PAR–LPTHE P03–13
RUNHETC-2003-06
hep-th/0304221

Gravitational Topological Quantum Field Theory Versus $N = 2$ $D = 8$ Supergravity and its lift to $N = 1$ $D = 11$ Supergravity

Laurent Baulieu
Laboratoire de Physique Théorique et Hautes Energies,
Université de Paris VI and Paris VII, France†
and
Department of Physics, Rutgers University, USA††

Abstract: In a previous work, it was shown that the 8-dimensional topological quantum field theory for a metric and a Kalb–Ramond 2-form gauge field determines $N = 1$ $D = 8$ supergravity. It is shown here that, the combination of this TQFT with that of a 3-form determines $N = 2$ $D = 8$ supergravity, that is, an untruncated dimensional reduction of $N = 1$ $D = 11$ supergravity. Our construction holds for 8-dimensional manifolds with $Spin(7) \subset SO(8)$ holonomy. We suggest that the origin of local Poincaré supersymmetry is the gravitational topological symmetry. We indicate a mechanism for the lift of the TQFT in higher dimensions, which generates Chern–Simons couplings.

Postal address:
†Laboratoire de Physique Théorique et des Hautes Energies, Unité Mixte de Recherche CNRS 7589, Université Pierre et Marie Curie, boîte postale 126. 4, place Jussieu, F–75252 PARIS Cedex 05.
†† Dept. of Physics, Rutgers University, New Brunswick, NJ 60637, USA

1 Introduction

Poincaré supersymmetry and topological supersymmetry are deeply related on manifolds with special holonomies. This was observed in [1] and [2], where the 8-dimensional Yang–Mills topological quantum field theory (TQFT) was constructed. [1] also suggested that the introduction of a TQFT for a 3-form in eight dimensions is of relevance, for a better understanding of supergravities and for determining effective actions for the M theory.

[3] indicated the existence of a formal link between topological sigma-models and all anomaly-free superstrings. For us, it was a signal that the origin of local Poincaré supersymmetry might be the gravitational topological symmetry. This raises the question whether $D = 11$ supergravity, which determines all known supergravities in lower dimensions as limits of superstrings, can be constructed in the context of a TQFT. Eventually, there is the possibility that there is a unique topological symmetry that can be represented in two possible phases, one which is purely topological and the other one which describes particles.

To concretely understand how supergravity can be deduced from topological gravity is not a trivial task. We first demonstrated that topological gravities in four and eight dimensions can be respectively untwisted into $N = 2$ $D = 4$ supergravity [4] and a truncated version of $N = 1$ $D = 8$ supergravity [5]. Then, we proved that the TQFT of a general tensor of rank two in eight dimensions, which is made of the symmetric metric and the Kalb–Ramond 2-form, determines in a twisted way the complete $N = 1$ $D = 8$ supergravity [6]. Introducing a TQFT for the 2-form in [6] has greatly clarified questions opened in our earlier work that were related to the interpretation of graviphotons and of Lorentz invariance. In all these cases, the result is that the TQFT can be identified with an Euclidean supergravity, around the solution of a gravitational instanton. The oversimplified case of $N = 1$ $D = 2$ supergravity was analyzed in [7]. Moreover, a geometrical insight on topological 2-dimensional gravitational invariance was given in [8].

We basically found in all these works that the gravitino is a topological ghost for the reparametrization symmetry, up to twist. This is an appealing feature, since the gravitino can then be identified as a curvature (in an enlarged space) and not as a connection. Moreover, we find that local supersymmetry is a ghost of ghost symmetry rather than a gauge symmetry. This changing point of view eliminates many of the puzzling questions that are related to the elusive construction of a gauge group for the Poincaré local supersymmetry. As a matter of fact, all elements of supergravity multiplets acquire robust geometrical definitions within the context of topological gravity coupled to TQFTs of forms.

In this paper, we reach our earlier goal, which was to show that the field spectrum of $N = 1$ $D = 11$ supergravity can be determined in the context of a 8-dimensional gravitational TQFT.

Using a freedom in the ghost sector of the Kalb–Ramond 2-form, which we had not exploited in [6], we enlarge the ghost of ghost symmetry of the twisted $N = 1$ $D = 8$ supergravity. This suggests adding new degrees of freedom. The very natural idea is to introduce a TQFT for a 3-form and to couple it to the TQFT of [6]. A second gravitino emerges, which is made of some of the topological ghosts of the 3-form. Remarkably, the resulting theory turns out to be $N = 2$ $D = 8$ supergravity, whose classical content is made of the metric, a 3-form, three 2-forms, seven scalars, two gravitinos and six Majorana spinors. This is nothing but an untruncated dimensional reduction of the spectrum $N = 1$ $D = 11$ supergravity, in a twisted form.

Our construction holds for 8-dimensional manifolds with $Spin(7) \subset SO(8)$ holonomy. (We could as well choose a smaller holonomy group, for instance G_2.) In fact, one covariantly constant spinor ϵ is needed in order to relate forms and spinors by "twist". It also allows one to construct self-duality equations, which are generally invariant only under the action of the holonomy group. Strictly speaking, we must thus think of a link between the gravitational TQFT and the supergravity theory that we expand around a gravitational instanton. The "twist" operation is what changes the fermions of the TQFT, which exactly balance all contributions of bosons, modulo zero modes, into fermions, which satisfy the physical spin–statistic relation and can be interpreted as particles.

After untwisting, we recover the full Lorentz invariance. Indeed, when we perform the untwisting from the TQFT toward supergravity, the explicit dependence of the TQFT on ϵ is absorbed in the change of variables that maps forms on spinors. Then, we get the supergravity action in its $SO(8)$ invariant form.

An intriguing question is whether the covariantly constant spinor, which makes the twist possible, has a physical origin, for instance, as an expectation value of some field. In the presence of a brane, a form may exist in higher dimensions with a constant flux φ through a hypersurface, which can generate the constant spinor ϵ, by relations like ${}^t\epsilon\gamma^a \dots \gamma^c \, \epsilon = \varphi^{a \dots c}$. Other possibilities exist.

The paper is organized as follows. The next section is devoted to string theory arguments, which further indicate that the determination of the complete spectrum of the maximal supergravity in the context of gravitational gravity is probably not coincidental. Section 3 details our precise arguments, which show the relationship between the 8-dimensional gravitational TQFT for a metric, a 2-form and a 3-form and $N = 2$ $D = 8$ supergravity. Section 4 sketches an argument that new TQFTs seem to exist in eight and seven dimensions, which involve as dynamical fields a twisted gravitino and forms, but not a metric. Section 5 shows a mechanism for lifting the theory in higher dimensions and generating Chern–Simons terms.

2 String theory argument

Superstring theory induces gravity, and is explicitly reparametrization invariant in target-space. The way target-space supersymmetry, and thus supergravity, emerges is less direct. Superconformal invariance on the world-sheet selects vertex operators that describe the gravitino in the quantum field theory limit. Eventually, one discovers target-space supersymmetry as a symmetry transformation between the graviton and the gravitino. The local supersymmetry transformations are given in an infinitesimal form. They build an open algebra that closes only modulo equations of motions, which makes their geometrical interpretation quite obscure.

Our suggestion is that the algebra of supergravity transformations does'nt correspond to a gauge symmetry group. It is basically supported by the observation that the gravitino is a curvature rather than a gauge field, since it can be defined as a combination of untwisted gravitational topological ghosts. The latter are actually defined from the action of exterior operators upon fields with status of connections. The whole gravitational TQFT construction suppresses well known difficulties that occur in all attempts at building a group out of infinitesimal supersymmetry transformations of supergravity

The idea that supergravity can be deduced from topological gravity is heuristically supported by the following world-sheet argument, in the NSR formalism. The string coordinate X^μ has world-sheet supersymmetric partners ψ^μ and $\bar{\psi}^\mu$, where μ labels the world index of target-space. Once a conformal structure has been chosen for the world-sheet of the string, both generators Q and $\bar{Q}$ of world-sheet supersymmetry determine a world-sheet spinor, with:

$$QX^\mu = \psi^\mu + \ldots \quad , \quad \bar{Q}X^\mu = \bar{\psi}^\mu + \ldots \tag{1}$$

The world-sheet spinor $(\psi^\mu, \bar{\psi}^\mu)$ has the physical spin–statistic relation on the world-sheet, but the unphysical one in target-space. Formally, Eq. (1) suggests that the induced metric $g_{\mu\nu}$ of the target-space quantum field theory limit has a symmetry of the type:

$$Qg_{\mu\nu} = \psi_{\mu\nu} + \ldots \tag{2}$$

The field $\psi_{\mu\nu}$ clearly looks as a target-space topological ghost. Its spin–statistic relation is unphysical, so that it contributes negatively to the energy, with contributions that are opposite to those of the metric. Our present understanding of TQFTs suggests to us that the symmetry in Eq. (2) must lead us to a quantum field theory limit that is topological gravity in target-space. Only indirectly can it lead us to supergravity, provided forms can be mapped on spinors in the manifold, and fermions can be extracted with the physical spin–statistic.

