, \bar{\nabla}_Y\Omega\rangle$, $\bar{\mathrm{Ric}}''$ is the J–anti–invariant part of the Ricci tensor $\bar{\mathrm{Ric}}$, ρ is the $(1,1)$–form associated to the J-invariant part of $\bar{\mathrm{Ric}}$, $\bar{\rho}^* := \bar{R}(\Omega)$ and $\bar{R}''$ denotes a certain component of the curvature tensor.

In our situation, since M^{2n+1} is Einstein with constant $2n$, (4) shows that $\bar{M}$ is Ricci–flat. So the formula above becomes

$$(11)\qquad \Delta^{\bar{M}}s^* - 8\delta^{\bar{M}}(\langle\bar{\rho}^*, \bar{\nabla}.\ \Omega\rangle) = -8|\bar{R}''|^2 - |\bar{\nabla}^*\bar{\nabla}\Omega|^2 - |\phi|^2$$

We now use Lemma 2.2 in order to express the left–hand side of this equality in terms of the codifferential and Laplacian on M. From (4) we get $\bar{\rho}^*(X, \partial_r) = 0$ and $\bar{\rho}^*(X, Y) = \bar{g}(\bar{R}(\frac{e_i}{r}, J\frac{e_i}{r})X, Y) = \rho^*(X, Y)$, for some 2–form ρ^* on M. Taking the scalar product with Ω yields

$$(12)\qquad s^* = \frac{1}{r^2}f$$

for some function f on M. Note that f is everywhere positive on M since $s^* = s^* - s = |\bar{\nabla}\Omega|^2$ on $\bar{M}$ (see [1], p. 777).

Now, from (2), (3) and (7) we get $\bar{\nabla}_{\partial_r}\Omega = 0$ and $\bar{\nabla}_X\Omega = r^2\omega + rdr \wedge \tau_X$ for some 2–form ω and 1–form τ_X on M. Consequently, the 1–form $\langle\bar{\rho}^*, \bar{\nabla}.\ \Omega\rangle$ on $\bar{M}$ is easily seen to be of the form

$$(13)\qquad \langle\bar{\rho}^*, \bar{\nabla}.\ \Omega\rangle = \frac{1}{r^2}\alpha$$

for some 1–form α on M. Using (12), (13) and Lemma 2.2, the equality (11) becomes

$$(14)\qquad \frac{1}{r^4}(\Delta^M f + 2(2n-2)f - 8\delta^M\alpha) = -8|\bar{R}''|^2 - |\bar{\nabla}^*\bar{\nabla}\Omega|^2 - |\phi|^2.$$

Integrating this last equation on each level set $M_r := \{r = constant\}$ of $\bar{M}$ yields

$$\int_{M_r} \frac{2(2n-2)}{r^4}f + 8|\bar{R}''|^2 + |\bar{\nabla}^*\bar{\nabla}\Omega|^2 + |\phi|^2.$$

In particular, since $f \geq 0$, ϕ vanishes identically on $\bar{M}$, hence $|\bar{\nabla}_X \Omega|^2 = -\phi(X, JX) = 0$ for every X on $\bar{M}$. Thus $\bar{M}$ is Kähler, so M is Sasakian.

4. A counterexample to further extension

As explained in the introduction, it was tempting to ask the following question, slightly more general than Theorem 1.1: *is every compact Einstein contact metric manifold Sasakian–Einstein?* This is however false, as the following simple example of D. Blair shows (see [2], p. 23, p. 68-69 & p. 52-53).

Example 4.1. *The 1–form* $\eta := \cos t\, dx + \sin t\, dy$ *defines a (non-regular) contact metric structure on the flat torus* T^3 *(where* t, x *and* y *are standard coordinates on* T^3 *of periods* 2π*), which is not K-contact (and hence not Sasakian).*

As a final remark, we note that constructing a counterexample *with positive scalar curvature* to the above question, amounts to finding eigenforms of degree 1 of the Laplace operator on the round sphere S^3 with constant norm (other than the dual of the Killing vector fields defining the Hopf fibrations). We have not studied the question of existence of such forms thoroughly.

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