As emphasized above, $\psi_{\mu\nu}$ has indeed a precise interpretation in target-space. As a topological ghost for the topological symmetry that is associated to the reparametrization group, $\psi_{\mu\nu}$ is the component of a curvature in the enlarged space that unifies the form-grading of fields and their ghost number. By no mean, can $\psi_{\mu\nu}$ be interpreted as a connection. In the topological BRST framework, $\psi_{\mu\nu}$ transforms under a ghost of ghost symmetry. The Faddeev–Popov spinor ghost for local supersymmetry will be obtained by untwisting the ghosts of ghosts. However, it is not expected that ghosts of ghosts correspond to "infinitesimal" anticommuting spinors, which would eventually be usable for giving a group structure by integration.

The proposed idea is so general, that it should apply to all known supergravities, and thus, to their essential parent, which is $N = 1$ $D = 11$ supergravity. And, indeed, after having achieved the details of the construction of the TQFT for a metric, a 2-form and a 3-form in 8 dimensions, we will have the desired identification between a topological BRST algebra and the infinitesimal symmetries of $N = 2$ $D = 8$ supergravity, which is an expression of the $N = 1$ $D = 11$ supergravity. It is striking that, in this way, all ingredients of supergravity will be described from geometrical considerations in the space of field configurations of a metric, a 2-form and a 3-form in 8 dimensions.

Another motivation for the description of supergravity as a gravitational TQFT is that Eq. (1) also indicates that the superstring theory can be twisted in a topological sigma-model. The operators Q and $\bar{Q}$ can be identified as BRST and anti-BRST operators for the topological sigma-model by shifting to zero value the conformal weight of the superstring fields ψ^μ and $\bar{\psi}^\mu$, and by doing a compensating twist that transforms the world-sheet $N = 1$ supergravity into 2-dimensional topological gravity, which ensures the conformal anomaly-free condition. Formally, the twist is a mere change of variables as for instance indicated in [3][7]. For a topological sigma-model, the dependence on the target-space local properties becomes very loose. This suggests to us that its quantum field theory limit is topological gravity instead of supergravity. For consistency, however, we must be able to recover supergravity from topological gravity directly, which is what we will achieve in this paper.

The proposal that the local details of target-space become a secondary notion is reinforced by observations presented in [3]. There, we have shown that, for a genuine world-sheet theory, i.e, a pure 2d-surface theory without matter, but with a rich enough 2-dimensional gravitational structure for enforcing the absence of a conformal anomaly, one can extract target-space coordinates from the world-sheet structure. Indeed, if we take 2d-supergravity with supersymmetry of rank larger than four, the sum of the contributions of all its ghosts to the conformal anomaly vanishes and it is thus inconsistent to introduce matter under the form of external string coordinates. The latter would generate an anomaly. On the other hand, we have shown that we can perform various twists of the 2d-ghosts, while keeping the conformal anomaly equal to zero and recover all possible superstring theories, with $N < 4$ worldsheeet supersymmetry, combined with decoupled TQFTs [3]. In this presentation, the physical string coordinates, i.e, the coordinates of the effective target-space, appear as bound states of the additional ghosts that are initially introduced for describing the extended supersymmetry on the world-sheet. The superstring coordinates are defined in an analogous way. An alternative scheme is to replace the 2d-supergravity of rank N by the topological W_N gravity, with the interesting limiting case of W_∞ gravity, which has a Lie algebra structure. Eventually, the vertex operators that describe the fields of the limiting supergravity theory are composites of 2-dimensional ghost fields that arise from a pure two-dimensional geometrical structure, with no early reference to a target-space.

These intriguing observations have been the support of our idea that supersymmetry in target-space is more of a topological origin than is usually expected. The next section is to show in detail how the maximal supergravity is actually related to a topological model of the Donaldson–Witten type [9].

3 Determination of the spectrum of N=2 D=8 supergravity

3.1 The TQFT for N=1 D=8 supergravity

In [6], we have constructed the TQFT for a tensor of rank 2 for manifolds with $Spin(7)$ holonomy. We obtained that the BRST topological multiplet is essentially:

$$e^a_\mu, B_{\mu\nu}, \sigma, A^{(2)}_\mu, A^{(-2)}_\mu, (\Psi^{(1)a}_\mu, \bar{\Psi}^{(-1)ab^-}_\mu, \bar{\Psi}^{(-1)}_\mu), (\bar{\chi}^{(-1)}_\mu, \Psi^{(1)}_{\mu\nu^-}, \chi^{(1)}) \tag{3}$$

This is nothing but the spectrum of $N = 1$ $D = 8$ supergravity, up to a twist, which is enabled by the existence of a covariantly constant spinor in the manifold.

In more detail, we constructed in [6] a TQFT for the vielbein e^a_μ, the spin connection ω^{ab}_μ and the Kalb–Ramond 2-form $B_{\mu\nu}$. We found that the topological BRST multiplets for these fields can be expressed as follows:

$$\begin{array}{ccccccc}
 & & & e^a_\mu & & & \\
 & & \swarrow & & & & \\
 & \Psi^{(1)a}_\mu & & & & (\bar{\Psi}^{(-1)ab^-}_\mu, \bar{\chi}^{(-1)}_\mu) & \\
 \swarrow & & & & \swarrow & & \\
\Phi^{(2)a} & & & \sigma, \Phi^{(0)ab^-}, (b^{(0)ab^-}_\mu, b^{(0)}_\mu) & & & \bar{\Phi}^{(-2)a} \\
 & & \swarrow & & & \swarrow & \\
 & \chi^{(1)}, \eta^{(1)ab^-} & & & & \bar{\eta}^{(-1)a} &
\end{array} \tag{4}$$

$$\begin{array}{ccccccc}
 & & & \omega^{ab}_\mu & & & \\
 & & \swarrow & & & & \\
 & \tilde{\Psi}^{(1)ab}_\mu & & & & \bar{\tilde{\Psi}}^{(-1)ab}_\mu & \\
 \swarrow & & & & \swarrow & & \\
\tilde{\Phi}^{(2)ab} & & & \tilde{\Phi}^{(0)ab^-}, \tilde{b}^{(0)ab^-}_\mu & & & \bar{\tilde{\Phi}}^{(-2)ab^+} \\
 & & & & & \swarrow & \\
 & & & & & \bar{\tilde{\eta}}^{(-1)ab^+} &
\end{array} \tag{5}$$

$$\begin{array}{ccccccc}
 & & & B_{\mu\nu} & & & \\
 & & \swarrow & & & & \\
 & & \Psi^{(1)}_{\mu\nu} & & \bar\Psi^{(-1)}_{\mu\nu^-} & & \\
 & \swarrow & & & \swarrow & & \\
 & A^{(2)}_\mu & (A^{(0)},A^{(0)}_{\mu\nu^-}) & & ,\, b^{(0)}_{\mu\nu^-} & A^{(-2)}_\mu & \\
\swarrow & & \swarrow & & & \swarrow & \\
R^{(3)} & S^{(1)},\ (\Psi^{(1)}_{\mu\nu^-},\Psi^{(1)}) & & \bar S^{(-1)},\bar\Psi^{(-1)}_\mu & & & \bar R^{(-3)}\\
 & \swarrow & & \swarrow & & & \swarrow\\
 & b^{(2)}_{S^{(1)}} & & b^{(0)}_{\bar S^{(-1)}} & & b^{(-2)}_{\bar R^{(-3)}} &
\end{array} \qquad (6)$$

To gain control on the gauge invariance of the graviphoton $A^{(-2)}_\mu$, we need a BRST quartet that includes an Abelian ghost $d^{(-1)}$. We thus extend the pair $(A^{(-2)}_\mu, \Psi^{(-1)}_\mu)$ in Eq.(6) as follows [1]:

$$\begin{array}{ccccccc}
 & A^{(-2)}_\mu & \hookrightarrow & & & & \\
\swarrow & & & & & A^{(-2)}_\mu & \\
\bar\Psi^{(-1)}_\mu & & & & \swarrow & & \\
 & & & \bar\Psi^{(-1)}_\mu, d^{(-1)} & & & \\
 & & \swarrow & & & & \\
 & & \Phi^{(0)} & & & & \bar\Phi^{(0)} \\
 & & & & & \swarrow & \\
 & & & & & \bar\eta^{(1)} &
\end{array} \qquad (7)$$

We also have the usual ghost systems for the reparametrization and Lorentz invariances:

$$\begin{array}{ccccccc}
\xi^\mu & & \bar\xi^\mu & & \Omega^{ab} & & \bar\Omega^{ab} \\
 & \swarrow & & & & \swarrow & \\
b^\mu & & & & b^{ab} & &
\end{array} \qquad (8)$$

In these equations, some 2-forms have been decomposed in a Spin(7) invariant way as $X_{\mu\nu} = X_{\mu\nu^-} + X_{\mu\nu^+}$, where $X_{\mu\nu^-}$ and $X_{\mu\nu^+}$ are self-dual and antiself-dual projections of $X_{\mu\nu}$, with dimensions 7 and 21 respectively, according to $28 = 7 \oplus 21$. We have slightly improved our general notation

[1]The BRST symmetry for this U(1) invariance is expressed by: $sA^{(-2)}_\mu = \bar\Psi^{(-1)}_\mu + \partial_\mu d^{(-1)}$, $s\bar\Psi^{(-1)}_\mu = -\partial_\mu \Phi^{(0)}$, $sd^{(-1)} = \Phi^{(0)}$, $s\Phi^{(0)} = 0$, $s\Phi^{(0)} = \bar\eta^{(1)}$, $s\bar\eta^{(1)} = 0$.

of [6] by adding southwest arrows, which indicate which fields are related by topological BRST transformations.

For each field, the upper index in parenthesis indicates the ghost number. The fields that are not on the left edge of each pyramid come into topological pairs that are made of a commuting or an anticommuting antighost $\bar{g}$ with its Lagrange multiplier λ of the opposite statistics. They satisfy BRST equations that can be basically written as $s\bar{g}^{(g)} = \lambda^{(g+1)}$, $s\lambda^{(g+1)} = 0$, i.e, that are of a trivial type. These BRST equations can be improved in a way that expresses the reparametrization symmetry and the symmetries of forms in an equivariant way.

The fields that carry the essential geometrical information are on the left edge of each pyramid. We refer to [6], where the explicit BRST transformations of all above fields have been displayed in great details.

All topological BRST equations actually derive from geometrical equations on extended curvatures, which are of the following type:

$$\begin{aligned} &(s+d)(B_{\mu\nu}dx^\mu \wedge dx^\nu + V^{(1)}_\mu dx^\mu + m^{(2)}) \\ &= \exp i_\xi \Big(dB_2 + \Psi^{(1)}_{\mu\nu} dx^\mu \wedge dx^\nu + A^{(2)}_\mu dx^\mu + R^{(3)} \Big) \end{aligned} \tag{9}$$

The operator $\exp i_\xi$ generically takes into account the reparametrization symmetry.

The expansion in ghost number of Eq.(9) and of its Bianchi identity defines the BRST symmetry. Eq.(9) also shows that the topological ghosts and ghosts of ghosts, which appear in its right hand-side, are components of a curvature. This turns out to be the important observation in view of a geometrical interpretation of the gravitino as curvature and not as a gauge field for a gauge symmetry. This interpretation generalizes to the antighost sector, and is allowed by the existence of a tri-grading, made of the ordinary form degree, the ghost number and the antighost number

It was found in [6] that the expression of the topological gravity action is:

$$\begin{aligned} \int_{M_8} \mathcal{L} \;=\; & \int_{M_8} s\Big[\bar{\Psi}^{(-1)ac^-}(b^{(0)cb^-} + \omega^{cb^-}(e) - G^{cb^-}_d e^d)\mathcal{V}_{ab} \\ & +\bar{\chi}^{(-1)a}\Big(b^{(0)}_a + \partial_a\sigma + \Omega_{abcd}G_{bcd}\Big)\Big] \\ & +s\Big[\partial_{[\mu} A^{(-2)}_{\nu]^-}(\Psi^{(1)a}_{[\mu} e^a_{\nu]} + \Psi^{(1)}_{\mu\nu})\ \Big] \\ & s\Big[e^\mu_c e^\nu_d\ \bar{\bar{\Phi}}^{(-2)cd^+}(\Psi^{(1)}_{\mu\nu} - \Psi^{(1)a}_{[\mu} e^a_{\nu]})\Big], \end{aligned} \tag{10}$$

where $\mathcal{V}_{a_1\ldots a_i} \equiv \frac{1}{(8-i)!}\epsilon_{a_1\ldots a_8} e^{a_{(i+1)}} \ldots e^{a_8}$. This topological action gives kinetic terms for the graviton, the gravitino, the 2-form $B_{\mu\nu}$, both two graviphotons $A_\mu^{\pm 2}$ and the dilaton and dilatino of the of $N = 1$ $D = 8$ supergravity, around an octonionic gravitational instanton and a given configuration of the dilaton. It was emphasized in [6] that, by suitable deformations of the topological gauge functions, it is possible to reproduce the exact supergravity action.

To gauge-fix the local supersymmetry, which emerges as an invariance of the topological ghosts in the action (10), we added in [6] the following s-exact action:

$$\int_{M_8} s\Big[\sqrt{g}(\bar{\Phi}^{(-2)a} D_\mu \Psi_\mu^{(1)a} + \Phi^{(0)ab^-} D_\mu \bar{\Psi}_\mu^{(-1)ab^-} + \bar{\Phi}^{(0)} \partial_\mu \bar{\Psi}_\mu^{(-1)})\Big] \tag{11}$$

This expression is quite instructive to us. It allows us to respectively identify $(\Phi^{(2)a}, \Phi^{(0)ab-}, \Phi^{(0)})$, $(\bar{\Phi}^{(-2)a}, \tilde{\bar{\Phi}}^{(0)ab-}, \bar{\Phi}^{(0)})$ and $(\bar{\eta}^{(-1)a}, \eta^{(1)ab^-}, \eta^{(1)})$ as twisted versions of the Faddeev–Popov spinor ghost and antighost for local N=1 supersymmetry, and of the fermionic Lagrange multipliers for the gauge-fixing of the gravitino, with a gauge function $(\gamma^\mu \partial_\mu)\gamma^\nu \lambda_\nu$.

The gauge-fixing of both graviphotons $A_\mu^{\pm 2}$ was obtained from:

$$\int_{M_8} s(S^1 \partial_\mu A_\mu^{-2} + R^{-3} \partial_\mu A_\mu^2). \tag{12}$$

Thus, (d^1, S^{-1}) and (R^{-3}, R^3) can be identified as the Faddeev–Popov ghosts and antighosts for the $U(1)$ invariances of both graviphotons.

The mapping between the fermionic degrees of freedom between the fermionic sector of the TQFT and of D=8, N=1 supergravity uses the covariantly constant spinor ε of the manifold with $Spin(7)$ holonomy. Call respectively $(\lambda, \bar{\lambda})$ and $(\chi, \bar{\chi})$ the chiral and antichiral parts of the gravitino and dilatino. Their relation with the topological ghosts of the TQFT was found to be [2]:

$$\begin{aligned}
\lambda &= \Psi^a \gamma_a \varepsilon \ , \\
\bar{\lambda} &= \bar{\Psi}\varepsilon + \bar{\Psi}^{ab^-} \gamma_{ab} \varepsilon \ , \\
\chi &= \bar{\chi}^a \gamma^a \varepsilon \ , \\
\bar{\chi} &= \chi\varepsilon + \chi^{ab^-} \gamma_{ab} \varepsilon \ .
\end{aligned} \tag{13}$$

[2]The 8-dimensional gamma matrices γ_a act on spinors of definite chirality. We have defined $\chi_{\mu\nu^-} = \Psi^{(1)}_{\mu\nu^-} - \Psi^{(1)a}_{[\mu} e^a_{\nu]^-}$ and $\chi_{ab^-} = e^\mu_a e^\nu_b \chi_{\mu\nu}$ [6].

3.2 Completion of the spectrum of N=1 D=8 supergravity by the TQFT of a 3-form

The previous section section was for recalling the link between $N = 1$ $D = 8$ supergravity and a TQFT that involves the fields in Eq. (3). We now show the novel result that the construction of a TQFT for a 3-form and its coupling to the TQFT that we have just written, determines $N = 2$ $D = 8$ supergravity.

In order to reach the $N = 1$ $D = 8$ supergravity, we have set to zero in a BRST invariant way the fields $A^{(0)}_{\mu\nu-}$ and $A^{(0)}$, which appear in the TQFT multiplet in Eq. (6). We used a term $s\Big[\bar\Psi^{(-1)}_{\mu\nu-}A^{(0)}_{\mu\nu-} + \bar S^{(-1)}A^{(0)}\Big]$ [6]. Here, we will use our freedom of possibly defining a propagation for these fields, by a change of the topological gauge functions, and reach eventually the $N = 2$ $D = 8$ supergravity.

We thus go back to the situation where the degrees of freedom in $A^{(0)}_{\mu\nu-}$ and $A^{(0)}$ are arranged as the elements of a vector ghost of ghost $A^{(0)}_\mu$. The latter will be interpreted shortly as the twisted expression of the chiral part of a commuting spinor, which turns out to be the Faddeev–Popov antighost for the second generator of N=2 supersymmetry. This suggests that we need new fields, for representing this symmetry. The natural idea is to introduce a 3-form gauge field $C_{\mu\nu\rho}$, together with its topological multiplet, and to complete the N=1 theory into the N=2 theory, by addition of a TQFT for the 3-form.

We now must find the way to write the BRST multiplet for a 3-form. Following the notation that is analogous to that we have used for the BRST multiplet of the 2-form $B_{\mu\nu}$, the BRST multiplet of the 3-form $C_{\mu\nu\rho}$ is:

$$
\begin{array}{ccccccccccccc}
 & & & & & & C_{\mu\nu\rho} & & & & & & \\
 & & & & & \swarrow & & & & & & & \\
 & & & & \psi^{(1)}_{\mu\nu\rho} & & & & \bar\psi^{(-1)}_{\mu\nu\rho\sigma+} & & & & \\
 & & & \swarrow & & & & \swarrow & & & & & \\
 & & B^{(2)}_{\mu\nu} & & & & B^{(0)}_{\mu\nu-}, b^{(0)}_{\mu\nu\rho\sigma+} & & & & B^{(-2)}_{\mu\nu} & & \\
 & \swarrow & & & & \swarrow & & & & \swarrow & & & \\
 R^{(3)}_\mu & & & & S^{(1)}_\mu, \psi^{(1)}_{\mu\nu-} & & & & S^{(-1)}_\mu, \psi^{(-1)}_{\mu\nu} & & & & R^{(-3)}_\mu \\
 \swarrow & & & \swarrow & & & & \swarrow & & & & \swarrow & \\
 M^{(4)} & & N^{(2)}, b^{(2)}_\mu & & & & N^{(0)}, b^{(0)}_\mu & & & & N^{(-2)}, b^{(-2)}_\mu & & M^{(-4)} \\
 & \swarrow & & & & \swarrow & & & & \swarrow & & & \swarrow \\
 & \eta^{(3)} & & & & \eta^{(1)} & & & & \eta^{(-1)} & & & \eta^{(-3)}
\end{array}
\qquad (14)
$$

The BRST equations for all these fields can be obtained in an analogous way as for those the 2-form, with equations as in Eq.(9).

In Eq.(14) the topological antighost of the 3-form has been already identified as a self-dual 4-form $\bar\psi_{\mu\nu\rho\sigma^+}$. This can be accepted as an input. Alternatively, we can start from an earlier stage and define the 56 components of the topological antighost of C_3 in a $Spin(7)$-invariant way as $35 \oplus 21$. Then, we can eliminate the 21-component against the component $B^{(0)}_{\mu\nu^+}$ of the ghost of ghost $B^{(0)}_{\mu\nu}$, with a term of the form $s(\bar\psi^{(-1)}_{\mu\nu^+} B^{(0)}_{\mu\nu^+})$. Eventually, this explains the absence of the forms $\bar\psi^{(-1)}_{\mu\nu^+}$, $B^{(0)}_{\mu\nu^+}$, $b^{(0)}_{\mu\nu^+}$ and $\psi^{(1)}_{\mu\nu^+}$ among the fields that we have displayed in the table (14). This reduction is natural, since we basically need a self-dual Lagrange multiplier 4-form $b^{(0)}_{\mu\nu\rho\sigma^+}$ for determining the square of the 4-form curvature $G_4 = dC_3$ of the 3-form C_3, using a standard self-duality gauge function $G_4 =^* G_4$, and $\int_{M^8} |G_4^+|^2 = \int_{M^8}(|G_4|^2 + G_4 \wedge G_4)$.

As a generalization of what we did for the vector ghost of ghost of the Kalb–Ramond 2-form, we must introduce the fields that are relevant to gain control of the gauge invariance of the 2-form ghost of ghost $B^{(-2)}_{\mu\nu}$. We thus replace the field $\bar\Psi^{(-1)}_{\mu\nu}$ within Eq. (14) by the following more refined set of fields:

$$
\begin{array}{cccccc}
\psi^{(-1)}_{\mu\nu} & \hookrightarrow & & \psi^{(-1)}_{\mu\nu}, \bar R^{(-1)}_{\mu} & & \\
 & & \swarrow & & & \\
 & \Phi^{(0)}_{\mu} & & & & \bar\Phi^{(0)}_{\mu} \\
 & & & & \swarrow & \\
 & & & \eta^{(1)}_{\mu} & &
\end{array}
\qquad (15)
$$

where

$$
\begin{array}{cccccc}
\Phi^{(0)}_{\mu} & \hookrightarrow & & \Phi^{(0)}_{\mu}, m^{(0)} & & \\
 & & \swarrow & & & \\
 & c^{(1)} & & & & \bar c^{(-1)} \\
 & & & & \swarrow & \\
 & & & \bar X^{(0)} & &
\end{array}
\qquad (16)
$$

$$
\begin{array}{cccccc}
\bar\Phi^{(0)}_{\mu} & \hookrightarrow & & \bar\Phi^{(0)}_{\mu}, \bar m^{(0)} & & \\
 & & \swarrow & & & \\
 & \bar c^{(1)} & & & & c^{(-1)} \\
 & & & & \swarrow & \\
 & & & X^{(0)} & &
\end{array}
\qquad (17)
$$

$$
\begin{array}{ccccc}
\eta^{(1)}_{\mu} & \hookrightarrow & & \eta^{(1)}_{\mu}, \bar\eta^{(1)}_{(X)} & \\
& & \swarrow & & \\
& X^{(2)} & & & \bar X^{(-2)} \\
& & & & \swarrow \\
& & & \eta^{(-1)}_{(X)} &
\end{array}
\tag{18}
$$

These substitutions amount to the addition of BRST quartets that count altogether for zero degrees of freedom, and can be cast in the general framework of unification between form degrees, ghost number and antighost number. The propagation of these fields will be given shortly, as well as their interpretation.

Obtaining the second gravitino of the theory from the BRST multiplet of the 3-form is less intuitive than the first gravitino. As shown in Eq.(13), the first gravitino is a rather obvious combination of $Spin(7)$ covariant topological ghosts of the metric and of the Kalb–Ramond 2-form. Our claim is that the chiral and antichiral components of the second gravitino are obtained by untwisting the following sets of forms, which come from the topological multiplet of the 3-form:

$$
\begin{array}{c}
(\bar\psi^{(-1)}_{\mu\nu\rho\sigma^+}, \psi^{(-1)}_{\mu\nu}, \eta^{(-1)}_{(X)}) \\
(\psi^{(1)}_{\mu\nu\rho}, \eta^{(1)}_{\mu}).
\end{array}
\tag{19}
$$

Each one of these combinations has 64 components that are irreducibly decomposed as $35 \oplus 21 \oplus 7 \oplus 1$ under $Spin(7)$ and as $56 \oplus 8$ under $SO(8)$. By appropriately multiplying these tensors by gamma matrices, and applying the resulting 8×8 matrices on the covariantly constant spinor ϵ of the $Spin(7)$ holonomy manifold, one gets chiral and antichiral spinors:

$$
\begin{array}{rcl}
\lambda'_{\mu} & = & \Big(\bar\psi^{(-1)}_{\mu\nu\rho\sigma^+}\gamma^{\nu\rho\sigma} + \psi^{(-1)}_{\mu\nu}\gamma^{\nu} + \eta^{(-1)}_{(X)}\gamma_{\mu}\Big)\epsilon \\
\bar\lambda'_{\mu} & = & \Big(\psi^{(1)}_{\mu\nu\rho}\gamma^{\nu\rho} + \eta^{(1)}_{\mu}\Big)\epsilon
\end{array}
\tag{20}
$$

These spinors determine a combination of a gauge-fixed gravitino with a Higgsino. We will be more precise on this decomposition shortly.

We now start to define the Lagrangian of all these fields, in the tree approximation. Eventually, this will allow for their physical interpretation. We determine the kinetic energy of the second gravitino of the action by the sum of the following BRST-exact terms:

$$
\int s\Big(\bar\psi^{(-1)}_{\mu\nu\rho\sigma^+}(\frac{1}{2}b^{(0)}_{\mu\nu\rho\sigma^+} + \partial_{[\mu}C_{\nu\rho\sigma]})\Big) \;\; = \;\; \int \frac{1}{2}|\partial_{[\mu}C_{\nu\rho\sigma]}|^2 - \bar\psi^{(-1)}_{\mu\nu\rho\sigma^+}\partial_{[\mu}\psi^{(1)}_{\nu\rho\sigma]}
$$

$$\begin{aligned}\int s\Big(\psi^{(1)}_{\mu\nu\rho}\partial_{[\mu}B^{(-2)}_{\nu\rho]}\Big) &= \int \partial_{[\mu}B^{(2)}_{\nu\rho]}\partial_{[\mu}B^{(-2)}_{\nu\rho]} - \bar\psi^{(1)}_{\mu\nu\rho}\partial_{[\mu}\psi^{(-1)}_{\nu\rho]}\\ \int s\Big(\psi^{(-1)}_{\mu\nu}\partial_{[\mu}\bar\Phi^{(0)}_{\nu]}\Big) &= \int \partial_{[\mu}\Phi^{(0)}_{\nu]}\partial_{[\mu}\bar\Phi^{(0)}_{\nu]} - \psi^{(-1)}_{\mu\nu}\partial_{[\mu}\eta^{(1)}_{\nu]}\\ \int s\Big(\bar X^{(-2)}\partial_\mu\eta^{(1)}_\mu\Big) &= \int \bar X^{(-2)}\partial^\mu\partial_\mu X^{(2)} - \eta^{(-1)}_{X^2}\partial_\mu\eta^{(1)}_\mu \quad (21)\end{aligned}$$

The sum of the fermionic terms is Eqs. (21) gives the Rarita–Schwinger action for the second gravitino, expressed in a twisted form, and with an algebraic gauge condition of the type $\gamma^\mu\lambda_\mu = 0$. The compensating Faddeev–Popov ghost dependent action will be determined shortly.

The bosonic terms that are present in the right-hand sides of Eqs. (21) also indicate that we are on the right track for determining $N = 2$ $D = 8$ supergravity. They provide gauge invariant kinetic energy for two 2-forms, $B^{(2)}_{\mu\nu}$ and $B^{(-2)}_{\mu\nu}$, two graviphotons, $\Phi^{(0)}_\mu$ and $\bar\Phi^{(0)}_\mu$, and two scalars, $\bar X^{(-2)}$ and $X^{(2)}$.

The dependence on the 3-form C_3 is equivariant with respect to the tensor gauge invariance $C_3 \sim C_2 + d\Lambda_2$, so it is only through the curvature dC_3. However, the topological field theory BRST construction provides all necessary fields for gauge-fixing the local symmetries of the topological ghosts ghosts of ghosts of C_3.

We must gauge-fix $B^{(2)}_{\mu\nu}$ and $B^{(-2)}_{\mu\nu}$, whose classical action has already been obtained in the second line of Eqs. (21). This is done from the following term:

$$\begin{aligned}&\int s\Big(S^{(1)}_\mu(\partial_\nu B^{(-2)}_{\mu\nu} + \partial_\mu N^{(-2)}) + R^{(-3)}_\mu(\partial_\nu B^{(2)}_{\mu\nu} + \partial_\mu N^{(2)}) + S^{(1)}_\mu b^{(-2)}_\mu\Big)\\ &\qquad= \int \Big(\partial_\nu B^{(-2)}_{\mu\nu}\partial_\rho B^{(2)}_{\mu\rho} + \bar N^{(-2)}\partial^\mu\partial_\mu N^{(2)}\\ &\qquad\quad +\partial_{[\mu}S^{(1)}_{\nu]}\partial_{[\mu}R^{(-1)}_{\nu]} + S^{(1)}_\mu(\partial_\nu\psi^{(-1)}_{\mu\nu} + \partial_\mu\eta^{(-1)})\\ &\qquad\quad +\partial_{[\mu}R^{(3)}_{\nu]}\partial_{[\mu}R^{(-3)}_{\nu]} + R^{(-3)}_\mu\partial_\mu\eta^{(3)}\Big) \qquad (22)\end{aligned}$$

This expression shows us that the propagators of vectors ghosts must be gauge-fixed, so we add the further term:

$$\begin{aligned}&\int s\Big(\bar M^{(-4)}\partial_\mu R^{(3)}_\mu + \bar m^{(0)}\partial_\mu R^{(-1)}_\mu\Big)\\ &= \int \Big(\bar M^{(-4)}\partial^\mu\partial_\mu M^{(4)} + \bar m^{(0)}\partial^\mu\partial_\mu m^{(0)} + \bar m^{(0)}\partial_\mu\Phi^{(0)}_\mu\\ &\qquad +\bar\eta^{(-3)}\partial_\mu R^{(3)}_\mu + \bar c^{(1)}\partial_\mu R^{(-1)}_\mu\Big) \qquad (23)\end{aligned}$$

We now understand that $(R^{(3)}_\mu, R^{(-3)}_\mu)$ and $(R^{(-1)}_\mu, S^{(1)}_\mu)$ are respectively standard Faddeev–Popov vector ghosts and antighosts for the covariant gauge-fixing of both 2-form gauge fields $B^{(\pm 2)}_{\mu\nu}$ [3]. The fields $M^{(\pm 4)}$, $N^{(\pm 2)}$ and $(m^{(0)}, m^{(0)})$ are the standard ghosts of ghosts of these vector ghosts. Thus, the sum of both actions in Eqs. (22) and (23) is nothing but the covariant gauge-fixing Lagrangian for both 2-forms $B^{(\pm 2)}_{\mu\nu}$ in a generalized Feynmann gauge, including all relevant ghosts for the vector gauge symmetry of 2-forms. This result shows the efficiency of the TQFT construction when it comes to encoding all gauge symmetries of the fields.

A remaining task is that of gauge-fixing the Abelian symmetries of $\bar{\Phi}^{(0)}_\mu$ and $\Phi^{(0)}_\mu$. This is done by writing:

$$\begin{aligned} &\int s\Big(\bar{c}^{(-1)}\partial_\mu \Phi^{(0)}_\mu + c^{(-1)}\partial_\mu \bar{\Phi}^{(0)}_\mu + \bar{c}^{(-1)} X^{(0)}\Big) \\ &= \int \Big(\bar{c}^{(-1)}\partial^\mu\partial_\mu c^{(1)} + c^{(-1)}\partial^\mu\partial_\mu \bar{c}^{(1)} + \partial_\mu \bar{\Phi}^{(0)}_\mu \partial_\nu \Phi^{(0)}_\nu\Big) \end{aligned} \tag{24}$$

In the last equation, $X^{(0)}$ and $\bar{X}^{(0)}$ have been eliminated by Gaussian integration. This gives the gauge-fixing action for $\bar{\Phi}^{(0)}_\mu$ and $\Phi^{(0)}_\mu$ in the Feynman gauge. $(\bar{c}^{(1)}, c^{(-1)})$ and $(c^{(1)}, \bar{c}^{(-1)})$ in Eq. (16) are thus identified respectively as the Fadeev–Popov ghosts and antighosts for the Abelian invariance of $\bar{\Phi}^{(0)}_\mu$ and $\Phi^{(0)}_\mu$.

The propagation of the fields $(B^{(0)}_{\mu\nu^-}, N^{(0)}, b^{(0)}_\mu, S^{(-1)}_\mu, \psi^{(1)}_{\mu\nu^-}, \eta^{(1)})$ of the 3-form TQFT multiplet remains to be defined, which we now do.

The fields $(B^{(0)}_{\mu\nu^-}, N^{(0)})$ determine up to twist the chiral part of a commuting Majorana spinor, which we will identify as the Faddeev–Popov ghost of the second generator of local supersymmetry in 8 dimensions. We consider the action:

$$\begin{aligned} &\int s\Big(S^{(-1)}_\mu(\partial_\nu B^{(0)}_{\mu\nu^-} + \partial_\mu N^{(0)})\Big) \\ &= \int S^{(-1)}_\mu(\partial_\nu \psi^{(1)}_{\mu\nu^-} + \partial_\mu \eta^{(1)}) + b^{(0)}_\mu(\partial_\nu B^{(0)}_{\mu\nu^-} + \partial_\mu N^{(0)}) \end{aligned} \tag{25}$$

The last term is identified as the ghost part of a Faddeev–Popov action, for a gauge condition that sets eight components of the second gravitino equal to zero. These eight components build the chiral part of the spin one-half component of the second gravitino. We are actually gradually unveiling the second local supersymmetry. The first term in Eq. (25) is a Dirac Lagrangian for the anticommuting Majorana spinor that one obtains by untwisting $(S^{(-1)}_\mu, \psi^{(1)}_{\mu\nu^-}, \eta^{(1)})$. This field is one Higgsino of the supergravity.

[3]We must redefine $\eta^{(1)}_\mu \to \eta^{(1)}_\mu + S^{(1)}_\mu$ to absorb the term $S^{(1)}_\mu \partial_\nu \psi^{(-1)}_{\mu\nu}$ in Eq. (22). Analogously, we redefine $\bar{X}^{(0)} \to \bar{X}^{(0)} + \bar{m}^{(0)}$ to absorb the term $\bar{m}^{(0)}\partial_\mu \Phi^{(0)}_\mu$.

The gauge-fixing term of the antichiral part of the second gravitino involves, as already suggested, the fields $(A^{(0)}_{\mu\nu^-}, A^{(0)}) \sim A^{(0)}_\mu$ of the spectrum of the 2-form B_2 in Eq.(4). These fields were set equal to zero in [6], so as to obtain $N = 1$ $D = 8$ supergravity. We now choose another gauge-fixing for defining the TQFT, and define a propagation for these commuting ghosts, by means of the action:

$$\begin{aligned} &\int s\Big(A^{(0)}_\mu(\partial_\nu \Psi^{(-1)}_{\mu\nu^-} + \partial_\mu \bar{S}^{(-1)})\Big) \\ &= \int \Psi^{(1)}_\mu(\partial_\nu \psi^{(-1)}_{\mu\nu^-} + \partial_\mu S^{(-1)}) + A^{(0)}_\mu(\partial_\nu b^{(0)}_{\mu\nu^-} + \partial_\mu b^{(0)}) \end{aligned} \tag{26}$$

The last term is indeed the desired Faddeev–Popov term for the antichiral part of the second supersymmetry. The first term in Eq. (26) is another twisted Higgsino term. The sum of both second terms in the actions (25) and (26) determine a complete Faddeev–Popov ghost action for a gauge condition $\Gamma^\mu \lambda_\mu = 0$, where λ stand for the chiral and antichiral ungauged-fixed components of the second gravitino. We conclude that, in a twisted form, the Faddeev-Popov ghosts and antighosts for the second component of local N=2 supersymmetry are:

$$\begin{aligned} &(b^{(0)}_\mu, B^{(0)}_{\mu\nu^-}, N^{(0)}) \\ &(A^{(0)}_\mu, b^{(0)}_{\mu\nu^-}, b^{(0)}) \end{aligned} \tag{27}$$

Of course, one may feel puzzled by the fact that the TQFT chooses different gauge conditions for the gravitinos of both sectors of $N = 2$ local supersymmetry. This is actually explainable, since both gravitinos originate as topological ghosts of different objects, a 2-form and a 3-form. This interesting inelegance might disappear, if we are able to covariantly formulate our equations in higher dimensions, perhaps in nine dimensions. In the untwisted model, it is however possible to readjust the gauge, in such a way that the symmetry of both supersymmetry sectors is recovered.

As for the number of degrees of freedom in the TQFT, the gauge-fixing that is revealed by the actions in Eqs. (25) and (26) suggests that only 56 components of each one of the multiplets in Eq. (19) are really part of the second gravitino.

We are now in the position of more precisely identifying the fields in Eq. (19). They must be understood as made of the twisted gauge-fixed second gravitino and of one independent Majorana spinor, $(\eta^{(1)}_\mu, \psi^{(-1)}_{\mu\nu^-}, \eta^{(-1)}_{(X)})$. The latter will be a third Higgsino of the supergravity, according to:

$$(\bar{\psi}^{(-1)}_{\mu\nu\rho\sigma^+}, \psi^{(-1)}_{\mu\nu}, \eta^{(-1)}_{(X)}) \;\; \rightarrow \;\; (\bar{\psi}^{(-1)}_{\mu\nu\rho\sigma^+}, \psi^{(-1)}_{\mu\nu^+}) \oplus (\psi^{(-1)}_{\mu\nu^-}, \eta^{(-1)}_{(X)})$$

$$(\psi^{(1)}_{\mu\nu\rho}, \eta^{(1)}_{\mu}) \quad \rightarrow \quad \psi^{(1)}_{\mu\nu\rho} \oplus \eta^{(1)}_{\mu} \tag{28}$$

We should now rest and contemplate the field content of the theory that we have constructed step by step.

The idea was to define a propagation for all fields of the BRST multiplet of a 3-form, and to complete the TQFT of [6]. The TQFT sector stemming from a 2-tensor (the metric and the Kalb–Ramond 2-form) yields a propagating theory for the vielbein, one scalar (the dilaton), the 2-form, two vectors (two Abelian graviphotons), one Majorana gravitino and one Majorana spinor, according to:

$$\begin{gathered} e^a_\mu, B_{\mu\nu} \rightarrow \\ e^a_\mu, B_{\mu\nu}, \sigma, A^{(2)}_\mu, A^{(-2)}_\mu, (\Psi^{(1)a}_\mu, \bar\Psi^{(-1)ab^-}_\mu, \bar\Psi^{(-1)}_\mu), (\bar\chi^{(-1)}_\mu, \Psi^{(1)}_{\mu\nu^-}, \chi^{(1)}) \end{gathered} \tag{29}$$

The TQFT sector stemming from the 3-form yields a second Majorana gravitino in a twisted form, two Abelian 2-forms, two vectors, two scalars and three Majorana spinors, according to:

$$\begin{gathered} C_{\mu\nu\rho} \rightarrow \\ C_{\mu\nu\rho}, B^{(\pm 2)}_{\mu\nu}, \bar\Phi^{(0)}_\mu, \Phi^{(0)}_\mu, X^{(2)}, \bar X^{(-2)}, (\bar\psi^{(-1)}_{\mu\nu\rho\sigma^+}, \psi^{(-1)}_{\mu\nu^+}, \psi^{(1)}_{\mu\nu\rho}), \\ (\eta^{(1)}_\mu, \psi^{(-1)}_{\mu\nu^-}, \eta^{(-1)}_{(X)}),\ (S^{(-1)}_\mu, \psi^{(1)}_{\mu\nu^-}, \eta^{(1)}),\ (\Psi^{(1)}_\mu, \psi^{(-1)}_{\mu\nu^-}, S^{(-1)}) \end{gathered} \tag{30}$$

The second gravitino is obtained in a gauge of the type $\gamma_\mu \Lambda_\mu = 0$.

The rest of the fields, which appear in the topological multiplets of the 2-form and 3-form multiplets, play the role of ordinary Faddeev–Popov ghosts for gauge-fixing all gauge symmetries of these fields that can be identified as classical fields of $N = 2$ $D = 8$ supergravity. We left aside the gauge-fixing of the 2-form $B_{\mu\nu}$ and 3-form $C_{\mu\nu\rho}$, for the symmetry $C_3 \sim C_3 + dc_2$ and $B_2 \sim B_2 + dc_1$, which is standard in the equivariant construction, as it is already well understood in the context of the topological Yang–Mills theory.

Putting everything together, we have therefore constructed a theory whose physical fields are, up to twist, a metric, a 3-form, three 2-forms, four 1-forms, three scalars, two gravitinos and four Majorana spinors.

This is not yet the complete $N = 2$ $D = 8$ supergravity. However, we left aside the possibility of completing the TQFT multiplet of the 3-form by topological sets that involve two 1-forms, $\bar\Phi^{(-2)}_\mu$ and $\Phi^{(2)}_\mu$, for gaining control to the possible gauge symmetry of $B^{(0)}_{\mu\nu}$. The fields $\bar\Phi^{(2)}_\mu$ and $\bar\Phi^{(-2)}_\mu$ play a role for $B^{(0)}_{\mu\nu}$ that is analogous to that played by $\bar\Phi^{(0)}_\mu$ and $\Phi^{(0)}_\mu$ for $B^{(-2)}_{\mu\nu}$ [4].

[4]A dissymmetry occurs however in the TQFT, because the anti-self part $B^{(0)}_{\mu\nu^+}$ of $B^{(0)}_{\mu\nu}$ is set equal to zero, while all components of $B^{(-2)}_{\mu\nu}$ propagate.

We introduce $\bar{\Phi}_\mu^{(-2)}$ and $\Phi_\mu^{(2)}$ as parts of topological multiplets, whose components count altogether for zero degree of freedom. Such multiplets are generically of the form $(A_\mu, \Psi_\mu, \eta, \kappa_{\mu\nu^-}, \phi, \bar{\phi})$. They must be completed with Faddeev–Popov ghost and antighost and a Lagrange multiplier, $(g, \bar{g}, h)$. The TQFT action that encodes the Abelian invariance is $\int s(\kappa_{\mu\nu^-}(H_{\mu\nu^-} + \partial_{[\mu} A_{\nu]^-}) + \eta \partial_\mu \Psi_\mu + \bar{g}(\partial_\mu A_\mu + h))$, which a twisted form of the eight dimensional supersymmetric Yang–Mills theory [1].

We can therefore incorporate in the topological multiplet of the 3-form, which is given in Eqs .(14-18), both following topological submultiplets;

$$
\begin{array}{ccccccc}
 & & & \Phi_\mu^{(-2)} & & & \\
 & & \swarrow & & & & \\
 & \psi_\mu^{(-1)}, g^{(-1)} & & & \chi^{(1)}_{\mu\nu^-}, \bar{g}^{(1)} & & \\
 & \swarrow & & & \swarrow & & \\
\phi^{(0)} & & & b^{(2)}_{\mu\nu^-}, h^{(2)} & & & \bar{\phi}^{(0)} \\
 & & & & & \swarrow & \\
 & & & & \bar{\eta}^{(1)} & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & \Phi_\mu^{(2)} & & & \\
 & & \swarrow & & & & \\
 & \psi_\mu^{(3)}, g^{(3)} & & & \chi^{(-3)}_{\mu\nu^-}, \bar{g}^{(-3)} & & \\
 & \swarrow & & & \swarrow & & \\
\phi^{(4)} & & & b^{(-2)}_{\mu\nu^-}, h^{(-2)} & & & \bar{\phi}^{(-4)}, \\
 & & & & & \swarrow & \\
 & & & & \bar{\eta}^{(-3)} & &
\end{array}
\qquad (31)
$$

The TQFT action of these fields is made of two twisted super Yang–Mills actions. We add each of them to the TQFT actions that we have already constructed.

We now face with a theory whose physical field content is made of the fields in Eq. (29-30), plus two graviphotons, two Higgsinos and four scalars, which stem from the fields that we just introduced in Eqs.(31). Thus, up to twist, our gravitational TQFT predicts the following set of fields, which are defined modulo ordinary gauge invariances:

$$
g_{\mu\nu}, C_{\mu\nu\rho}, 3B_{\mu\nu}, 6A_\mu, 7S, 2\lambda, 6\chi. \qquad (32)
$$

The notation is that S, λ and χ respectively denote scalar fields, Majorana gravitinos and Higgsinos. This set of fields fields is nothing, but the spectrum

[10] of $N = 2$ $D = 8$ supergravity. The Lagrangians of the TQFT and of supergravity coincide in the quadratic approximation for the fermions.

All other fields that have occurred in our construction can be understood as conventional ghosts and Lagrange multipliers for fixing the gauge symmetries of the fields in Eq. (32). The way all fields transform under the TQFT symmetry has been determined from the tri-complex structure that one associates to the gauge symmetries of gravity coupled to a 2-form and a 3-form. The expression of the topological BRST transformations can be reinterpreted to reproduce the symmetries of $N = 2$ $D = 8$ supergravity, including local supersymmetry.

The fields that are displayed in Eq. (32) can also be understood as the spectrum of N=1, D=11 supergravity. Indeed, the straightforward dimensional reduction of N=1, D=11 supergravity gives:

$$(g_{\mu\nu}, C_{\mu\nu\rho}, \lambda)_{D=11} \sim (g_{\mu\nu}, C_{\mu\nu\rho}, 3B_{\mu\nu}, 6A_\mu, 7S, 2\lambda, 6\kappa)_{D=8} \tag{33}$$

To conclude this section, let us insist on the following points. As for the identification of the TQFT action and the supergravity action, our demonstration holds at the quadratic level for the fermions. Suitable modifications of the gauge functions by higher order terms should reveal the exact laws of supergravity. Moreover, the word identification means that the gravitational TQFT reproduces the supergravity around a $Spin(7) \subset SO(8)$ invariant background. The $SU(2)$ internal symmetry of the $N = 2$ $D = 8$ supergravity multiplet is not explicit in our construction. This is a direct consequence of the fact that, in the 8-dimensional TQFT, both gravitinos have different origins, as topological ghosts of a 2-form of a 3-form.

4 A digression about the TQFT for the 3-form

It is actually an interesting question to investigate whether a TQFT for the 3-form alone has some interest, in particular for the study of 8-dimensional manifolds with $Spin(7)$ holonomies and of 7-dimensional manifolds with G_2 holonomies. This was suggested as early as in [1]. We would like to briefly sketch new ideas relative to this question and indicate a possible hint for getting an interacting theory for the 3-form, when topological gravity is decoupled.

We have seen in the previous section that the TQFT multiplet for a 3-form is basically made of the following fields:

$$C_3, 2B_2, 2A_1, 2S, \lambda^\alpha_\mu, 2\chi^\alpha \tag{34}$$

These fields are those that propagate in the TQFT, provided one does a suitable gauge-fixing. Spinorial notations have already been used for the Fermi fields that are topological ghosts.

We understood in the previous section that the gravitino λ^α_μ (a 1-form which a spinorial index) is made of topological ghosts, including the self-dual antighost $\chi_{\mu\nu\rho\sigma^+}$ of C_3 The multiplet (34) does not involve a metric. If we follow the conventional pattern for building a TQFT, we introduce an external metric $g_{\mu\nu}$, and do a topological BRST invariant gauge-fixing. In this way, we can get a free TQFT action for the fields in Eq. (34). The action contains the term:

$$\begin{aligned} &\int d^8x \; s(\chi^{\mu\nu\rho\sigma^+}(\partial_{[\sigma}B_{\mu\nu\rho]} + H_{\mu\nu\rho\sigma^+})) \\ &\qquad = \int d^8x \; (|dB_3 +^* dB_3|^2 + \ldots) \\ &= \int d^8x \; \sqrt{g}(g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}g^{\sigma\sigma'}\partial_{[\sigma}B_{\mu\nu\rho]}\partial_{[\sigma'}B_{\mu'\nu'\rho']} + \ldots) \end{aligned} \tag{35}$$

The dots are easy to compute. We can dimensionally reduce the fields and get a TQFT in seven dimensions, for a 3-form and a 2-form. The self-duality condition in eight dimensions is replaced by the following one in seven dimensions, which also stands for 35 conditions:

$$dC_3 +^* dB_2 = 0 \tag{36}$$

Imposing these conditions in a BRST-invariant way is allowed by the fact that C_3 and B_2 carry 20 and 15 degrees of freedom modulo gauge transformations in seven dimensions. Eq. (36) is a generalization of the well-known Bogomolny equation.

If we can define a metric $g_{ij}(B_3)$ in seven dimensions, which depends on the 3-form, and use it for the * Hodge operation, we can trigger interactions between the forms, by defining the following action:

$$\begin{aligned} &\int d^7x s(\chi^{ijk}((dB_2 +^* dC_3)_{ijk} + H_{ijk}) \\ &\qquad = \int d^7x(|dB_2 +^* dC_3|^2 + \ldots) \end{aligned}$$

$$= \int d^7x\sqrt{g}(g^{ii'}g^{jj'}g^{kk'}\ \partial_{[i}B_{jk]}\ \partial_{[i'}B_{j'k']}$$
$$+g^{ii'}g^{jj'}g^{kk'}g^{ll'}\ \partial_{[i}C_{jkl]}\partial_{[i'}C_{j'k'l']} + \ldots) \tag{37}$$

The dots involve the metric dependence of the 3-form. This action is analogous in spirit to that of supersymmetric mechanics in curved space. The subtlety is to find the relevant configuration space of the 3-form for defining the path integral.

A further reduction in six dimensions determines theories with self-dual 2-forms. Such TQFTs have been directly studied in six dimensions, and interesting correspondences with supersymmetry have already been found [12].

There is another topological field theory, which we can obtain by directly starting from the topological action $\int_{M_7} C_3 \wedge dC_3$. For certain manifolds, one can give a special role to the direction x^7 and write:

$$\int_{M_7} C_3 \wedge dC_3 = \int_{M_7} d^7x(C_{7ab}\epsilon^{abcdef}\partial_{[c}C_{def]} + \epsilon^{abcdef}C_{abc}\partial_7 C_{def}) \tag{38}$$

Here, a, b, c, d, e, f stand for 6-dimensional indices.

Generalizing the analysis of the Chern–Simons theory in three dimensions [11], formal manipulations suggest that the 7-dimensional theory with the above action might also describe a theory in six dimensions, depending of a 2-form, with an action of the type:

$$\int_{M_6} d^6x\sqrt{g}\ g^{aa'}g^{bb'}g^{cc'}\ \partial_{[a}B_{bc]}\ \partial_{[a'}B_{b'c']} + \text{WZZ action} \tag{39}$$

A heuristic argument can be done by formally integrating out C_{7ab}, solving the constraint $\partial_{[a}B_{bc]} = 0$, and picking a slice at $x^7 = cte$. Eventually, the 6d-metric depends on the 2-form, $g^{aa'} = g^{aa'}(dB_2)$, and we have a non trivial model. The observables in the seven dimensional theory theory are obtained from the flux of the 3-form on a 3-cycle:

$$exp \int\int\int_{\Gamma_3} C_{ijk}dx^i \wedge dx^j \wedge dx^k \tag{40}$$

and

$$exp \int\int_{\Gamma_2} B_{ij}dx^i \wedge dx^j \tag{41}$$

We leave the investigation of these theories for future work.

5 Higher dimensions and N=1 D=11 supergravity

We have shown in section 3 that the 8-dimensional gravitational TQFT reproduces, up to twist, the $N = 2$ $D = 8$ supergravity theory in the quadratic approximation in the gravitino fields, around a $Spin(7)$ invariant vacuum. However, we left open the question of precisely determining the mechanism according to which Chern-Simons interactions are built in the TQFT.

We wish to look in more details how we can lift in higher dimensions the 8-dimensional TQFT that we have built in section 3.

Let us recall that $N = 1$ $D = 11$ supergravity can be reduced to type IIA 10-dimensional supergravity, which, in turns, can be reduced to a 9-dimensional model and, eventually, to $N = 2$ $D = 8$ supergravity theory.

This 9-dimensional model can also be deduced from type IIB 10-dimensional supergravity. The 9-dimensional theory is interesting due the remarkable properties of $Spin(9)$ [16].

To relate the 8-dimensions TQFT with another one in 9 dimensions, we tentatively refer to the mechanism explained in [15]. There, it is shown on general grounds that, if one has a TQFT in d dimensions, with an action of the type $\mathcal{I}_d = \int_{M_d}(d(...) + \{Q, ...\})$, where Q^2 vanishes modulo some gauge transformations, then, it exists another operator Q_{d+1} in $d + 1$ dimensions, such that

$$Q^2_{d+1} = \partial_{x^{d+1}}, \tag{42}$$

modulo gauge transformations. Q_{d+1} acts on d+1 dimensional fields which are in one to one correspondence with those of the d-dimensional theory. Q_{d+1} is identical to Q when fields are taken at equal values of x^{d+1}. The existence of the Q-exact action $\mathcal{I}_d$ implies that of an action $\mathcal{I}_{d+1}$ in $d + 1$ dimensions, which is of the type [15]:

$$\mathcal{I}_{d+1} = \int_{M_{d+1}} (\Delta_{d+1} + \{Q_{d+1}, ...\}\ ) \tag{43}$$

Δ_{d+1} contains Chern–Simons terms and is Q_{d+1}-invariant but not Q_{d+1}-exact. It descends from the cocycles that one can construct for the cohomology of the Q symmetry [15]. It contains terms that are only $SO(d) \subset SO(d + 1)$ invariant. These general properties have been analyzed in great details in

[15], and geometrical arguments have been given for explaining such a delicate mechanism that relates theories in d and d+1 dimensions.

The second term in the r.h.s of Eq. (43) gives back the action $\mathcal{I}_d$ we started from, by standard dimensional reduction, up to Chern–Simons terms. However, as emphasized in in [15], one may obtain deformations of the original d-dimensional theory. For instance, compactification on a circle provides a radius R that may be used as a parameter for the deformation, when one descends from $d+1$ to d dimensions.

In our case, we start from the the 8 dimensional Q-exact action that we have introduced in section 3. Using the method of [15], we can construct 9-dimensional fields from the 8-dimensional ones and introduce the following Q_9-invariant term in 9 dimensions:

$$\begin{aligned}\int \Delta_9 = \int (A \wedge dC_3 \wedge dC_3 + B_2^{(-2)} \wedge dB_2 \wedge dC_3 \\ +dx^9(\varphi_0^{(2)} dC_3 \wedge dC_3 + \varphi_1^{(0)} \wedge dB_2 \wedge dC_3 + \\ \psi_3^{(1)} \wedge \psi_3^{(1)} \wedge dA + dB_2^{(-2)} \wedge \Psi_2^{(1)} \wedge \Psi_3^{(1)})\ )\end{aligned} \tag{44}$$

We can add to Δ_9 a Q_9-exact action, which is inspired from the one we have constructed in section 3. Eventually, it is likely that we will determine in this way the 9-dimensional supergravity action, including Chern-Simons terms.

The first terms in (44) are the dimensional reduction, modulo d-exact terms, of the 11-dimensional Chern-Simons term

$$\Delta_{11} = C_3 \wedge dC_3 \wedge dC_3 + dx^{11}(\varphi_2^{(2)} \wedge dC_3 \wedge dC_3 + \psi_3^{(1)} \wedge \psi_3^{(1)} \wedge dC_3) \tag{45}$$

We thus postulate the existence of a TQFT in 11 dimensions of a TQFT with an action of the following type:

$$\int (C_3 \wedge dC_3 \wedge dC_3 + dx^{11}(\varphi_2^{(2)} dC_3 \wedge dC_3 + \psi_3^{(1)} \wedge \psi_3^{(1)} \wedge dC_3) + \{Q_{11}, Z_{11}\}) \tag{46}$$

The first terms are Q_{11}-closed but not Q_{11}-exact. The last term, which is Q_{11}-exact, determines propagators and regularizes the theory. We conjecture that the existence of the gauge fermion Z_{11} on 11-dimensional manifolds with $Spin(7)$ holonomy follows from that of the TQFT in 8 dimensions. The action (46) should be closely related to that of $N=1,\ D=11$ supergravity on special manifolds. Various types of compactifications from 11 to 8 and 7 dimensions may produce interesting deformations.

Our suggestion is that the TQFT point of view is likely to to single out the Chern–Simon part of the supergravity action in 11 dimensions as a Q-invariant but not Q-exact terms, while the rest of the action can be cast in a Q-exact form. The latter part, which includes the Einstein–Hilbert and Rarita–Schwinger actions, is important to regularize the theory and the path integral; however, in the topological phase, only the the Chern–Simons part is truly relevant.

6 Conclusion and discussion

This paper shows that, in eight dimensions, the TQFT for a metric, two 1-forms, a 2-form and a 3-form reproduces in a twisted form (at least in the quadratic approximation) the $N = 2$ $D = 8$ supergravity in a $Spin(7)$-invariant gravitational background, which is the dimensional reduction of $N = 1$ $D = 11$ supergravity. We found the relevant self-duality equations for defining the TQFT.

This result is quite interesting. In particular, it gives an alternative definition for local supersymmetry. It circumvates the challenge of closing the algebra of infinitesimal transformations of supergravities by mean of auxiliary fields and of getting a group structure for the transformations of supergravity. We now understand the fact that the latter "close modulo equations of motions" as a consequence of the determination of a TQFT by enforcing self–duality equations in a BRST invariant way.

Many dimensional reductions of the TQFT that we have exhibited can be thought of. We foresee theories in seven and six dimensions, where $G_2 \subset Spin(7) \subset SO(8)$ plays an important role [17], as well as potentially interesting models in 3, 2, 1 and 0 dimensions. In the latter case, there could be a generalization of the DVV matrix model [13] by gravitational terms, stemming from the topological 8-dimensional action. They might be relevant to the recent works in [14].

The 8-dimensional gravitational theory can be also coupled to a non Abelian Yang–Mills theory, which allows us to consider other types of models by dimensional reduction, and perhaps, to find a description of the octonionic superstring [19].

A more speculative idea is that, since $N = 1$ $D = 11$ supergravity seems of a topological origin, it might encode by dimensional reduction the supersymmetric expression of stochastically quantized gauge theories, which have also the structure of a TQFT, and efficiently regularize many of the problems that one encounters in the quantization of 4-dimensional Yang–Mills theories [18].

We added the following remarks.

The construction of a TQFT for a 3-form in eight dimensions can be done independently of the context of topological gravity. It still needs $Spin(7)$ holonomy manifolds, but the metric is no longer an independent field. One gets models of the topological type that depend on a gravitino and seem to contain topological observables. Such models are related to the"Chern–Simons action" $\int C_3 \wedge dC_3$ in seven dimensions.

We also suggested that, on special manifolds, the supergravity action in eleven dimensions can be separated into a Q-closed but not Q-exact term, which is the Chern–Simons part, and a Q-exact term, which defines the propagation of the fields.

Acknowledgments: I thank all members of RUNHETC, where most of this work has been done. I am grateful to M. Douglas, G. Moore, E. Rabinovici, P. Ramond and IM Singer for interesting discussions. I am indebted to E. Witten for an enlighting discussion, which has lead me to add section 4 to this paper.

References

[1] L. Baulieu, H. Kanno and I.M. Singer, *Special Quantum Field Theories In Eight And Other Dimensions,* Commun.Math.Phys.**194** (1998) 149, [hep-th/9704167] and [hep-th/9705127].

[2] B.S.Acharya, M. O'Loughlin and B. Spence, *Higher Dimensional Analogues of Donaldson-Witten Theory,* Nucl. Phys. **B503** (1997) 657, hep-th/9705138.

[3] L. Baulieu, M.B. Green, and E. Rabinovici, *Superstrings from theories with $N > 1$ world-sheet supersymmetry* , Nucl. Phys. **B498** (1997) 119 , [hep-th/9611136]; *A Unifying Topological Action for Heterotic and Type II Superstring Theories*, Phys.Lett. **B386** (1996) 91, [hep-th/9606080].

[4] L. Baulieu and A. Tanzini, *Topological gravity versus supergravity on manifolds with special holonomy,* JHEP **0203** (2002) 015, [hep-th/0201109], D. Anselmi, P. Frè, Nucl. Phys. **B392** (1993) 401, [hep-th/9208029].

[5] L. Baulieu, M. Bellon and A. Tanzini, *Eight-dimensional topological gravity and its correspondence with supergravity,* Phys. Lett. **B543** (2002) 291, [hep-th/0207020], P. de Medeiros and B. Spence, *Four-dimensional topological Einstein-Maxwell gravity,* [hep-th/0209115].

[6] L. Baulieu, M. Bellon and A. Tanzini, *Supergravity and the Knitting of the Kalb–Ramond Two-Form in Eight-Dimensional Topological Gravity,* to appear in Phys. Lett. B, [hep-th/0303165].

[7] L. Baulieu, *Transmutation of Pure 2-D Supergravity Into Topological 2-D Gravity and Other Conformal Theories*, Phys. Lett. **B288** (1992) 59, [hep-th/9206019].

[8] L. Baulieu and I. M. Singer, *Conformally Invariant Gauge Fixed Actions For* 2d *Topological Gravity,* Commun. Math. Phys. **135** (1991) 253-265.

[9] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. **117** (1988) 353.

[10] A. Salam and E. Sezgin, *D=8 Supergravity: matter couplings, gauging and Minkowski compactification,* Phys. Lett. **B154**, 37; *D=8 Supergravity,* Nucl. Phys. **B258** 284, 1984.

[11] E. Witten, *Quantum Field Theory and the Jones Polynomials,* Commun.Math.Phys.121:351,1989.

[12] L. Baulieu and P. West, *Six-Dimensional TQFTs and Twisted Supersymmetry*, Phys. Lett. **B436** (1998) 97, [hep-th/9805200].

[13] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix String Theory*, [hep-th/9703030], Nucl. Phys. B500 (1997) 43-61.

[14] C. Vafa and H. Ooguri, *Gravity Induced C-Deformation*, [hep-th/0303063]; R. Dijkgraaf and C. Vaffa, *N=1 Supersymmetry, Deconstruction, and Bosonic Gauge Theories* , [hep-th/0302011]; F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, *Chiral Rings and Anomalies in Supersymmetric Gauge Theory* , [hep-th/0211170]; R. Dijkgraaf, A. Sinkovics and M. Temurhan *Matrix Models and Gravitational Corrections*, [hep-th/0211241].

[15] L. Baulieu, A. Losev and N. Nekrasov, *Chern-Simons and Twisted Supersymmetry in Higher Dimensions,* Nucl. Phys. **B 522** (1998) 82-104, [hep-th/9707174]; L. Baulieu and E. Rabinovici, *Selfduality And New TQFTs For Forms,* JHEP 9806 (1998) 006, [hep-th/9805122].

[16] L. Brink, P. Ramond, X. Xiong, *Supersymmetry and Euler Multiplets,* JHEP 0210 (2002) 058 , [hep-th/0207253].

[17] A. Bilal, J. P. Derendinger and K. Sfetsos, *(Weak) G(2) holonomy from self-duality, flux and supersymmetry,* Nucl. Phys. **B628** (2002) 112, [hep-th/0111274]; A. Bilal and S. Metzger, *Compact weak G(2)-manifolds with conical singularities,* [hep-th/0302021].

[18] L. Baulieu, P. A. Grassi and D. Zwanziger, ' *Gauge and topological symmetries in the bulk quantization of gauge theories*, Nucl. Phys. **B597** (2001) 583, [hep-th/0006036].

[19] J. A. Harvey and A. Strominger, *Octonionic Superstring Solitons,* Phys. Rev. Lett. **66** (1991) 549.